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TOPOLOGICAL TRANSITIVITY OF ALGEBRAICALLY RECURRENT SETS

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ABSTRACT. In this paper we will discuss the connection between topological transitivity and recurrence of G-flows acting on a compact metric space X. We will prove that the $\mathcal{TT}$-property of the set of all algebraically recurrent points $\mathcal{AR}(\varphi)$ implies chain recurrent properties of the whole space and hence improve some of the results from [6].

1. INTRODUCTION

The area of topological dynamics is large and some of the more technical results have found applications. We will concentrate on those basic aspects which provide a foundation for dynamical system theory in general. Recurrence behavior is surely one of them being one of the most important concepts in topological dynamics. It is a fundamental feature indicating different types of dynamics which serves as a basis for a broad classification of dynamical systems according to their recurrence properties. The best insight to the general framework of this concept is given in [5].

We are going to investigate the properties of sets with different levels of recurrence in the framework of continuous (discrete) dynamical systems employing techniques from topological dynamics. Recall that a topological group is a set G on which two structures are given, a group structure and a topology such that the group operations are continuous. Specifically, a topology τ on G is said to be a group topology if the map $f : G\times G \to G$ defined by $f(x,y) = xy^{-1}$ is continuous. A topological group is a pair (G,τ) of a group G and a group topology τ on G. For every group G the discrete topology and the indiscrete topology on G are trivial examples of group topologies. Non trivial examples are provided by the additive group $\mathbb{R}$ of reals and by the multiplicative group $\mathbb{S}$ of complex numbers z with $|z| = 1$, equipped both with their usual topology. A dynamical system is a topological group G together with a topological space X and a continuous group action $\varphi : X \times G \to X$ of G on X. Sometimes we say that X together with the G-action is a G-flow. In what follows the relating topological group G will be the additive reals $\mathbb{R}$ or integers $\mathbb{Z}$ in the discrete

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case. The trajectory (orbit) of a point x is the set $\gamma(x) = \{\varphi(x,t) | t \in G\}$. By replacing the set G with $G^+ \cup \{0\}$ or $G^- \cup \{0\}$ we obtain the corresponding notions of positive and negative semi trajectory. A subset $Y \subseteq X$ is topologically transitive if it contains a trajectory $\gamma(x)$ which is dense in Y.

Many questions concerning flows involve their long term behavior. One of the important problems in dynamical systems concerns the asymptotic behavior of trajectories as time goes to plus or minus infinity. Limit sets are fundamental tools for this problem.

The positive limit set of a point x is the set:

$$\omega(x) = \{y \in X | \exists t_n \to \infty, \varphi(x,t_n) \to y\}.$$

Analogously, the negative limit set of a point x is the set:

$$\alpha(x) = \{y \in X | \exists t_n \to -\infty, \varphi(x,t_n) \to y\}.$$

A fixed point of a dynamical system φ, exhibits the simplest type of recurrence. We denote by $Fix(\varphi)$ the set of all fixed points of φ. A point carried back to itself by a dynamical system φ exhibits the next most elementary type of recurrence. For a positive $T \in G$ a point $x \in X$ is called T-periodic if $\varphi(x,T) = x$. We denote by $Per_T(\varphi)$ the set of all T-periodic points of φ and we set $Per(\varphi) = \bigcup_{T>0} Per_T(\varphi)$. A set $A \subset X$ is said to be positively recursive with respect to a set $B \subset X$ if for each $T \in G$ there is a $t > T$ and an $x \in B$ such that $\varphi(x,t) \in A$. We will say that a set A is self positively recursive whenever it is positively recursive with respect to itself. A point $x \in X$ is said to be non-wandering if every neighborhood U of x is self positively recursive. We denote by $\Omega(\varphi)$ the set of all non-wandering points of φ. The next level of recurrence for a point $x \in X$ is chain recurrence. Let $(x,y) \in X \times X$ and $\varepsilon > 0$, $t > 0$. (ε,t,φ)-chain from x to y is a collection $\{x = x_1, x_2, ..., x_n, x_{n+1} = y; t_1, t_2, ..., t_n\}$ such that for all $i \in \{1,2,...,n\}$, $t_i \geq t$ and $d(\varphi(x_i,t_i), x_{i+1}) < \varepsilon$. Let

$$P(\varphi) = \{(x,y) | \forall \varepsilon, t > 0, \text{ there exists } (\varepsilon,t,\varphi)\text{-chain from } x \text{ to } y\}.$$

Now $\mathcal{CR}(\varphi) = \{x | (x,x) \in P(\varphi)\}$ is the set of all chain recurrent points for φ. It is known that $Fix(\varphi) \subseteq Per_T(\varphi) \subseteq Per(\varphi) \subseteq \Omega(\varphi) \subseteq \mathcal{CR}(\varphi)$.

