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CONVERGENCE OF THE GENERALIZED STEFFENSEN METHOD IN RIEMANNIAN MANIFOLDS

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ABSTRACT. In this article, we present semilocal convergence of the generalized Steffensen method in Riemannian manifolds to find the singular points of a vector field. We establish the convergence under the Kantarovich–Ostrowski's conditions. Finally, two examples are given to show the applicability of our convergence analysis.

1. INTRODUCTION

A lot of problems in applied sciences and others which includes engineering, optimization, dynamic economic system, physics, biological problems that are abstract can be formulated as finding the solution of nonlinear equations [1–5]. The exact solution of these equations is difficult to find. Therefore one can use iterative methods to solve these equations. There are so many iterative methods used to find zeros of nonlinear equations in Banach spaces and Newton method is one such highly used method. As in some cases the existence of invertibility of the derivative of the operator can't be guaranteed, some authors considered Kurchatov method [15, 16], generalized Steffensen method and many other's in Banach spaces and studied their convergence. Some study the convergence of iterative methods usually based on semilocal and local convergence analysis. If the convergence analysis which gives information about a solution and estimates the radius of the convergence ball, then it is said to be local convergence where as if the convergence analysis tells information about an initial point, then it is said to be semilocal convergence. The generalized Steffensen method [13] in Banach space is :

$$\left.\begin{aligned} &l_0 \in \Omega, \\ &m_n = l_n + \beta_n E(l_n), \\ &l_{n+1} = m_n - [l_n, m_n; E]^{-1} E(m_n), \text{ for each } n = 0, 1, 2, \ldots, \end{aligned}\right\} \tag{1.1}$$

where E is a mapping from $\Omega \subset B$ to B which is a nonlinear operator, B is a Banach space, Ω is an open convex set and $\beta_n \in (0, 1]$. Unlike Newton's method, this method does not contain derivatives of the operator. Here the authors studied semilo-

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cal convergence under Lipschitz-type convergence conditions. Recently, there has been a growing interest in studying numerical iterative methods in Riemannian manifolds(see [7] and references there in), since there are many numerical problems in manifolds that may arise in many contexts. Some examples which include the eigenvalue problem, minimization problems with orthogonality constraints, optimization problems with equality constraints, invariant subspace computations etc. For such types of problems, one has to find solutions of equations or to find zeros of a vector field in Riemannian manifolds. Now in Riemannian manifolds the study of convergence analysis of iterative methods began with the work of Ferreira and Svaiter [7] in 2002, where the Kantorovich's method (Newton) was extended into this context. Dedieu et.al. [6, 11] generalized the Smale's α-theory and γ-theory in intrinsic Newton's method in a complete manifold by using the work of Ferreira and Svaiter. The assumptions for the convergence of Newton's method in Riemannian manifolds has been weakened by Argyros [8]. Later the studies of Newton's method on Lie groups has been studied by Argyros [9] and He et.al. [10]. Sometimes the extension of iterative methods into Riemannian manifolds creates new difficulty. For example, the vector field defined on a Banach space is a mapping from a Banach space into itself, while in manifolds, it is a mapping from the manifold to the tangent bundle. Sometimes, it can be difficult to find the radius of convergence and uniqueness of the solution in the open ball which is defined by such a radius [14].

In this article, we study the generalized Steffensen method (1.1) in Riemannian manifolds. Such a study was presented in a Banach space setting in [13].

This article is divided into five sections : Section 1 is the introduction. In Section 2, we discuss all the necessary background on fundamental properties and notation of Riemannian manifolds. In Section 3, we present the semilocal convergence of the generalized Steffensen method in Riemannian manifolds. In Section 4, two examples are provided. Section 5, forms the conclusion.

2. PRELIMINARIES

In this section, we discuss some basic definitions and necessary results in Riemannian manifolds.

Let W be a real $n-$ dimensional Riemannian manifold. The tangent space of W at e is denoted by T_eW. The inner product $\langle .,. \rangle_e$ on T_eW induces the norm $\|.\|_e$. The tangent bundle of W is denoted by TW and is defined by

$$TW := \{(e,v); e \in W \text{ and } v \in T_eW\} = \bigcup_{e\in W} T_eW.$$

Let $e,t \in W$ and $\phi : [0,1] \to W$ be a piecewise smooth curve joining the points e and t. Then the arc length of ϕ is defined by

$$l(\phi) = \int_0^1 \|\phi'(x)\| dx = \int_0^1 \left\langle \frac{d\phi}{dx}, \frac{d\phi}{dx} \right\rangle^{\frac{1}{2}} dx$$

and the Riemannian distance from e to t is defined by $d(e,t) = \inf_{\phi} l(\phi)$, where the infimum is taken over all the piecewise smooth curves ϕ connecting e and t.

Let $\chi(W)$ be the set of all vector fields of class C^∞ on W and $D(W)$ be the ring of real-valued functions of class C^∞ defined on W. An affine connection ∇ on W is a mapping from $\chi(W)\times\chi(W)$ to $\chi(W)$ defined by $(X,H)\mapsto \nabla_X H$, that satisfies the following properties

$$\text{(i)}\quad \nabla_{fX+gH}\mathfrak{V} = f\nabla_X\mathfrak{V} + g\nabla_H\mathfrak{V}.$$
$$\text{(ii)}\quad \nabla_X(H+\mathfrak{V}) = \nabla_X H + \nabla_X\mathfrak{V}.$$
$$\text{(iii)}\quad \nabla_X(fH) = f\nabla_X H + X(f)H,$$

where $X,\ H,\ \mathfrak{V}\in\chi(W)$ and $f\ ,g\in D(W)$.

