SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.15.02.01
Vol.15 (28), No.2, (2019), 129 – 154

GLOBAL DYNAMICS OF CERTAIN MIX MONOTONE DIFFERENCE EQUATION VIA CENTER MANIFOLD THEORY AND THEORY OF MONOTONE MAPS

MUSTAFA R.S. KULENOVIĆ, MEHMED NURKANOVIĆ, AND ZEHRA NURKANOVIĆ

Dedicated to the memory of Accademicians Harry I. Miller and Fikret Vajzović, our teachers and supporters.

ABSTRACT. We investigate the global dynamics of the following rational difference equation of second order

$$x_{n+1}=\frac{Ax_n^2+Ex_{n-1}}{x_n^2+f}, \quad n=0,1,\dots,$$

where the parameters A and E are positive real numbers and the initial conditions x_{-1} and x_0 are arbitrary non-negative real numbers such that $x_{-1}+x_0>0$. The transition function associated with the right-hand side of this equation is always increasing in the second variable and can be either increasing or decreasing in the first variable depending on the parametric values. The unique feature of this equation is that the second iterate of the map associated with this transition function changes from strongly competitive to strongly cooperative. Our main tool for studying the global dynamics of this equation is the theory of monotone maps while the local stability is determined by using center manifold theory in the case of the nonhyperbolic equilibrium point.

1. INTRODUCTION

In this paper, we investigate the global dynamics of the following difference equation

$$x_{n+1}=\frac{Ax_n^2+Ex_{n-1}}{x_n^2+f}, \quad n=0,1,\dots, \tag{1}$$

where $A,E,f\in(0,\infty)$ and where the initial conditions x_{-1} and x_0 are arbitrary non-negative real numbers such that $x_{-1}+x_0>0$.

Equation (1) is a special case of equations

$$x_{n+1}=\frac{Ax_n^2+Ex_{n-1}+F}{ax_n^2+ex_{n-1}+f}, \ n=0,1,2,\dots \tag{2}$$

2010 *Mathematics Subject Classification.* 39A10; 39A20; 39A23, 39A30.

Key words and phrases. Difference equations, global dynamics, center manifold theory, monotone maps, equilibrium, period-two solutions.

and

$$x_{n+1}=\frac{Ax_n^2+Bx_nx_{n-1}+Cx_{n-1}^2+Dx_n+Ex_{n-1}+F}{ax_n^2+bx_nx_{n-1}+cx_{n-1}^2+dx_n+ex_{n-1}+f},\quad n=0,1,\dots\,. \tag{3}$$

The dynamics of Equation (2) was investigated in [10] and some special cases were considered in [10, 11]. It was shown that Equation (2) has very rich dynamics ranging from global attractivity of the equilibrium to global period doubling bifurcation to non-conservative chaos. Some special cases of Equation (3) have been considered in the series of papers [8–10, 19, 21].

The special case of Equation (1) when $A=1$ and $E=0$, is the well-known sigmoid Beverton-Holt or Thomson equation

$$x_{n+1}=\frac{x_n^2}{ax_n^2+f},\quad n=0,1,\dots, \tag{4}$$

which is used in the modelling of fish populations [25].

Notice that Equation (1) is an example of a rational difference equation with an associated map that is always strictly increasing with respect to the second variable and that changes monotonicity with respect to the first variable, i.e. can be increasing or decreasing depending on the corresponding parametric space. There are not many papers that study in detail the dynamics of second order rational difference equations with quadratic terms that have associated maps that change monotonicity with respect to their parameters (see [12, 13, 19]).

Consider the difference equation

$$x_{n+1}=f(x_n,x_{n-1}),\quad x_{-1},x_0\in I,\quad n=0,1,\dots, \tag{5}$$

where $f\in C[I\times I,I]$ and I is some interval of real numbers. Some of our results will be based on the following theorem, see [1, 5].

Theorem 1.1. *Let I be a set of real numbers and $f:I\times I\to I$ be a function which is either non-increasing or non-decreasing in the first variable and non-decreasing in the second variable. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of Equation (5) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually monotonic sequences.*

The consequence of Theorem 1.1 is that every bounded solution of (5) converges to either an equilibrium solution or period-two solution or to the singular point, and the most important question becomes finding the basins of attraction of these solutions as well as any unbounded solutions. The answer to this question follows from the theory of monotone maps in the plane. As we have shown in a sequence of previous papers the boundaries of the basins of attraction of locally asymptotically stable equilibrium solutions or period-two solutions are the global stable manifolds of neighboring saddle points or stable type non-hyperbolic equilibrium solutions or period-two solutions, see [2–4, 16]. The major difference between the cases when $f(u,v)$ is non-decreasing in u and when $f(u,v)$ is non-increasing in u, is the orientation of the stable manifolds, which are decreasing functions in the first case and increasing functions in the second case. Consequently, one may assume that all solutions of such difference equation will eventually enter an invariant interval where $f(u,v)$ has specific monotonic behavior with respect to u. In other words, the existence of solutions which oscillate between the regions where the function $f(u,v)$ changes monotonicity with respect to u seems to be not feasible. In this paper we give an example of such a case when $E = f$ and study the global dynamics of that case by using semicycle analysis [14, 15].