2. ALGEBRAIC RECURRENCE

We will discuss the connection between topological transitivity and recurrence of G-flows acting on a compact metric space X. Recall that topological transitivity indicates existence of complicated (dense) orbits. Quite naturally, the transitivity and the frequency at which the neighborhoods are visited play a role in relating dynamic situations. We will need the following definition from [7] introducing a class of fast recurrent points:

Definition 2.1. *A point $a \in X$ is said to be algebraically recurrent for a dynamical system $\varphi : X \times G \to X$ if for every neighborhood U of it there exists a sequence*

$(k(n))_n$ of elements from G^+ such that $\varphi(a,k(n)) \in U$, for every $n \in \mathbb{N}, k(n) \to \infty$ and one of the difference sets:

$$A_0 = \{k(n)|n \in \mathbb{N}\},\ A_n = \{s-t|for\,all\ t < s\ in\ A_{n-1}\}, n \geq 1$$

has bounded gaps i.e. for a suitable $L > 0$, every interval $[\alpha,\beta] \subset \mathbb{R}^+$ with $\beta - \alpha > L$ contains an element in that set. We denote by $\mathcal{AR}(\varphi)$ the set of all algebraically recurrent points of φ.

Roughly speaking, algebraic recurrence means that each neighborhood of the point under attention is visited at polynomial frequency. Every periodic point $a \in Per_T(\varphi)$ is algebraically recurrent. Namely, for $k(n) = nT$ the set $A_0 = \{nT|n \in \mathbb{N}\}$ has bounded gaps, hence the claim follows.

We will prove that the non-wandering set $\Omega(\varphi)$ possess algebraically recurrent points which are not necessarily from $Per(\varphi)$. The following simple recurrence theorem and corollary from [4] will help us in that matter.

Theorem 2.1. *Let φ be a G-flow in a compact metric space X and let $\mathcal{U}$ be an open cover of X. Then there exists an open set $U \in \mathcal{U}$ such that for infinitely many $g_n \in G, g_n \to \infty$, $U \cap \varphi(U, g_n) \neq \emptyset$.*

Proof. Since X is compact we may assume that $\mathcal{U}$ is finite. Consider the action $(g,x) \to \varphi(x,g)$ of G on X. By the infinite pigeonhole principle, for every $x \in X$ there is $U \in \mathcal{U}$ such that for infinitely many $g_n \in G$, $\varphi(x,g_n) \in U$. Let us consider the set $S = \{g \in G|\varphi(x,g) \in U\}$ which according to the previous remark is infinite. Let $g_0 \in S$. Now $\varphi(x,g_0) \in U$ and for every $g \in S$, $\varphi(\varphi(x,g_0), g-g_0) = \varphi(x,g) \in U$ hence $U \cap \varphi(U, g-g_0) \neq \emptyset$. □

Corollary 2.1. *Let φ be a G-flow in a compact metric space X. Then the non-wandering set $\Omega(\varphi)$ is not empty.*

Proof. Let us assume that $\Omega(\varphi) = \emptyset$. Then for every point $x \in X$ there exists a neighborhood U_x of x which is not positively self recursive. The collection $\mathcal{U} = \{U_x|x \in X\}$ is an open cover of X. According to the previous theorem 2.1 there exists a neighborhood $U_p \in \mathcal{U}$ such that for infinitely many $g_n \in G, g_n \to \infty$, $U_p \cap \varphi(U_p, g_n) \neq \emptyset$. Hence U_p is positively self recursive which is a contradiction. The claim follows. □

Remark 2.1. Let us note that the non-wandering set is closed and invariant. Hence in a compact metric space $\Omega(\varphi)$ is a nonempty compact invariant set.

Let us recall that a subset $M \subseteq X$ is called minimal, if it is non-empty, closed and invariant and no proper subset of M has these properties. If $M = X$ then we say that the flow φ is minimal. The following theorem is a slight adaptation of a theorem in [2].