Definition 2.1. *Let H be a vector field of class C^1 on W, the covariant derivative of H is determined by the connection ∇ which defines on each $e\in W$ a linear application of T_eW itself*

$$DH(e): T_eW \to T_eW$$
$$v\mapsto DH(e)(v) = \nabla_X H(e),$$

where X is a vector field satisfying $X(e) = v$.

Definition 2.2. *A parametrized curve $\phi: I\subseteq\mathbb{R}\to W$ is a geodesic at $p_0\in I$, if $\nabla_{\phi'(p)}\phi'(p) = 0$ at the point p_0. If ϕ is a geodesic for all $p\in I$, we say ϕ is a geodesic. If $[x,y]\subseteq I$, then ϕ is a geodesic segment joining $\phi(x)$ to $\phi(y)$. A basic property of geodesics is that, $\phi'(p)$ is parallel along $\phi(p)$ and therefore $\|\phi'(p)\|$ is constant. Let $U(e,s)$ and $U[e,s]$ be an open and a closed geodesic ball with centre e and radius s respectively. By the Hopf-Rinow theorem, if W is a complete metric space, then for any $e,t\in W$ there exists a geodesic ϕ called the minimizing geodesic joining e to t with*

$$l(\phi) = d(e,t).$$

If $v\in T_eW$, then there exists a unique minimizing geodesic ϕ such that $\phi(0) = e$ and $\phi'(0) = v$. The point $\phi(1)$ is called the image of v by the exponential map at e, i.e.

$$\exp_e : T_eW \to W$$

such that $\exp_e(v) = \phi(1)$ *and* $\phi(p) = \exp_e(pv)$ *for any* $p\in[0,1]$.

Definition 2.3. *Let ϕ be a piecewise smooth curve. Then for any $x,y\in\mathbb{R}$, the parallel transport along ϕ is denoted by $S_{\phi,\cdot,\cdot}$ and given by*

$$S_{\phi,x,y}: T_{\phi(x)}W \to T_{\phi(y)}W$$
$$v\mapsto V(\phi(y)),$$

where V is the unique vector field along ϕ such that $\nabla_{\phi'(p)}V = 0$ and $V(\phi(x)) = v$.

Definition 2.4. *Let $j\in\mathbb{N}$ and H be a vector field of class C^k. The covariant derivative of order j of H is denoted by D^jH and defined as the multilinear map*

$$D^jH : \underbrace{C^k(TW) \times C^k(TW) \times \cdots \times C^k(TW)}_{j\text{-times}} \to C^{k-j}(TW)$$

which is given by

$$\begin{aligned} D^jH(A_1,A_2,\ldots,A_{j-1},A) &= \nabla_A D^{j-1}H(A_1,A_2,\ldots,A_{j-1}) \\ &- \sum_{i=1}^{j-1} D^{j-1}H(A_1,A_2,\ldots,\nabla_A A_i,\ldots,A_{j-1}) \end{aligned} \tag{2.1}$$

for all $A_1,A_2,\ldots,A_{j-1} \in C^k(TW)$.

Definition 2.5. *Let* W *be a Riemannian manifold,* $\Omega \subseteq W$ *be an open convex set and* $H \in \chi(W)$. *The covariant derivative* $DH = \nabla_{(.)}H$ *is Lipschitz with constant* $M > 0$, *if for any geodesic* ϕ *and* $x, y \in \mathbb{R}$ *such that* $\phi[x,y] \subseteq \Omega$, *it satisfies the inequality*

$$\|S_{\phi,y,x}DH(\phi(y))S_{\phi,x,y} - DH(\phi(x))\| \leq M\int_x^y \|\phi'(p)\|dp$$

and we write $DH \in Lip_M(\Omega)$. *If* $W = \mathbb{R}^n$, *then the above definition is the same as the usual Lipschitz condition for the operator* $DH : W \to W$.

Definition 2.6. *Let* W *be a Riemannian manifold and* $\Omega \subseteq W$ *be an open convex set. Let* ϕ *be a curve in* W *such that* $[b,d] \subset dom(\phi)$ *and* H *be a vector field of class* C^0 *on* W. *The divided difference of first order for* H *on the points* $\phi(e)$ *and* $\phi(e+z)$ *in the direction* $\phi'(e)$ *is given by*

$$[\phi(e+z),\phi(e);H]\phi'(e) = \frac{1}{z}\big(S_{\phi,e+z,e}H(\phi(e+z)) - H(\phi(e))\big). \tag{2.2}$$

If W *is a Banach space and* ϕ *is a geodesic joining the points* e_1 *and* e_2 *such that* $\phi(a) = e_1 + a(e_2 - e_1)$, $a \in \mathbb{R}$, *then, we have*

$$[e_2,e_1;H](e_2-e_1) = H(e_2) - H(e_1)$$

and also if $DH(l)$ *exists, then* $DH(l) = [l,l;H]$.

Proposition 2.1. Let ϕ be a curve in W and H be a C^1 vector field on W. The covariant derivative of H in the direction of $\phi'(t)$ is defined as

$$DH(\phi(t))\phi'(t) = \nabla_{\phi'(t)}H_{\phi(t)} = \lim_{\delta x \to 0}\frac{1}{\delta x}\big(S_{\phi,t+\delta x,t}H(\phi(t+\delta x)) - H(\phi(t))\big).$$

If $W = \mathbb{R}^n$, then it is same as the directional derivative in $\mathbb{R}^n$.