By using center manifold theory we investigate the local stability of nonhyperbolic equilibrium points (similar to that in [6, 15, 22, 23]). Our investigation of the global dynamics of Equation (1) makes use of the theory of monotone maps (see [7, 16–18, 20, 24]) since we show that in all cases except $E = f$, all solutions of Equation (1) will eventually be attracted to an invariant interval where the transition function $f(u,v)$ is either increasing in both variables or decreasing in the first and increasing in the second variable. As we know from [4, 7, 16] the second iterate T^2 of the map associated with Equation (5) will be either strongly cooperative or competitive. This fact has been used to obtain several global dynamics results for this type of second order difference equation. For the sake of completeness we list two such results from [7]:

Theorem 1.2. *Let $I \subset \mathbb{R}$ be an interval. Consider the Eq.(5) where $f(x,y) : I \times I \to I$ is continuous. Suppose that:*

(a_1) $f(\operatorname{int} I \times \operatorname{int} I) \subset \operatorname{int} I$, and $f(x,y)$ is strictly decreasing in x and strictly increasing in y in $\operatorname{int} I \times \operatorname{int} I$.

(a_2) There exists an equilibrium point $\bar{x} \in \operatorname{int} I$ and the region of initial conditions $(Q_1(\bar{x},\bar{x}) \cup Q_3(\bar{x},\bar{x})) \cap \operatorname{int} I \times \operatorname{int} I$ contains no period-two solutions or equilibria other than $(\bar{x},\bar{x})$. In addition, f is continuously differentiable on a neighborhood of $\bar{x}$, and $\bar{x}$ is a saddle.

Then the global stable manifold of the equilibrium $\mathcal{W}^s(\bar{x},\bar{x})$ is a curve in $I \times I$ which is the graph of a continuous and increasing function that passes through $(\bar{x},\bar{x})$ and that has endpoints in $\partial(I \times I)$. Furthermore,

(ii.1) If I is compact or if there exists a compact set $\mathcal{K} \subset I$ such that $f(I \times I) \subset \mathcal{K}$, then there exist minimal-period two solutions (ϕ,ψ) and (ψ,ϕ).

(ii.2) If minimal period two solutions (ϕ,ψ) and (ψ,ϕ) exist in $Q_2(\bar{x},\bar{x}) \cap I \times I$ and $Q_4(\bar{x},\bar{x}) \cap I \times I$ respectively, and if they are the only minimal period-two solutions there, then every solution $\{x_n\}$ with initial condition in the complement of the global stable manifold of the equilibrium is attracted to one of the period-two solutions. That is, whenever $x_n \nrightarrow \bar{x}$, either $x_{2n} \to \phi$ and $x_{2n+1} \to \psi$, or, $x_{2n} \to \psi$ and $x_{2n+1} \to \phi$.

(ii3) If there are no minimal period-two solutions in I^2, then every solution $\{x_n\}$ of Eq. (5), with initial condition in $\mathcal{W}_-$ is such that the subsequence $\{x_{2n}\}$ eventually leaves any given compact subset of I, and every solution $\{x_n\}$ of Eq.

(5), with initial condition in $\mathcal{W}_+$ is such that the subsequence $\{x_{2n+1}\}$ eventually leaves any given compact subset of I.