Theorem 2.2. *Let φ be a minimal G-flow in a compact metric space X. Then every trajectory in X is algebraically recurrent.*

Proof. Let us assume that the point $x \in X$ is not algebraically recurrent. Then there is a neighborhood U of x such that the set $A_0 = \{g \in G | \varphi(x,g) \in U\}$ has unbounded gaps. Thus for every $n \in \mathbb{N}$ there is an interval $[\alpha_n, \beta_n] \subset G^+$ with $\beta_n - \alpha_n > n$ such that for all $g \in [\alpha_n, \beta_n], \varphi(x,g) \notin U$. Using the compactness of X we can assume that the sequence $p_n = \varphi(x, \alpha_n)$ is convergent, i.e. $p_n \to y$. Now for arbitrary $g \in G$ we have that $\varphi(p_n, g) = \varphi(\varphi(x, \alpha_n), g) = \varphi(x, \alpha_n + g) \notin U$, for sufficiently large n. But $\varphi(p_n, g) \to \varphi(y, g)$ which implies that x is not in the closure of the trajectory $\gamma(y)$. This contradicts the minimality of X. □

Now using the following theorem (see [2], pp.41)

Theorem 2.3. *Every non-empty compact invariant set contains a compact minimal set.*

we conclude that the previous claim holds true. Namely,

Corollary 2.2. *Let φ be a G-flow in a compact metric space X. Then the non-wandering set $\Omega(\varphi)$ contains algebraically recurrent trajectories.*

Example 2.1. *Consider a dynamical system defined on a torus by means of the planar differential system*

$$\frac{d\phi}{dt} = 1, \frac{d\theta}{dt} = \alpha.$$

Let $\alpha > 0$ be irrational. Then every trajectory is dense in the torus and moreover the torus is also the positive and negative limit set of each point. This example describes algebraically recurrent trajectories which are not periodic. Also note that this is an example of a topologically transitive manifold.

Example 2.2. *Consider the dynamical system defined on the unit square*

$$X = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$$

by means of the planar system of differential equations

$$\frac{dx_1}{dt} = 0, \frac{dx_2}{dt} = -x_1 x_2 (1 - x_1)(1 - x_2).$$

Then X is a compact metric space and the phase portrait is shown in Figure 1 below. The fixed points consist of the set Q,

$$Q = \{(x_1, x_2) \mid x_1 = 1\} \cup \{(x_1, x_2) \mid x_2 = 1\} \cup \{(x_1, x_2) \mid x_1 = 0\} \cup \{(x_1, x_2) \mid x_2 = 0\}.$$

Let us note that every point in this $\mathbb{R}$-flow is chain recurrent, i.e. $X = \mathcal{CR}(\varphi)$. But the only algebraically recurrent points are the fixed points from Q, which coincide with the non-wandering set $\Omega(\varphi)$.

Recall that a point $x \in X$ is called recurrent if $x \in \omega(x)$. The set of all recurrent points is denoted by $\mathcal{R}(\varphi)$. It is known that $Per(\varphi) \subseteq \mathcal{AR}(\varphi) \subseteq \mathcal{R}(\varphi) \subseteq \Omega(\varphi)$.

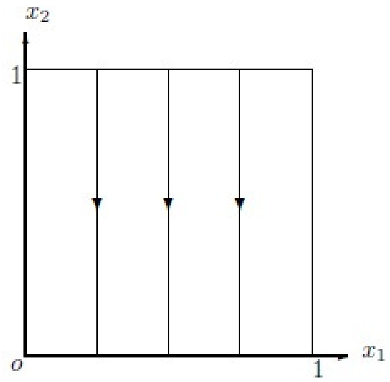

FIGURE 1. Phase portrait of the flow φ on the unit square X

Theorem 2.4. *Let φ be a $\mathbb{R}$-flow in a compact metric space X and let the non-wandering set be the whole space, i.e. $\Omega(\varphi) = X$. If the trajectories $\gamma(x)$ are locally compact in $\gamma(x)$ for every $x \in Q$, where Q is some subset of $\mathcal{R}(\varphi)$, then $Q \subseteq Per(\varphi)$. Furthermore if Q is such that $\overline{Q} = \overline{\mathcal{R}(\varphi)}$, then the set of all periodic trajectories $Per(\varphi)$ is dense in X.*