Next, we will prove Taylor's type expansions in Riemannian manifolds that are useful to prove our convergence theorem.

Theorem 2.1. *Let* ϕ *be a geodesic in* W *such that* $[0,1] \subseteq Dom(\phi)$ *and* H *be a* C^1*-vector field on* W. *Then,*

$$S_{\phi,t,0}H(\phi(t)) = H(\phi(0)) + \int_0^t S_{\phi,z,0}DH(\phi(z))\phi'(z)dz.$$

Proof. See [12] □

3. Generalized Steffensen method in Riemannian manifolds

In this section, we will provide convergence analysis of the generalized Steffensen method (1.1) in Riemannian manifolds. The given method in Riemannian manifolds has the form

$$\left.\begin{aligned} u_n &= \beta_n H(e_n),\\ i_n &= \exp_{e_n}(u_n),\\ v_n &= -[e_n, i_n; H]^{-1} H(i_n),\\ e_{n+1} &= \exp_{i_n}(v_n), \text{ for each } n = 0,1,2,\ldots, \end{aligned}\right\} \tag{3.1}$$

where $\beta_n \in (0,1]$. We define the following scalar sequences in order to study convergence of the method (3.1) given as follows:

$$\left.\begin{aligned} \mathfrak{b}_n &= \mathfrak{c}_n + \beta_n g(\mathfrak{c}_n),\\ \mathfrak{c}_{n+1} &= \mathfrak{b}_n - \frac{g(\mathfrak{b}_n)}{g'(\mathfrak{b}_n) - z_n}, \end{aligned}\right\} \tag{3.2}$$

where $\mathfrak{c}_0 = 0,\ z_n > 0,\ g(\mathfrak{c}) = \frac{K}{2}\mathfrak{c}^2 - \mathfrak{c} + b,\ b, K > 0$. Let a_1 and a_2 be the two positive roots of $g(\mathfrak{c})$, $Kb < \frac{1}{2}$. Assume that $a_1 < a_2$ and $H(e)$ satisfies the following conditions:

(1) $\|H(e_0)\| \leq b$,
(2) $\|DH(e_0)^{-1}\| \leq 1$,
(3) $\|S_{\Phi,b,a} DH(\Phi(b)) S_{\Phi,a,b} - DH(\Phi(a))\| \leq K \int_a^b \|\Phi'(x)\| dx$,
where Φ is the geodesic such that $\Phi[a,b] \subseteq \Omega$,
(4) $\|\frac{I_{e_n}}{\beta_n} + S_{\eta_n,1,0} DH(e_0) S_{\eta_n,0,1}\| \leq \frac{1}{\beta_n} - 1$,
(5) $U(e_0, a_1) \subset \Omega$,

where η_n is the family of minimizing geodesics such that $\eta_n(0) = e_n$ and $\eta_n(1) = e_0$ for each $n = 0,1,2,\ldots$, and I_{e_n} is the identity operator on $T_{e_n}W$. To prove the convergence of the iterative method first we shall prove some Lemmas given below.

Lemma 1. Let $e, i \in \Omega$ and assume that $d(e,i) \leq |\mathfrak{c} - \mathfrak{b}|$. Then,

$$\|S_{\rho,1,0} DH(e) S_{\rho,0,1} - DH(i)\| \leq |g'(\mathfrak{c}) - g'(\mathfrak{b})|,$$

where $\rho : [0,1] \to W$ is the minimizing geodesic such that $\rho(0) = i$ and $\rho(1) = e$.

Proof. We have

$$\|S_{\rho,1,0} DH(e) S_{\rho,0,1} - DH(i)\| \leq K \int_0^1 \|\rho'(0))\| ds = K d(e,i) \tag{3.3}$$

and

$$g'(\mathfrak{c}) - g'(\mathfrak{b}) = \int_0^1 g''(\mathfrak{b} + t(\mathfrak{c} - \mathfrak{b}))(\mathfrak{c} - \mathfrak{b}) dt.$$

Taking the modulus on both sides, we get

$$|g'(\mathfrak{c}) - g'(\mathfrak{b})| = \int_0^1 |g''(\mathfrak{b} + t(\mathfrak{c} - \mathfrak{b}))| |(\mathfrak{c} - \mathfrak{b})| dt = K|\mathfrak{c} - \mathfrak{b}|. \tag{3.4}$$

From (3.3) and (3.4), we get

$$\|S_{\rho,1,0}DH(e)S_{\rho,0,1} - DH(i)\| \leq |g'(\mathfrak{c}) - g'(\mathfrak{b})|. \qquad \square$$

Lemma 2. Suppose the sequences $\{\mathfrak{c}_n\}$ and $\{\mathfrak{b}_n\}$ are generated by (3.2). If $Kb < \frac{1}{2}$, then the sequences $\{\mathfrak{c}_n\}$ and $\{\mathfrak{b}_n\}$ are increasing and bounded from above. Hence they converge to a_1.