Theorem 1.3. *Consider Equation (5) where f is continuous function and f is decreasing in first argument and increasing in its second argument. Assume that $\overline{x}$ is a unique equilibrium point which is locally asymptotically stable and assume that (ϕ,ψ) and (ψ,ϕ) are minimal period-two solutions which are saddle points such that*

$$(\phi,\psi) \preceq_{se} (\overline{x},\overline{x}) \preceq_{se} (\psi,\phi).$$

Then the basin of attraction $\mathcal{B}((\overline{x},\overline{x}))$ of $(\overline{x},\overline{x})$ is the region between the global stable manifolds $\mathcal{W}^s((\phi,\psi))$ and $\mathcal{W}^s((\psi,\phi))$. More precisely

$$\mathcal{B}((\overline{x},\overline{x})) = \{(x,y) : \exists y_u, y_l : y_u < y < y_l,\ (x,y_l) \in \mathcal{W}^s((\phi,\psi)), (x,y_u) \in \mathcal{W}^s((\psi,\phi))\}.$$

The basins of attraction $\mathcal{B}((\phi,\psi)) = \mathcal{W}^s((\phi,\psi))$ and $\mathcal{B}((\psi,\phi)) = \mathcal{W}^s((\psi,\phi))$ are exactly the global stable manifolds of (ϕ,ψ) and (ψ,ϕ).

If $(x_{-1},x_0) \in \mathcal{W}_+((\psi,\phi))$ or $(x_{-1},x_0) \in \mathcal{W}_-((\phi,\psi))$ then $T^n((x_{-1},x_0))$ converges to the other equilibrium point or to the other minimal period-two solutions or to the boundary of the region $I \times I$.

Remark 1.1. The analogous results hold in the case where $f(u,v)$ is increasing in both arguments with the only change being that the orientations of the stable and unstable manifolds are reversed, that is the stable mani-folds are decreasing functions while the unstable manifolds are increasing functions, see [3, 4].

Remark 1.2. The dynamics of Equation (1) can be described in terms of bifurcation theory as the parameter $E - f$ passess through several critical values. The first critical value is $-A^2/4$ where transcritical bifurcation occurs and two additional fixed points $\overline{x}_- < \overline{x}_+$ appear. The second critical value is 0 where two fixed points $\overline{x}_-$ and 0 coincide and the zero fixed point changes in local stability from locally asymptotically stable to repeller. The third critical value is $3A^2/4$ where the locally asymptotically stable period-two point (ϕ,ψ) appears and the positive fixed point changes in local stability from locally asymptotically stable to saddle point, giving an example of period-doubling bifurcation. The fourth critical value is A^2 where the period-two point (ϕ,ψ) disappears.

2. Linearized Stability Analysis

In this section, we present the local stability of the equilibrium points of Equation (1). The equilibrium points of Equation (1) are the positive solutions of the equation

$$\overline{x} = \frac{A\overline{x}^2 + E\overline{x}}{\overline{x}^2 + f}, \tag{6}$$

or equivalently

$$\overline{x}\left(\overline{x}^2 - A\overline{x} - (E - f)\right) = 0, \tag{7}$$

from which we obtain

$$\overline{x}_1 = 0,$$

and

$$\bar{x}_{\pm} = \frac{A \pm \sqrt{A^2 + 4(E - f)}}{2},$$

where $\bar{x}_+ > 0$ if $E \geq f - \frac{1}{4}A^2$ and $\bar{x}_- > 0$ when $f - \frac{1}{4}A^2 < E < f$ or $0 < E = f - \frac{1}{4}A^2$.

The existence of the equilibrium points of Equation (1) is cataloged in Table 1.

Now, set

$$f(u,v) = \frac{Au^2 + Ev}{u^2 + f}.$$

Then Equation (1) has a linearized equation $z_{n+1} = pz_n + qz_{n-1}$, where

$$\frac{\partial f}{\partial u} = \frac{2u(Af - vE)}{(u^2 + f)^2}, \quad \frac{\partial f}{\partial v} = \frac{E}{u^2 + f}.$$

Lemma 2.1. *The equilibrium point $\bar{x}_1 = 0$ of Equation (1) is:*

i) locally asymptotically stable if $E < f$,
ii) a repeller if $E > f$,
iii) a nonhyperbolic point if $E = f$.

Proof. Since

$$p = \frac{\partial f}{\partial u}(0,0) = 0, \quad q = \frac{\partial f}{\partial v}(0,0) = \frac{E}{f},$$

from the corresponding characteristic equation $\lambda^2 - p\lambda - q = 0$ we obtain $\lambda_{1,2} = \pm\sqrt{\frac{E}{f}}$. Then

$$|\lambda_{1,2}| \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} 1 \Leftrightarrow E \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} f. \qquad \square$$

The partial derivatives at a positive equilibrium point $\bar{x}$ satisfy:

$$\left\{ \begin{array}{l} p = \frac{\partial f}{\partial u}(\bar{x},\bar{x}) = \frac{2\bar{x}(Af - \bar{x}E)}{(\bar{x}^2+f)^2} = \frac{2(A\bar{x}f - \bar{x}^2E)}{(\bar{x}^2+f)^2} = \frac{2((\bar{x}^2+f-E)f - \bar{x}^2E)}{(\bar{x}^2+f)^2} = \frac{2(f-E)}{f+\bar{x}^2}, \\ q = \frac{\partial f}{\partial v}(\bar{x},\bar{x}) = \frac{E}{\bar{x}^2+f}. \end{array} \right. \tag{8}$$