Proof. In what follows we assume that there exists a point $x \in Q$ such that x is not periodic. First let us note that from $x \in \mathcal{R}(\varphi)$ we have that $\gamma(x) \subset \omega(x)$ using the invariance property of limit sets. We will need the following lemma:

Lemma 2.1. *The set $\varphi(x,[n,n+1])$ has an empty interior in $\gamma(x)$ for every $n \in \mathbb{Z}$.*

Proof. Let $n \in \mathbb{Z}$ and $p \in \varphi(x,[n,n+1])$ be arbitrary and let V_p be an arbitrary neighborhood of p. Then according to the previous remark there exists a sequence $(t_m)_m, t_m \to \infty$ such that $\varphi(x,t_m) \to p$. Hence for sufficiently large $m \in \mathbb{N}$ we have that $n+1 < t_m$ and $\varphi(x,t_m) \in V_p$ (having in mind the convergence of the sequence $(\varphi(x,t_m))_m$). From the assumption, the point x is not periodic so we can conclude that the map $t \to \varphi(x,t)$ is injective. Hence $\varphi(x,t_m) \notin \varphi(x,[n,n+1])$ which means that $V_p \nsubseteq \varphi(x,[n,n+1])$. The choice of $p \in \varphi(x,[n,n+1])$ was arbitrary so we can conclude that $int_{\gamma(x)}(\varphi(x,[n,n+1])) = \emptyset$. □

Now from the conditions in the theorem the trajectory $\gamma(x)$ is a locally compact Hausdorff space. On the other hand for arbitrary $n \in \mathbb{Z}$, $\varphi(x,[n,n+1])$ is a compact set as a continuous image of a compact set $[n,n+1]$ into a metric space X. Hence the sets $\varphi(x,[n,n+1])$ are closed for arbitrary $n \in \mathbb{Z}$. This means that the sets $\varphi(x,[n,n+1])$ are closed and with an empty interior for arbitrary $n \in \mathbb{Z}$ according to the previous lemma 2.1. Now from the equality $\gamma(x) = \bigcup_{n \in \mathbb{Z}} \varphi(x,[n,n+1])$
we have that $\gamma(x)$ is a countable union of closed sets with an empty interior. But from the Baire category theorem a Hausdorff locally compact space $\gamma(x)$ is of second category. This is a contradiction with the previous equality. Hence x is indeed a

periodic point. This means that $Q \subseteq Per(\varphi)$. This concludes the proof of the first part of the theorem. Let us note that from the inclusion $Per(\varphi) \subseteq \mathcal{R}(\varphi)$ if $\overline{Q} = \overline{\mathcal{R}(\varphi)}$, the following equality holds $\overline{Q} = \overline{Per(\varphi)} = \overline{\mathcal{R}(\varphi)}$. Now let us discuss the second part. We will introduce a sequence of sets which mimic recurrence behavior for a fixed neighborhood. We will have two free control indexes for the time evolution and the size of the neighborhood. Let us consider the following sets:

$$V_{k,n} = \bigcup_{t \geq n} \{x \in X | d(x, \varphi(x,t)) < \tfrac{1}{k}\}.$$

The set of recurrent points is exactly the intersection of the sets $V_{k,n}$, for all $k, n \geq 1$, i.e. the following holds: $\mathcal{R}(\varphi) = \bigcap_{k,n \geq 1} V_{k,n}$. Let us note that the sets $V_{k,n}$ are open for all $k, n \geq 1$. Now from the conditions in the theorem there are no wandering points, i.e. every point is non-wandering. We will prove that the sets $V_{k,n}$ are dense in X for all $k, n \geq 1$. Let us choose an arbitrary point $p \in X$ and let us consider the set $V_{k,n}$ for arbitrary $k, n \geq 1$. We will choose an arbitrary neighborhood U of p such that $U \subseteq B(p, \frac{1}{2k})$ (open ball centered at p). Now from the fact that the point p is non-wandering there exists $t \geq n$ such that $\varphi(U, t) \cap U \neq \emptyset$. Hence there exists $z \in U$ such that $\varphi(z, t) \in U$. Now from the triangle inequality it easily follows that $z \in V_{k,n}$. Hence $U \cap V_{k,n} \neq \emptyset$, i.e. the set $V_{k,n}$ is dense in X. Let us note that every compact metric space is a Baire space hence from the Baire category theorem we conclude that $\mathcal{R}(\varphi)$ is dense in X. Finally from the equalities $\overline{Q} = \overline{\mathcal{R}(\varphi)} = \overline{Per(\varphi)}$ the proof is complete. □

Remark 2.2. The density of the recurrent set $\mathcal{R}(\varphi)$ is also mentioned in [3].