Proof. We define the function $\mathfrak{K}$ by

$$\mathfrak{K}(\mathfrak{c}) = \mathfrak{c} + \beta_n g(\mathfrak{c}) - \frac{g(\mathfrak{c} + \beta_n g(\mathfrak{c}))}{g'(\mathfrak{c} + \beta_n g(\mathfrak{c})) - z_n}.$$

As $g(\mathfrak{c}) > 0,\ g''(\mathfrak{c}) > 0,\ g'(\mathfrak{c}) < 0$ in $[0, a_1)$ we get $\mathfrak{K}'(\mathfrak{c}) > 0\ \forall\ \mathfrak{c} \in [0, a_1)$. Therefore the function $\mathfrak{K}$ is increasing on $[0, a_1)$. So, if $\mathfrak{c}_k \in [0, a_1)$ for some k, then

$$\mathfrak{c}_k < \mathfrak{c}_k + \beta_n g(\mathfrak{c}_k) - \frac{g(\mathfrak{c}_k + \beta_n g(\mathfrak{c}_k))}{g'(\mathfrak{c}_k + \beta_n g(\mathfrak{c}_k)) - z_n} = \mathfrak{c}_{k+1}$$

and

$$\mathfrak{c}_{k+1} = \mathfrak{c}_k + \beta_n g(\mathfrak{c}_k) - \frac{g(\mathfrak{c}_k + \beta_n g(\mathfrak{c}_k))}{g'(\mathfrak{c}_k + \beta_n g(\mathfrak{c}_k)) - z_n} < a_1. \qquad \square$$

Lemma 3. Let δ_n be the family of minimizing geodesics such that $\delta_n(0) = e_n$ and $\delta_n(1) = i_n$, η_n be given as above. Then, for all $n \geq 0$, we have

$$\begin{aligned} S_{\delta_n,1,0}H(i_n) &= \int_0^1 \big(S_{\delta_n,t,0}DH(\delta_n(t))S_{\delta_n,0,t} - DH(e_n)\big)u_n dt \\ &\quad + \Big(\frac{I_{e_n}}{\beta_n} + S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}\Big)u_n + \big(DH(e_n) - S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}\big)u_n. \end{aligned}$$

Proof. Since $\delta_n(0) = e_n,\ \delta_n(1) = i_n,\ \eta_n(0) = e_n,\ \eta_n(1) = e_0$. By Theorem 2.1 and (3.1), we have

$$\begin{aligned} S_{\delta_n,1,0}H(i_n) &= S_{\delta_n,1,0}H(i_n) - H(e_n) + H(e_n) + DH(e_n)u_n - DH(e_n)u_n \\ &\quad + S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}u_n - S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}u_n \\ &= \int_0^1 S_{\delta_n,t,0}DH(\delta_n(t))S_{\delta_n,0,t}u_n dt - DH(e_n)u_n + H(e_n) + DH(e_n)u_n \\ &\quad - S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}u_n + S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}u_n \\ &= \int_0^1 \big(S_{\delta_n,t,0}DH(\delta_n(t))S_{\delta_n,0,t} - DH(e_n)\big)u_n dt + \frac{u_n}{\beta_n} \\ &\quad + \big(DH(e_n) - S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}\big)u_n + S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}u_n \\ &= \int_0^1 \big(S_{\delta_n,t,0}DH(\delta_n(t))S_{\delta_n,0,t} - DH(e_n)\big)u_n dt \\ &\quad + \Big(\frac{I_{e_n}}{\beta_n} + S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}\Big)u_n \\ &\quad + \big(DH(e_n) - S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}\big)u_n. \qquad \square \end{aligned}$$

Lemma 4. Let ϕ_n be the family of minimizing geodesics such that $\phi_n(0) = i_{n-1}$ and $\phi_n(1) = e_n$, δ_{n-1} be given as above. Then, for all $n \in \mathbb{N}$, we have

$$S_{\phi_n,1,0}H(e_n) = \int_0^1 \left(S_{\phi_n,t,0}DH(\phi_n(t))S_{\phi_n,0,t} - S_{\delta_{n-1},t,0}DH(\delta_{n-1}(t))S_{\delta_{n-1},0,t}\right)dt v_{n-1}.$$

Proof. From (2.2) and (3.1), we have

$$\begin{aligned} S_{\phi_n,1,0}H(e_n) &= S_{\phi_n,1,0}H(e_n) - H(i_{n-1}) + H(i_{n-1}) \\ &= S_{\phi_n,1,0}H(e_n) - H(i_{n-1}) - [e_{n-1}, i_{n-1}; H]v_{n-1} \\ &= [e_n, i_{n-1}; H]v_{n-1} - [e_{n-1}, i_{n-1}; H]v_{n-1} \\ &= \int_0^1 \left(S_{\phi_n,t,0}DH(\phi_n(t))S_{\phi_n,0,t} - S_{\delta_{n-1},t,0}DH(\delta_{n-1}(t))S_{\delta_{n-1},0,t}\right)dt v_{n-1}. \quad \square \end{aligned}$$

Now, we can demonstrate the convergence of the method; we begin by showing that $\{\mathfrak{c}_n\}_{n\in\mathbb{N}}$ is a majorizing sequence of $\{e_n\}_{n\in\mathbb{N}}$.

Theorem 3.1. *Let W be a complete Riemannian manifold and $\Omega \subseteq W$ be an open convex set. Assume that $H \in \chi(W)$ satisfies the conditions C with $Ka_1 < \frac{1}{2}$, then, for all $z_n > 0$ there are $\beta_n \in (0,1]$ such that*

$$d(e_{n+1}, e_n) \leq \mathfrak{c}_{n+1} - \mathfrak{c}_n,$$

where $\{e_n\}, \{\mathfrak{c}_n\}$ are defined by (3.1) *and* (3.2) *respectively.*

Proof. First, we shall prove the result for $n \geq 0$, by mathematical induction.