Lemma 2.2. *The equilibrium point $\bar{x}_+$ of Equation* (1) *is:*

i) a nonhyperbolic point if $0 < E = f - \frac{1}{4}A^2$ (when $\bar{x}_- = \bar{x}_+ = \frac{A}{2}$) or $E = f + \frac{3}{4}A^2$ (when $\bar{x}_- = \bar{x}_+ = \frac{3A}{2}$),
ii) locally asymptotically stable if $0 \leq f - \frac{1}{4}A^2 < E < f$ or $E = f$ or $f < E < f + \frac{3}{4}A^2$,
iii) a saddle point if $f + \frac{3}{4}A^2 < E$.

Proof. Notice that $0 < 1 - q = \frac{f - E + \bar{x}_+^2}{f + \bar{x}_+^2} < 2$.

i) If $E = f - \frac{A^2}{4}$ then for $\bar{x}_+ = \dfrac{A}{2}$ we have that

$$p = \frac{2(f-E)}{f+\overline{x}_+^2} = \frac{2\left(\frac{A^2}{4}\right)}{f+\left(\dfrac{A}{2}\right)^2} = \frac{2A^2}{4f+A^2},$$

$$q = \frac{E}{\overline{x}_+^2+f} = \frac{E}{\left(\dfrac{A}{2}\right)^2+f} = \frac{4E}{4f+A^2} = \frac{4f-A^2}{4f+A^2},$$

and the corresponding characteristic equation has the following form

$$(\lambda-1)\left(\left(A^2+4f\right)\lambda - A^2+4f\right) = 0,$$

from which

$$\lambda_1 = 1, \quad -1 < \lambda_2 = \frac{A^2-4f}{A^2+4f} < 0.$$

This means that the equilibrium point $\overline{x}_+ = \frac{A}{2}$ is nonhyperbolic of the stable type.

Suppose that $E = f + \frac{3}{4}A^2$. Then $\overline{x}_+ = \frac{3A}{2}$ and

$$|p| = |1-q| \Leftrightarrow \overline{x}_+^2 = 3(E-f) \Leftrightarrow E = f + \frac{3}{4}A^2.$$

ii) If $0 \leq f - \frac{1}{4}A^2 < E < f$, then

$$|p| < 1-q \Leftrightarrow \frac{2(f-E)}{f+\overline{x}_+^2} < \frac{f-E+\overline{x}_+^2}{f+\overline{x}_+^2} \Leftrightarrow \overline{x}_+^2 > f-E,$$

which is true because

$$\overline{x}_+^2 = \left(\frac{A+\sqrt{A^2+4(E-f)}}{2}\right)^2 > \frac{A^2}{4} > f-E.$$

If $f = E$, then $\overline{x}_+ = A$ and

$$p = \frac{2(f-E)}{f+\overline{x}^2} = 0, \quad q = \frac{E}{\overline{x}^2+f} = \frac{E}{A^2+E},$$

so that $1-q = 1-\frac{E}{A^2+E} = \frac{A^2}{E+A^2} < 2$.

Assume that $f < E < f + \frac{3}{4}A^2$. Then

$$\begin{aligned}
|p| < 1-q \Leftrightarrow & \frac{2(E-f)}{f+\overline{x}_+^2} < \frac{\overline{x}_+^2+f-E}{\overline{x}_+^2+f} \\
& \Leftrightarrow \overline{x}^2 > 3(E-f) \Leftrightarrow A^2+2A\sqrt{A^2+4(E-f)}+A^2+4(E-f) > 12(E-f) \\
& \Leftrightarrow A\sqrt{A^2+4(E-f)} > 4(E-f)-A^2 \Leftrightarrow A^4+4A^2(E-f) > \left(4(E-f)-A^2\right)^2 \\
& \Leftrightarrow (E-f)\left(4f-4E+3A^2\right) > 0 \Leftrightarrow E < f+\frac{3}{4}A^2.
\end{aligned}$$

iii) If $f + \frac{3}{4}A^2 < E$, then (similarly as in ii))

$$|p| > 1-q \Leftrightarrow \frac{2(E-f)}{f+\overline{x}_+^2} > \frac{\overline{x}_+^2+f-E}{\overline{x}_+^2+f}$$
$$\Leftrightarrow \overline{x}^2 < 3(E-f) \Leftrightarrow A^2+2A\sqrt{A^2+4(E-f)}+A^2+4(E-f) < 12(E-f)$$
$$\Leftrightarrow 4(E-f)\left(4f-4E+3A^2\right) < 0 \Leftrightarrow f+\frac{3}{4}A^2 < E. \qquad \square$$