Remark 2.3. Let us note that a similar claim does not hold for the chain recurrent set $\mathcal{CR}(\varphi)$. Namely, in example 2.2 the whole space is chain recurrent, i.e. $X = \mathcal{CR}(\varphi)$. Also for arbitrary $x \in \mathcal{R}(\varphi)$ the trajectory $\gamma(x)$ is locally compact in $\gamma(x)$ but $Per(\varphi)$ is not dense in X.

The following theorem from [7] provides sufficient conditions for minimality or sensitivity. In the next section we will discuss this theorem in the case of $\mathcal{TT}$-property of $\mathcal{AR}(\varphi)$.

Theorem 2.5. *Let $\varphi : X \times G \to X$ be a G-flow satisfying the following two conditions:*

($\mathcal{TT}$) *φ is topologically transitive*;
($\mathcal{AR}$) *The set of all algebraically recurrent points is dense.*

Then either φ shows sensitive dependence on initial conditions or φ is nonsensitive and minimal.

Example 2.3. *The dynamical system from example 2.1 is topologically transitive and the set of all algebraically recurrent points $\mathcal{AR}(\varphi) = X$ is dense in X. Hence it is nonsensitive and minimal.*

3. Topological Transitivity of $\mathcal{AR}(\varphi)$, $\Omega(\varphi)$ and $\mathcal{CR}(\varphi)$

Topological transitivity is a global characteristic of a dynamical system. Although the local structure of topologically transitive dynamical system fulfills certain conditions, there is variety of such systems. Some of them have dense periodic points while some of them may be minimal and without any periodic points. We will present a few results connecting the last level of recurrence, chain recurrence with the topological transitivity of the non-wandering set $\Omega(\varphi)$ and the algebraically recurrent set $\mathcal{AR}(\varphi)$. We will show the connection between $\mathcal{TT}$-property of these sets with the recurrent property of the whole phase space X.

We will discuss first some special cases of topological transitivity of the non-wandering set $\Omega(\varphi)$ or the chain recurrent set $\mathcal{CR}(\varphi)$ and note that the $\mathcal{TT}$-property of these sets implies certain recurrent properties of the whole space X. We will generalize this claim in what follows for the case $G = \mathbb{R}$.

Theorem 3.1. *Let φ be a G-flow in a compact metric space X. If $\Omega(\varphi) = \gamma(x)$ for some $x \in X$, then $\Omega(\varphi) = Per(\varphi)$ and $X = \mathcal{CR}(\varphi)$. Consequently if $\mathcal{CR}(\varphi) = \gamma(x)$, then $\mathcal{CR}(\varphi) = Per(\varphi) = X$.*