(I_n) $e_n \in U(e_0, a_1)$ (II_n) $i_n \in U(e_0, a_1)$.

For $n = 0$, (I_0) is trivial and (II_0) holds, since $d(i_0, e_0) = \|\beta_0 H(e_0)\| \leq \beta_0 b \leq b < a_1$. By Lemma 3, we have

$$\begin{aligned} \|S_{\delta_0,1,0}H(i_0)\| =& \|H(i_0)\| \\ \leq& \int_0^1 \|S_{\delta_0,t,0}DH(\delta_0(t))S_{\delta_0,0,t} - DH(e_0)\|\|u_0\|dt \\ &+ \|\frac{I_{e_0}}{\beta_0} + S_{\eta_0,1,0}DH(e_0)S_{\eta_0,0,1}\|\|u_0\| \\ &+ \|S_{\eta_0,1,0}DH(e_0)S_{\eta_0,0,1} - DH(e_0)\|\|u_0\| \\ \leq& \frac{K}{2}(\mathfrak{b}_0 - \mathfrak{c}_0)^2 + (\frac{1}{\beta_0} - 1)(\mathfrak{b}_0 - \mathfrak{c}_0) = g(\mathfrak{b}_0). \end{aligned}$$

Now, we will show that the operator $[e_0, i_0; H]^{-1}$ is bounded. Let ρ_n be the family of minimizing geodesics such that $\rho_n(0) = e_0$ and $\rho_n(1) = i_n$ for each $n = 0, 1, 2\ldots$, δ_0 be given as above. Then, we have

$$\begin{aligned} &\|I_{e_0} - DH(e_0)^{-1}[e_0, i_0; H]\| \\ =&\|DH(e_0)^{-1}(DH(e_0) - [e_0, i_0; H])\| \\ =&\|DH(e_0)^{-1}(DH(e_0) - S_{\rho_0,1,0}DH(i_0)S_{\rho_0,0,1} + S_{\rho_0,1,0}DH(i_0)S_{\rho_0,0,1} - [e_0, i_0; H])\| \\ \leq&|g'(\mathfrak{c}_0) - g'(\mathfrak{b}_0)| + g'(\mathfrak{b}_0) - g'(\mathfrak{b}_0) + z_0 = |g'(\mathfrak{c}_0) - g'(\mathfrak{b}_0)| + z_0 \end{aligned}$$

and

$$\begin{aligned}\|I_{e_0} - DH(e_0)^{-1}[e_0,i_0;H]\| &= \|DH(e_0)^{-1}\| \int_0^1 \|S_{\delta_0,t,0}DH(\delta_0(t))S_{\delta_0,0,t} - DH(e_0)\|dt \\ &\leq \int_0^1 K \int_0^t \|\delta_0'(l)\|dldt = K\|\delta_0'(0)\| \int_0^1 tdt \\ &= \frac{1}{2}K\|u_0\| = \frac{1}{2}K\|\beta_0 H(e_0)\| \leq \frac{1}{2}\beta_0 Kb \leq \beta_0 Kb < 1,\end{aligned}$$

as $Kb < \frac{1}{2}$. Therefore the operator $[e_0,i_0;H]$ is invertible by Banach's lemma and

$$\begin{aligned}\|[e_0,i_0;H]^{-1}\| &\leq \frac{\|DH(e_0)^{-1}\|}{1-\|I_{e_0} - DH(e_0)^{-1}[e_0,i_0;H]\|} \\ &= \frac{\|DH(e_0)^{-1}\|}{1-\|DH(e_0)^{-1}(DH(e_0)-[e_0,i_0;H])\|} \\ &\leq \frac{1}{1-(|g'(\mathfrak{c}_0)-g'(\mathfrak{b}_0)|+z_0)} \leq \frac{-1}{g'(\mathfrak{b}_0)-z_0}.\end{aligned}$$

Also

$$\begin{aligned}d(e_1,e_0) &\leq d(e_1,i_0)+d(i_0,e_0) = \|\beta_0 H(e_0)\| + \|[e_0,i_0;H]^{-1}H(i_0)\| \\ &\leq \beta_0 g(\mathfrak{c}_0) - \frac{g(\mathfrak{b}_0)}{g'(\mathfrak{b}_0)-z_0} = \mathfrak{c}_1 - \mathfrak{c}_0 < a_1.\end{aligned}$$

Assume that the above relations hold for $0 < j \leq n$ that is, $d(e_j,e_{j-1}) \leq \mathfrak{c}_j - \mathfrak{c}_{j-1}$. Then,

$$\begin{aligned}d(e_n,e_0) &\leq d(e_n,e_{n-1})+d(e_{n-1},e_{n-2})+\cdots+d(e_1,e_0) \\ &\leq \mathfrak{c}_n - \mathfrak{c}_{n-1} + \mathfrak{c}_{n-1} - \mathfrak{c}_{n-2} + \cdots + \mathfrak{c}_1 - \mathfrak{c}_0 = \mathfrak{c}_n - \mathfrak{c}_0 < a_1.\end{aligned}$$

Therefore $e_n \in U(e_0,a_1)$ and it holds for $n \geq 0$.