Lemma 2.3. *The equilibrium point $\overline{x}_-$ of Equation* (1) *is:*

i) a nonhyperbolic point if $0 < E = f - \frac{1}{4}A^2$ (when $\overline{x}_- = \overline{x}_+ = \frac{A}{2}$) or $E = f$ (when $\overline{x}_- = \overline{x}_1 = 0$),

ii) a saddle point if $0 \le f - \frac{1}{4}A^2 < E < f$,

iii) a repeller if $f < E < f + \frac{3}{4}A^2$ (when is $\overline{x}_- = \overline{x}_1 = 0$).

Proof. i) This was shown in the proofs of Lemmas 2.1 and 2.2.

ii) Assume that $0 \le f - \frac{1}{4}A^2 < E < f$, then

$$|p| > |1-q| \Leftrightarrow \frac{2(f-E)}{f+\overline{x}_-^2} > \frac{f-E+\overline{x}_-^2}{f+\overline{x}_-^2} \Leftrightarrow \overline{x}_-^2 < f-E$$
$$\Leftrightarrow \left(A-\sqrt{A^2+4(E-f)}\right)^2 < 4(f-E)$$
$$\Leftrightarrow A^2+4(E-f) < A\sqrt{A^2+4(E-f)}$$
$$\Leftrightarrow (E-f)\left(A^2-4f+4E\right) < 0,$$

which is true when $0 \le f - \frac{1}{4}A^2 < E < f$.

iii) If $f < E < f + \frac{3}{4}A^2$, then $\overline{x}_- = 0$ and is a repeller. This was shown in the proof of Lemma 2.1. $\square$

	The conditions	The equilibrium points	P2-solutions	
1)	$0<E<f-\frac{1}{4}A^2$	$\overline{x}_1=0$ (LAS)	-	
2)	$0<E=f-\frac{1}{4}A^2$	$\begin{cases}\overline{x}_1=0 \text{ (LAS)}\\ \overline{x}_2=\frac{A}{2}\\ \text{(NH, } \lambda_1=1, \lambda_2\in(-1,0))\end{cases}$	-	$\overline{x}_-=\overline{x}_+=\frac{A}{2}<A<\frac{Af}{E}$
3)	$0\le f-\frac{1}{4}A^2<E<f$	$\begin{cases}\overline{x}_1=0 \text{ (LAS)}\\ \overline{x}_2=\overline{x}_- \text{ (SP)}\\ \overline{x}_3=\overline{x}_+ \text{ (LAS)}\end{cases}$	-	$\frac{A}{2}<\overline{x}_-<\overline{x}_+<A<\frac{Af}{E}$
4)	$E=f$	$\begin{cases}\overline{x}_1=\overline{x}_-=0 \text{ (NH, } \lambda_{1,2}=\pm 1)\\ \overline{x}_2=\overline{x}_+=A \text{ (LAS)}p\end{cases}$	-	$\overline{x}_+=A=\frac{Af}{E}$
5)	$f<E<f+\frac{3}{4}A^2$	$\begin{cases}\overline{x}_1=\overline{x}_-=0 \text{ (R)}\\ \overline{x}_2=\overline{x}_+ \text{ (LAS)}p\end{cases}$	-	$\overline{x}_+>A>\frac{Af}{E}$
6)	$E=f+\frac{3}{4}A^2$	$\begin{cases}\overline{x}_1=0 \text{ (R)}\\ \overline{x}_2=\overline{x}_+=\frac{3}{2}A\\ \text{(NH, } \lambda_1=-1, \lambda_2\in(0,1))\end{cases}$	-	$\overline{x}_+>A>\frac{Af}{E}$
7)	$f+\frac{3}{4}A^2<E<f+A^2$	$\begin{cases}\overline{x}_1=0 \text{ (R)}\\ \overline{x}_2=\overline{x}_+ \text{ (SP)}\end{cases}$	$\begin{cases}(\phi,\psi) \text{ (LAS)}\\ (\psi,\phi) \text{ (LAS)}\end{cases}$	$\psi>\overline{x}_+>\phi>A>\frac{Af}{E}$
8)	$E\ge f+A^2$	$\begin{cases}\overline{x}_1=0 \text{ (R)}\\ \overline{x}_2=\overline{x}_+ \text{ (SP)}\end{cases}$	-	$\overline{x}_+>A>\frac{Af}{E}$

Table 1: Existence and local stability of equilibrium and period-two solutions.