Proof. We shall discuss the first part of the claim first. Although the restriction of the flow to the non-wandering set does not admit necessarily a non-wandering set which coincides with the non-wandering set from the whole space X (see example 3.1 for instance) we shall prove that in this case the previous claim holds i.e. $\Omega_X(\varphi) = \Omega_{|\Omega(\varphi)}(\varphi)$. Let us note that $\Omega_X(\varphi)$ is a non-empty compact invariant set. Hence according to corollary 2.2 the non-wandering set $\Omega_X(\varphi)$ contains an algebraically recurrent point. It follows that $\gamma(x) = \mathcal{AR}(\varphi) = \mathcal{R}(\varphi)$ using the invariance of the algebraically recurrent set $\mathcal{AR}(\varphi)$. Now let us discuss the equality $\Omega_X(\varphi) = \Omega_{|\Omega(\varphi)}(\varphi)$. Let $z \in \Omega_X(\varphi) = \gamma(x)$ be an arbitrary point. It follows that for an arbitrary neighborhood U of z in X and arbitrary $g_0 \in G^+$ there exists $t > g_0$ such that $\varphi(U,t) \cap U \neq \emptyset$. Now let us choose an arbitrary neighborhood V of z in $\Omega_X(\varphi)$. Then there exists an open neighborhood V_z of z in X such that $V = V_z \cap \Omega_X(\varphi)$. Note that z is a recurrent point which implies that there exists a $t > g_0$ such that $\varphi(z,t) \in V_z$. But the non-wandering set $\Omega_X(\varphi)$ is invariant which implies that $\varphi(z,t) \in \Omega_X(\varphi)$ as well. Hence $\varphi(z,t) \in V_z \cap \Omega_X(\varphi) = V$. This means that $\varphi(V,t) \cap V \neq \emptyset$ so we can conclude that $\Omega_X(\varphi) \subseteq \Omega_{|\Omega(\varphi)}(\varphi)$. Note that the opposite inclusion $\Omega_{|\Omega(\varphi)}(\varphi) \subseteq \Omega_X(\varphi)$ trivially holds. Hence the equality follows. Now this fact combined with the fact that the trajectory $\gamma(x)$ is locally compact in $\gamma(x)$ for every $x \in \mathcal{R}(\varphi)$ enables us to use theorem 2.4 which implies that the set of periodic points $Per(\varphi)$ is dense in $\gamma(x)$. Hence $\Omega(\varphi) = \gamma(x) = Per(\varphi)$. The equality $X = \mathcal{CR}(\varphi)$ follows from the fact that the limit sets $\alpha(y)$ and $\omega(y)$ are contained in $\Omega(\varphi) = \gamma(x) = Per(\varphi)$ for arbitrary $y \in X$ (see [2], pp.35). The details of the proof are given in the second part. This completes the first part of the proof. For the second part let us note that $\mathcal{CR}(\varphi) = \gamma(x)$ implies that $\Omega(\varphi) = \mathcal{CR}(\varphi) = \gamma(x)$, by use of the invariance property of $\Omega(\varphi)$, which explains the relation $\mathcal{CR}(\varphi) = Per(\varphi)$. If we assume that there exists $y \in X \setminus \mathcal{CR}(\varphi)$, then $\alpha(y) = \omega(y) = Per(\varphi)$ having in mind that $\mathcal{CR}(\varphi)$ contains the limit sets. But then for arbitrary $\varepsilon > 0, t > 0$ from the point y we can get arbitrary close to $Per(\varphi)$ by going with the flow in positive time $T_1 \geq t$. Then we can make a small ε-jump on the periodic trajectory and make enough rotation to obtain time evolution $T_2 \geq t$. Then again we can make a small ε-jump back on the trajectory of y but now on the negative tail which is arbitrary close to $\alpha(y)$. Finally again going with the flow for arbitrary large $T_3 \geq t$ we can go back to y. Hence $y \in \mathcal{CR}(\varphi)$ which is a contradiction as well. The proof is complete. □

Remark 3.1. Let us note that topological transitivity of the non-wandering set $\Omega(\varphi)$ does not imply that $\Omega(\varphi) = X$. Consider for example a $\mathbb{R}$-flow on the sphere $\mathbb{S}^1$ that contains exactly one rest point p. For the points $x \in \mathbb{S}^1, x \neq p$ let $\alpha(x) = \omega(x) = p$. We can easily conclude that the non-wandering set $\Omega(\varphi) = \{p\}$ and hence that it is topologically transitive. Nevertheless, $\Omega(\varphi) = \{p\} \neq \mathbb{S}^1 = X$.

Example 3.1. *Consider the following differential system defined in the Euclidean plane* $\mathbb{R}^2$ *by* (*polar coordinates*)

$$\frac{dr}{dt} = r(1-r), \frac{d\theta}{dt} = \begin{cases} sin^2(\theta) + \frac{1}{log 3}, 0 < r \leq \frac{3}{4}, \\ sin^2(\theta) + \frac{1}{log \frac{r}{1-r}}, \frac{3}{4} < r < 1, \\ sin^2(\theta), r = 1, \\ sin^2(\theta) + \frac{1}{log \frac{r}{r-1}}, r > 1. \end{cases}$$

The phase portrait of this system is given in Figure 2 below. Notice that if we restrict the flow on the unit disk, then $\Omega(\varphi) = \mathbb{S}^1 \cup \{(0,0)\}$. *But if we consider the flow on* $\Omega(\varphi) = \mathbb{S}^1 \cup \{(0,0)\}$, *then* $\Omega_{|\Omega(\varphi)} = \{(0,0),(1,0),(1,\pi)\}$. *Hence* $\Omega(\varphi) \neq \Omega_{|\Omega(\varphi)}(\varphi)$.