By Lemma 4, we have

$$\begin{aligned}\|S_{\phi_n,1,0}H(e_n)\| =& \|H(e_n)\| \\ =& \Big\| \Big(\int_0^1 \big(S_{\phi_n,t,0}DH(\phi_n(t))S_{\phi_n,0,t} - DH(i_{n-1})\big)dt \\ &+ \int_0^1 \big(DH(i_{n-1}) - S_{\delta_{n-1},t,0}DH(\delta_{n-1}(t))S_{\delta_{n-1},0,t}\big)dt\Big) v_{n-1} \Big\| \\ \leq& \Big(\frac{K}{2}d(e_n,i_{n-1}) + \frac{K}{2}d(i_{n-1},e_{n-1})\Big)\|v_{n-1}\| \\ =& \Big(\frac{K}{2}d(e_n,i_{n-1}) + \frac{K}{2}\|u_{n-1}\|\Big)d(e_n,i_{n-1}) \\ =& \frac{K}{2}[d(e_n,i_{n-1})]^2 + \frac{K}{2}\|\beta_{n-1}H(e_{n-1})\|d(e_n,i_{n-1}) \\ \leq& \frac{K}{2}(\|H(i_{n-1})[e_{n-1},i_{n-1};H]^{-1}\|)^2 \\ &+ \frac{K}{2}\beta_{n-1}g(\mathfrak{c}_{n-1})\|H(i_{n-1})[e_{n-1},i_{n-1};H]^{-1}\|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{K}{2}\Big(\frac{-g(\mathfrak{b}_{n-1})}{g'(\mathfrak{b}_{n-1})-z_{n-1}}\Big)^2+\frac{K}{2}\beta_{n-1}g(\mathfrak{c}_{n-1})\Big(\frac{-g(\mathfrak{b}_{n-1})}{g'(\mathfrak{b}_{n-1})-z_{n-1}}\Big)\\
&=\frac{K}{2}(\mathfrak{c}_n-\mathfrak{b}_{n-1})^2+\frac{K}{2}\beta_{n-1}g(\mathfrak{c}_{n-1})(\mathfrak{c}_n-\mathfrak{b}_{n-1})=g(\mathfrak{c}_n).
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
d(i_n,e_0) \leq& d(i_n,e_n)+d(e_n,e_0) \leq \|\beta_n H(e_n)\|+d(e_n,e_0)\\
\leq& \beta_n g(\mathfrak{c}_n)+d(e_n,e_0) < \beta_n g(a_1)+a_1=a_1.
\end{aligned}
$$

Therefore $i_n \in U(e_0,a_1)$ and it holds for $n\geq 0$. Again by Lemma 3, we have

$$
\begin{aligned}
\|S_{\delta_n,1,0}H(i_n)\| =&\|H(i_n)\|\\
\leq& \int_0^1 \|S_{\delta_n,t,0}DH(\delta_n(t))S_{\delta_n,0,t}-DH(e_n)\|\|u_n\|dt\\
&+\|\frac{I_{e_n}}{\beta_n}+S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}\|u_n\|\\
&+\|S_{\eta_n,1,0}DH(e_0)S_{\eta_n,0,1}-DH(e_n)\|\|u_n\|\\
\leq& \frac{K}{2}(\mathfrak{b}_n-\mathfrak{c}_n)^2+K(\mathfrak{b}_n-\mathfrak{c}_n)(\mathfrak{c}_n-\mathfrak{c}_0)+(\frac{1}{\beta_n}-1)(\mathfrak{b}_n-\mathfrak{c}_n)=g(\mathfrak{b}_n).
\end{aligned}
$$

Next, we will show that the operator $[e_n,i_n;H]^{-1}$ is bounded. Let τ be the minimizing geodesic such that $\tau(0)=e_0$ and $\tau(1)=e_n$, δ_n and ρ_n be given as above. Then, we have

$$
\begin{aligned}
&\|I_{e_0}-DH(e_0)^{-1}S_{\tau,1,0}[e_n,i_n;H]S_{\tau,0,1}\|\\
=&\|DH(e_0)^{-1}(DH(e_0)-S_{\tau,1,0}[e_n,i_n;H]S_{\tau,0,1})\|\\
=&\|DH(e_0)^{-1}(DH(e_0)-S_{\rho_n,1,0}DH(i_n)S_{\rho_n,0,1}\\
&+S_{\rho_n,1,0}DH(i_n)S_{\rho_n,0,1}-S_{\tau,1,0}[e_n,i_n;H]S_{\tau,0,1})\|\\
\leq&|g'(\mathfrak{c}_0)-g'(\mathfrak{b}_n)|+g'(\mathfrak{b}_n)-g'(\mathfrak{b}_n)+z_n=|g'(\mathfrak{c}_0)-g'(\mathfrak{b}_n)|+z_n
\end{aligned}
$$

and

$$
\begin{aligned}
&\|DH(e_0)^{-1}S_{\tau,1,0}[e_n,i_n;H]S_{\tau,0,1}-I_{e_0}\|\\
\leq&\|DH(e_0)^{-1}S_{\tau,1,0}([e_n,i_n;H]-DH(e_n))S_{\tau,0,1}\|\\
&+\|DH(e_0)^{-1}(S_{\tau,1,0}DH(e_n)S_{\tau,0,1}-DH(e_0))\|\\
\leq&\|DH(e_0)^{-1}\|\int_0^1\|S_{\delta_n,t,0}DH(\delta_n(t))S_{\delta_n,0,t}-DH(e_n)\|dt\\
&+\|DH(e_0)^{-1}\|\|S_{\tau,1,0}DH(e_n)S_{\tau,0,1}-DH(e_0)\|\\
\leq&Kd(e_n,e_0)+\int_0^1\|S_{\delta_n,t,0}DH(\delta_n(t))S_{\delta_n,0,t}-DH(e_n)\|dt\\
\leq&\frac{K}{2}\big(2(\mathfrak{c}_n-\mathfrak{c}_0)+d(e_n,i_n)\big)\leq\frac{K}{2}\big(2(\mathfrak{c}_n-\mathfrak{c}_0)+\beta_n g(\mathfrak{c}_n)\big)<1.
\end{aligned}
$$