3. Period-two solutions

In this section, we investigate the existence and the local stability of the minimal period-two solution of Equation (1).

Theorem 3.1. *If $f+\frac{3}{4}A^2<E<f+A^2$, then Equation* (1) *has the minimal period-two solution*

$$\{\ldots,\phi,\psi,\phi,\psi,\ldots\}, \quad (\phi\neq\psi \text{ and } \phi>0 \text{ and } \psi>0), \tag{9}$$

where

$$\phi=\frac{(E-f)\left(A-\sqrt{4(E-f)-3A^2}\right)}{2(f-E+A^2)}, \ \ \psi=\frac{(E-f)\left(A+\sqrt{4(E-f)-3A^2}\right)}{2(f-E+A^2)}. \tag{10}$$

Proof. Assume that $\{\ldots,\phi,\psi,\phi,\psi,\ldots\}$ is a minimal period-two solution of (1) with $\phi,\psi\in[0,+\infty)$ and $\phi\neq\psi$. Then

$$\phi=\frac{A\psi^2+E\phi}{\psi^2+f}, \quad \psi=\frac{A\phi^2+E\psi}{\phi^2+f},$$

from which

$$\phi\left(\psi^2+f\right)=A\psi^2+E\phi, \tag{11}$$

$$\psi\left(\phi^2+f\right)=A\phi^2+E\psi. \tag{12}$$

By subtracting (12) from (11) we have

$$(\phi-\psi)(f-\phi\psi-E+A\phi+A\psi)=0,$$

i.e.,

$$f-\phi\psi-E+A(\phi+\psi)=0. \tag{13}$$

Similarly, by adding (12) to (11) we obtain

$$(\phi+\psi)(f+\phi\psi-E-A(\phi+\psi))+2A\phi\psi=0. \tag{14}$$

If we denote that $\phi+\psi=s$ (>0) and $\phi\psi=t$ (>0), then from (13) and (14):

$$f-E=t-As,$$
$$s(f+t)=sE+As^2-2At,$$

which implies that

$$s\left(f-E+A^2\right)=A(E-f).$$

Now,

i) if $f=E$, then $s=t=0$, which means that there is no minimal period-two solution;

ii) if $f>E$, then $s<0$, and there is no positive minimal period-two solution;

iii) if $f<E$, then for $E<f+A^2$

$$s=\frac{A(E-f)}{f-E+A^2}>0, \quad t=f-E+A\left(\frac{A(E-f)}{f-E+A^2}\right)=\frac{(f-E)^2}{f-E+A^2}>0,$$

and

$$\phi+\psi=\frac{A(E-f)}{f-E+A^2}=s, \quad \phi\psi=\frac{(f-E)^2}{f-E+A^2}=\frac{E-f}{A}s,$$

from which

$$\phi=\frac{1}{2}s-\frac{1}{2}\sqrt{\left(s-\frac{4(E-f)}{A}\right)s}, \quad \psi=\frac{1}{2}s+\frac{1}{2}\sqrt{\left(s-\frac{4(E-f)}{A}\right)s},$$

i.e.,

$$\phi=\frac{(E-f)\left(A-\sqrt{4(E-f)-3A^2}\right)}{2(f-E+A^2)}>0, \ \psi=\frac{(E-f)\left(A+\sqrt{4(E-f)-3A^2}\right)}{2(f-E+A^2)}>0,$$

for

$$f+\frac{3}{4}A^2<E<f+A^2.$$

□

Lemma 3.2. *If* $f+\frac{3}{4}A^2<E<f+A^2$, *then*

$$\psi>\overline{x}_+>\phi>A>\frac{Af}{E}.$$

Proof. Indeed i)

$$\begin{aligned}\phi=&\frac{(E-f)\left(A-\sqrt{4(E-f)-3A^2}\right)}{2(f-E+A^2)}>A\\ &\Leftrightarrow (E-f)\left(A-\sqrt{4(E-f)-3A^2}\right)>2A\left(f-E+A^2\right)\\ &\Leftrightarrow -(E-f)\left(\sqrt{4(E-f)-3A^2}\right)>A\left(3f-3E+2A^2\right)\\ &\Leftrightarrow (E-f)\left(\sqrt{4(E-f)-3A^2}\right)<A\left(-3f+3E-2A^2\right)\\ &\Leftrightarrow (E-f)^2\left(4(E-f)-3A^2\right)<A^2\left(-3f+3E-2A^2\right)^2\\ &\Leftrightarrow (E-f)^2\left(4(E-f)-3A^2\right)-A^2\left(-3f+3E-2A^2\right)^2<0\\ &\Leftrightarrow -4\left(f-E+A^2\right)^3<0,\end{aligned}$$

which is true if $E<f+A^2$.