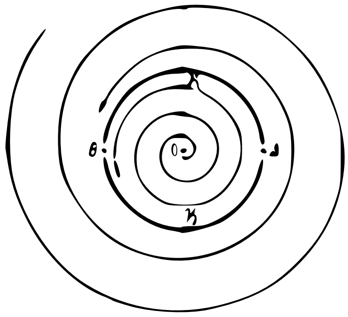

FIGURE 2. Phase portrait of the plane flow φ

We introduce the notation $\mathcal{AR}(\varphi) = \bigcup_{n=0}^{\infty} \mathcal{AR}_n(\varphi)$, where $a \in \mathcal{AR}_n(\varphi)$ if and only if the set A_n has bounded gaps.

Definition 3.1. *If an algebraically recurrent point x is algebraically recurrent for the reverse flow* $\varphi^-(x,t) = \varphi(x,-t)$ *as well, we say that x is algebraically recurrent in both directions.*

Now a question is imposed. Can we expect a certain level of recurrence behavior of the whole space X when discussing the general case of topological transitivity of $\Omega(\varphi)$ or $\mathcal{CR}(\varphi)$? Let us include the topological transitivity of $\mathcal{AR}(\varphi)$ in the discussion. So the question is "Can we expect a certain level of recurrence behavior of the whole space X when discussing the general case of topological transitivity of $\Omega(\varphi)$, $\mathcal{CR}(\varphi)$ or $\mathcal{AR}(\varphi)$"? The answer is affirmative but we shall prove that the $\mathcal{TT}$-property of these sets actually is not essential for the recurrent behavior of the whole space X. The following claim makes some general conclusions regarding the recurrent properties of X for the case $G = \mathbb{R}$.

Theorem 3.2. *Let φ be a G-flow in a compact metric space X. The set of all algebraically recurrent points $\mathcal{AR}(\varphi)$ is contained in a connected component of $\mathcal{CR}(\varphi)$ if and only if $X = \mathcal{CR}(\varphi)$ and X is connected. Consequently if $\mathcal{AR}(\varphi)$ has the $\mathcal{TT}$-property i.e. $\mathcal{AR}(\varphi) = \overline{\gamma(a)}$ for some $a \in X$, then the whole space is chain recurrent i.e. $X = \mathcal{CR}(\varphi)$. Furthermore if $a \in \mathcal{AR}_0(\varphi) \cap \mathcal{AR}_0(\varphi^-)$, then $\mathcal{AR}(\varphi)$ is minimal.*