Therefore $S_{\tau,1,0}[e_n,i_n;H]S_{\tau,0,1}$ is invertible by Banach's lemma and we have

$$\begin{aligned}\|[e_n,i_n;H]^{-1}\| &= \|S_{\tau,1,0}[e_n,i_n;H]^{-1}S_{\tau,0,1}\| \\ &\le \frac{\|DH(e_0)^{-1}\|}{1-\|DH(e_0)^{-1}(S_{\tau,1,0}[e_n,i_n;H]S_{\tau,0,1}-DH(e_0))\|} \\ &\le \frac{1}{1-(|g'(\mathfrak{c}_0)-g'(\mathfrak{b}_n)|+z_n)} \le \frac{-1}{g'(\mathfrak{b}_n)-z_n}.\end{aligned}$$

Thus, we have

$$\begin{aligned}d(e_{n+1},e_n) &\le d(e_{n+1},i_n)+d(i_n,e_n) = \|\beta_n H(e_n)\| + \|[e_n,i_n;H]^{-1}H(i_n)\| \\ &\le \beta_n g(\mathfrak{c}_n) - \frac{g(\mathfrak{b}_n)}{g'(\mathfrak{b}_n)-z_n} = \mathfrak{c}_{n+1}-\mathfrak{c}_n.\end{aligned}$$ □

Theorem 3.2. *Suppose that all the assumptions of Theorem 3.1 hold, then the method given by* (3.1) *converges to a singular point e_* of the vector field H in $U(e_0,a_1)$ and the singular point e_* is unique in $U(e_0,a_1)$.*

Proof. Since $d(e_{n+1},e_n) \le \mathfrak{c}_{n+1}-\mathfrak{c}_n \le \mathfrak{c}_{n+1}$, this shows that $\{e_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete Riemannian manifold W and hence it converges to say $e_* \in U(e_0,a_1)$. It is clear that e_* is a singularity of H because for all $n\in\mathbb{N}$, $\|H(e_n)\| \le g(\mathfrak{c}_n)$ and taking the limit as $n\to\infty$, we get

$$\|H(e_*)\| \le g(a_1) \to 0.$$

Next, we will show that the singularity is unique in $U(e_0,a_1)$. For this let us prove that $\|DH(e_j)^{-1}\| \le \frac{-1}{g'(\mathfrak{c}_j)}$ for $j\ge 0$. For $j=0$, it is trivial. Suppose it is true for $j=1,\ldots,n-1$. Then, we will prove it for $j=n$. Let τ be the minimizing geodesic given as above. We have

$$\|S_{\tau,1,0}DH(e_n)S_{\tau,0,1}-DH(e_0)\| \le K\int_0^1 \|\tau'(0))\|ds \le Kd(e_n,e_0) < Ka_1.$$

So

$$\|DH(e_0)^{-1}\|\|S_{\tau,1,0}DH(e_n)S_{\tau,0,1}-DH(e_0)\| \le Ka_1 < 1,$$

as $Ka_1 < \frac{1}{2}$. Therefore $S_{\tau,1,0}DH(e_n)S_{\tau,0,1}$ is invertible by Banach's lemma and we have

$$\begin{aligned}\|DH(e_n)^{-1}\| =& \|S_{\tau,1,0}DH(e_n)^{-1}S_{\tau,0,1}\| \\ \le& \frac{\|DH(e_0)^{-1}\|}{1-\|DH(e_0)^{-1}\|\|S_{\tau,1,0}DH(e_n)S_{\tau,0,1}-DH(e_0)\|} \\ \le& \frac{1}{1-Kd(e_n,e_0)} \le \frac{1}{1-K(\mathfrak{c}_n-\mathfrak{c}_0)} = \frac{1}{1-K\mathfrak{c}_n} = \frac{-1}{g'(\mathfrak{c}_n)}.\end{aligned}$$

Hence, it holds for $j\ge 0$. Let m_* be the another singularity of H in $U(e_0,a_1)$ and ν be the minimizing geodesic such that $\nu(0)=e_*$ and $\nu(1)=m_*$. Then, we have

$$\begin{aligned}\|S_{\nu,t,0}DH(\nu(t))S_{\nu,0,t}-DH(e_*)\| \leq & K\int_0^t \|\nu'(0)\|ds = Kt\|\nu'(0)\| \\ & = Ktd(e_*,m_*) \leq Kt\Big(d(e_0,e_*)+d(e_0,m_*)\Big).\end{aligned}$$

Hence

$$\begin{aligned}&\|DH(e_*)^{-1}\|\int_0^1 \|S_{\nu,t,0}DH(\nu(t))S_{\nu,0,t}dt-DH(e_*)\|dt \\ \leq&(1-Ka_1)^{-1}\int_0^1 Kt\Big(d(e_0,e_*)+d(e_0,m_*)\Big)dt \\ \leq&(1-Ka_1)^{-1}\frac{K}{2}(a_1+a_1)<1.\end{aligned}$$