ii)

$$\phi<\overline{x}_+=\frac{A+\sqrt{A^2+4(E-f)}}{2}\Leftrightarrow\frac{(E-f)\left(A-\sqrt{4(E-f)-3A^2}\right)}{2(f-E+A^2)}<\frac{A+\sqrt{A^2+4(E-f)}}{2}$$

$$\Leftrightarrow (E-f)\left(A-\sqrt{4(E-f)-3A^2}\right)<\left(f-E+A^2\right)\left(A+\sqrt{A^2+4(E-f)}\right)$$

$$\Leftrightarrow A\left(2(E-f)-A^2\right) < \left(f-E+A^2\right)\sqrt{A^2+4(E-f)}+(E-f)\sqrt{4(E-f)-3A^2}$$
$$\Leftrightarrow 4(E-f)-3A^2 < \sqrt{(A^2+4(E-f))(4(E-f)-3A^2)}$$
$$\Leftrightarrow 4A^2\left(4f-4E+3A^2\right) < 0,$$

which is true because $f+\frac{3}{4}A^2 < E$.

iii)

$$\overline{x}_+ = \frac{A+\sqrt{A^2+4(E-f)}}{2} < \psi \Leftrightarrow \frac{A+\sqrt{A^2+4(E-f)}}{2} < \frac{(E-f)\left(A+\sqrt{4(E-f)-3A^2}\right)}{2(f-E+A^2)}$$
$$\Leftrightarrow \left(f-E+A^2\right)\sqrt{A^2+4(E-f)} < A\left(2(E-f)-A^2\right)+(E-f)\sqrt{4(E-f)-3A^2}$$
$$\Leftrightarrow 2A^2(f-E)\left(4(E-f)-3A^2\right) < 2A\left(2(E-f)-A^2\right)(E-f)\sqrt{4(E-f)-3A^2},$$

which is true because $f+\frac{3}{4}A^2 < E < f+A^2$. □

By the substitution $x_{n-1}=u_n$, $x_n=v_n$, Equation (1) becomes the system of equations

$$\left.\begin{array}{c} u_{n+1}=v_n \\ v_{n+1}=\dfrac{Av_n^2+Eu_n}{v_n^2+f} \end{array}\right\}. \tag{15}$$

The map T corresponding to Equation (15) is of the form

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ h(v,u) \end{pmatrix}, \tag{16}$$

where $h(v,u)=\frac{Av^2+Eu}{v^2+f}$. Since (ψ,ϕ) and (ϕ,ψ) are the fixed points of the second iteration $T^2=T\circ T$ of the map T, i.e.,

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = T\begin{pmatrix} v \\ h(v,u) \end{pmatrix} = \begin{pmatrix} h(v,u) \\ H(v,u) \end{pmatrix}, \tag{17}$$

where

$$H(v,u)=\frac{Ah^2(v,u)+Ev}{h^2(v,u)+f},$$

the Jacobian matrix of the map T^2 is of the form

$$J_{T^2}(u,v) = \begin{pmatrix} \frac{\partial h(v,u)}{\partial u} & \frac{\partial h(v,u)}{\partial v} \\ \frac{\partial H(v,u)}{\partial u} & \frac{\partial H(v,u)}{\partial v} \end{pmatrix}.$$

Since,

$$\begin{aligned}
\frac{\partial h(v,u)}{\partial u} &= \frac{E}{v^2+f}, \quad \frac{\partial h(v,u)}{\partial v} = 2v\frac{Af-uE}{(v^2+f)^2},\\
\frac{\partial H(v,u)}{\partial u} &= \frac{\partial h(v,u)}{\partial u}\frac{2h(v,u)(Af-Ev)}{(h^2(v,u)+f)^2},\\
\frac{\partial H(v,u)}{\partial v} &= \frac{\partial h(v,u)}{\partial v}\frac{2h(v,u)(Af-Ev)}{(h^2(v,u)+f)^2}+\frac{E}{h^2(v,u)+f}.
\end{aligned}$$

and

$$\phi = \frac{A\psi^2+E\phi}{\psi^2+f}, \quad \psi = \frac{A\phi^2+E\psi}{\phi^2+f}, \quad h(v,u) = \frac{Av^2+Eu}{v^2+f},$$