Proof. Let us assume that the set of all algebraically recurrent points $\mathcal{AR}(\varphi)$ is contained in a connected component of $\mathcal{CR}(\varphi)$. Then using the fact that the connected components of $\mathcal{CR}(\varphi)$ coincide with the chain transitive components (see [1]) we can conclude that there exists a chain transitive component M_1 such that $\mathcal{AR}(\varphi) \subseteq M_1$ (chain transitive component means that any two points $x, y \in M_1$ can be connected by a finite $(\varepsilon, t, \varphi)$-chain from x to y in M_1). Now let us assume that there exists another non-empty chain transitive component $M_2 \neq M_1$ different from M_1. It follows that M_2 is a non-empty compact invariant set disjoint from M_1 i.e. $M_1 \cap M_2 = \emptyset$. Hence using theorems 2.3 and 2.2 we conclude that there exists an algebraically recurrent point in M_2. But this means that $\mathcal{AR}(\varphi) \cap M_2 \neq \emptyset$ which yields that $M_1 \cap M_2 \neq \emptyset$. Hence a contradiction. So we can conclude that there exists only one chain transitive component and hence only one connected chain recurrent component. Now imitating the discussion from the last part of the previous Theorem 3.1 it easily follows that $X \setminus \mathcal{CR} = \emptyset$ (instead of rotations just use $(\varepsilon, t, \varphi)$-chains from the chain transitive component). The conclusion follows. The opposite direction is trivial. Now let us assume that $\mathcal{AR}(\varphi)$ has the $\mathcal{TT}$-property. This implies that $\mathcal{AR}(\varphi)$ is connected. Hence it is contained in a connected component of the chain recurrent set $\mathcal{CR}(\varphi)$. Now the claim follows from the previous discussion. For the second part first let us note that $\mathcal{AR}(\varphi)$ is an invariant set. From the assumption of topological transitivity we can conclude that $\mathcal{AR}(\varphi)$ is a compact invariant set. We choose an arbitrary neighborhood U of a in $\mathcal{AR}(\varphi)$. Let us also note that if $a \in \mathcal{AR}_0(\varphi) \cap \mathcal{AR}_0(\varphi^-)$ is algebraically recurrent in both directions, then there exists an $L \geq 0$ such that the interval $[-L, L]$ contains an element in A_0 (and any arbitrary interval with length at least $2L$). Now from the compactness of $\overline{\gamma(a)}$ it follows that for arbitrary $\varepsilon > 0$, there is a $\delta > 0$ such that $d(\varphi(x,t), \varphi(y,t)) < \varepsilon$, for $t \in [-L, L]$ whenever $x, y \in \overline{\gamma(a)}$ and $d(x,y) < \delta$. Now for arbitrary $z \in \overline{\gamma(a)}$ we can easily conclude that the trajectory of z will eventually enter the neighborhood U. Hence $\varphi(U,G) = \overline{\gamma(a)}$. Now using the compactness of $\overline{\gamma(a)}$ there exists a finite set $F \subset G$ such that $\varphi(U,F) = \overline{\gamma(a)}$. Let us suppose that $\mathcal{AR}(\varphi)$ is not minimal and let $Y \subseteq \mathcal{AR}(\varphi)$ be a non-empty proper closed G-invariant subset of $\mathcal{AR}(\varphi)$. Note that $a \in Y^c$. Let us choose a non-empty open neighborhood U of a such that $U \subseteq \overline{U} \subseteq Y^c$. Also note that the inclusion $\overline{U} \subseteq Y^c$ in normal spaces can always be done. Now from the previous discussion there exists a finite set $F \subset G$ such that $\varphi(U,F) = \overline{\gamma(a)} = \mathcal{AR}(\varphi)$. Hence $\overline{\varphi(U,F)} = \varphi(\overline{U},F) = \mathcal{AR}(\varphi)$ which contradicts that $\overline{U}$ is a subset of Y^c. The conclusion follows. □

Now as a corollary of our previous result we obtain the results from [6] concerning the topological transitivity of $\Omega(\varphi)$ and $\mathcal{CR}(\varphi)$.

Corollary 3.1. *If the non-wandering set $\Omega(\varphi)$ or the chain recurrent set $\mathcal{CR}(\varphi)$ has the $\mathcal{TT}$-property, then the whole space is chain recurrent i.e. $X = \mathcal{CR}(\varphi)$.*

Proof. If the non-wandering set $\Omega(\varphi)$ or the chain recurrent set $\mathcal{CR}(\varphi)$ has the $\mathcal{TT}$-property, then they are connected hence the set of all algebraically recurrent points $\mathcal{AR}(\varphi)$ as their subset is contained in a connected component of $\mathcal{CR}(\varphi)$. Now the claim follows from Theorem 3.2. □

Remark 3.2. In example 2.2 none of the sets $\Omega(\varphi), \mathcal{CR}(\varphi)$ or $\mathcal{AR}(\varphi)$ are $\mathcal{TT}$ but $\mathcal{AR}(\varphi)$ is contained in a connected component of $\mathcal{CR}(\varphi)$. Hence $X = \mathcal{CR}(\varphi)$.

The following corollary involves another recurrent concept of almost periodic points. For the definition of almost periodic points see [2], pp.106.

Corollary 3.2. *Let $\mathcal{AR}(\varphi) = \mathcal{CR}(\varphi)$. If $\mathcal{AR}(\varphi) = \overline{\gamma(a)}$, for some almost periodic point $a \in X$, then φ is nonsensitive and minimal.*

Proof. From the previous Theorem 3.2 we can conclude that $X = \mathcal{CR}(\varphi)$ and hence the whole space is topologically transitive. Furthermore $\mathcal{AR}(\varphi)$ is dense in X so according to Theorem 2.5 either φ shows sensitive dependence on initial conditions or φ is nonsensitive and minimal. But since a is an almost periodic point it follows that $a \in \mathcal{AR}_0(\varphi) \cap \mathcal{AR}_0(\varphi^-)$ (see [2], pp.107), so according to the previous Theorem 3.2 $X = \mathcal{AR}(\varphi)$ is minimal. Hence φ is nonsensitive and minimal (see [2], pp.108). □

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