By Banach's lemma, the operator $T \to \int_0^1 S_{\nu,t,0}DH(\nu(t))S_{\nu,0,t}(T)dt$ is invertible and we have

$$0 = S_{\nu,1,0}H(e_*)-H(m_*) = \int_0^1 S_{\nu,t,0}DH(\nu(t))S_{\nu,0,t}(\nu'(0))dt.$$

Therefore $\nu'(0)=0$. As $0=\|\nu'(0)\|=d(e_*,m_*)$, we get $e_*=m_*$. □

Theorem 3.3. *Suppose there is a singularity e_* of H in $U(e_0,a_1)$. If $U(e_0,a_2)\subseteq\Omega$, then e_* is the unique singular point of H in $U(e_0,p)$, where $a_1<p\leq a_2$.*

Proof. Let m_* be the singularity of H in $U(e_0,p)$ and γ be the minimizing geodesic such that $\gamma(0)=e_0$ and $\gamma(1)=m_*$. Then by Theorem 2.1, we have

$$\begin{aligned}S_{\gamma,1,0}H(m_*) =& S_{\gamma,1,0}H(m_*)-H(e_0)+H(e_0)+DH(e_0)\gamma'(0)-DH(e_0)\gamma'(0) \\ =& \int_0^1 S_{\gamma,t,0}DH(\gamma(t))S_{\gamma,0,t}\gamma'(0)dt-DH(e_0)\gamma'(0)+H(e_0)+DH(e_0)\gamma'(0) \\ =& \int_0^1 \big(S_{\gamma,t,0}DH(\gamma(t))S_{\gamma,0,t}-DH(e_0)\big)\gamma'(0)dt+H(e_0)+DH(e_0)\gamma'(0).\end{aligned}$$

Thus

$$\begin{aligned}\frac{Kd(e_0,m_*)^2}{2} \geq& \|H(e_0)+DH(e_0)\gamma'(0)\| \geq \frac{1}{\|DH(e_0)^{-1}\|}\|DH(e_0)^{-1}H(e_0)+\gamma'(0)\| \\ \geq& \Big(\|\gamma'(0)\|-\|DH(e_0)^{-1}H(e_0)\|\Big) \geq \Big(d(e_0,m_*)-a\Big).\end{aligned}$$

Therefore

$$g(d(e_0,m_*)) = \frac{Kd(e_0,m_*)^2}{2}-d(e_0,m_*)+a \geq 0.$$

Since $d(e_0,m_*)\leq p\leq a_2$, we have $d(e_0,m_*)\leq a_1$, hence by Theorem 3.2, $e_*=m_*$. □

4. NUMERICAL EXAMPLES

In this section, two examples are given to show the applicability of our convergence analysis.

Example 4.1. *Let us consider the vector field* $H : (-1,1)^3 \subseteq \mathbb{R}^3 \to (-1,1)^3$ *given by*

$$H(\boldsymbol{e}) = H(e_1,e_2,e_3)^T = (e^{e_1}-1, e_2^2+e_2, e_3)^T$$

with the norm $\|.\|_\infty$. *The first and second Fréchet derivatives of H are:*

$$DH(\boldsymbol{e}) = \begin{bmatrix} e^{e_1} & 0 & 0 \\ 0 & 2e_2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D^2H(\boldsymbol{e}) = \left[\begin{array}{ccc|ccc|ccc} e^{e_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Initially for $\boldsymbol{e}_0 = (0.11, 0.11, 0.11)^T$, *we get* $\|H(\boldsymbol{e}_0)\| = \max(0.12, 0.12, 0.11) = 0.12 = b$, $\|DH(\boldsymbol{e}_0)^{-1}\| = 1$, *and* $\|D^2H(\boldsymbol{e})\| = \max(1.12, 2, 0) = 2 = K$. *All the assumptions of the convergence theorem are satisfied and the generalized Steffensen method can be applied to get the desired singular point.*

Example 4.2. *Let us consider the vector field* $H : \mathbb{R}^2 \to \mathbb{R}^2$ *given by*

$$H(e) = H(e_1,e_2)^T = \Big(\frac{\cos e_1 + 20e_1}{20}, e_2\Big)^T \tag{4.1}$$

with the norm $\|.\|_\infty$. *The first and second Fréchet derivatives of H are:*

$$DH(e) = \begin{bmatrix} \frac{-\sin e_1+20}{20} & 0 \\ 0 & 1 \end{bmatrix}, D^2H(e) = \left[\begin{array}{cc|cc} \frac{-\cos e_1}{20} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Initially for $e_0 = (0,0)^T$, *we get* $\|H(e_0)\| = \max(\frac{1}{20}, 0) = \frac{1}{20} = b$, $\|DH(e_0)^{-1}\| = 1$, *and* $\|D^2H(e)\| = \max(|\frac{-\cos e_1}{20}|, 0) = |\frac{\cos e_1}{20}| \leq \frac{1}{20} = K$. *Again all the assumptions of the convergence theorem are satisfied and one can apply the generalized Steffensen method to get the desired singular point.*

5. CONCLUSION

In this article, we have presented the generalized Steffensen method in Riemannian manifolds to approximate the zeros of a vector field. We have established the convergence theorem under the Kantarovich–Ostrowski's conditions. Finally, two examples are given to show the applicability of our convergence analysis. In the future, we will extend this method for $\beta_n \in \mathbb{R}^+$.

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