we obtain

$$h(\psi,\phi) = \frac{A\psi^2+E\phi}{\psi^2+f} = \phi.$$

By Lemma 3.2 we have

$$\begin{aligned}
\frac{\partial h(\psi,\phi)}{\partial u} &= \frac{E}{\psi^2+f} > 0, \quad \frac{\partial h(\psi,\phi)}{\partial v} = \frac{2\psi(Af-\phi E)}{(\psi^2+f)^2} < 0,\\
\frac{\partial H(\psi,\phi)}{\partial v} &= \frac{\partial h(\psi,\phi)}{\partial v}\frac{2h(\psi,\phi)(Af-E\psi)}{(h^2(\psi,\phi)+f)^2}+\frac{E}{h^2(\psi,\phi)+f},\\
\frac{\partial H(\psi,\phi)}{\partial v} &= \frac{2\psi(Af-\phi E)}{(\psi^2+f)^2}\frac{2\phi(Af-E\psi)}{(\phi^2+f)^2}+\frac{E}{\phi^2+f} > 0\\
\frac{\partial H(\psi,\phi)}{\partial u} &= \frac{\partial h(\psi,\phi)}{\partial u}\frac{2h(\psi,\phi)(Af-E\psi)}{(h^2(\psi,\phi)+f)^2} = \frac{E}{\psi^2+f}\frac{2\phi(Af-E\psi)}{(\phi^2+f)^2}.
\end{aligned}$$

It follows that the Jacobian matrix of the map T^2 at the point (ψ,ϕ) is of the form

$$J_{T^2}\begin{pmatrix}\psi\\ \phi\end{pmatrix} = \begin{pmatrix} \frac{E}{\psi^2+f} & \frac{2\psi(Af-\phi E)}{(\psi^2+f)^2} \\ \frac{E}{\psi^2+f}\frac{2\phi(Af-E\psi)}{(\phi^2+f)^2} & \frac{2\psi(Af-\phi E)}{(\psi^2+f)^2}\frac{2\phi(Af-E\psi)}{(\phi^2+f)^2}+\frac{E}{\phi^2+f}\end{pmatrix}.$$

The corresponding characteristic equation is

$$\lambda^2 - p\lambda + q = 0,$$

where

$$\begin{aligned}
p &= TrJ_{T^2}(\psi,\phi) = \frac{E}{\psi^2+f}+\frac{2\psi(A-\phi)}{\psi^2+f}\frac{2\phi(A-\psi)}{\phi^2+f}+\frac{E}{\phi^2+f} > 0,\\
q &= DetJ_{T^2}(\psi,\phi) = \frac{E^2}{(f+\psi^2)(f+\phi^2)} > 0,
\end{aligned}$$

since $Af-\phi E = (A-\phi)\left(\psi^2+f\right)$ and $Af-\psi E = (A-\psi)\left(\phi^2+f\right)$ by Lemma 3.2.

Theorem 3.3. *If $f+\frac{3}{4}A^2 < E < f+A^2$, then the minimal period-two solution (9) of Equation* (1) *is locally asymptotically stable.*

Proof. i) By Lemma 3.2 we have that

$$1+q<2 \Leftrightarrow \frac{E^2}{(f+\psi^2)(f+\phi^2)} < 1 \Leftrightarrow \frac{E^2}{(f+\psi^2)(f+\phi^2)} < \frac{E^2}{(f+A^2)(f+A^2)} < 1$$
$$\Leftrightarrow E^2-\left(f+A^2\right)^2 = \left(E-f-A^2\right)\left(E+f+A^2\right) < 0,$$

which is true because $E < f+A^2$.

ii) Since

$$\begin{aligned}
p &= \frac{E}{\psi^2+f}+\frac{E}{\phi^2+f}+\frac{4\phi\psi\,(Af-\phi E)\,(Af-E\psi)}{\left(\psi^2+f\right)^2\left(\phi^2+f\right)^2}\\
&= \frac{E\left(2f+(\phi+\psi)^2-2\phi\psi\right)}{(f+\psi^2)(f+\phi^2)}+\frac{4\phi\psi\left(A^2f^2+E^2\phi\psi-AfE\,(\phi+\psi)\right)}{\left(\psi^2+f\right)^2\left(\phi^2+f\right)^2}\\
&= \frac{E\left(2f+\left(\frac{A(E-f)}{f-E+A^2}\right)^2-2\left(\frac{(f-E)^2}{f-E+A^2}\right)\right)}{\frac{f^2E^2+E^4-2fE^3+A^2f^3+A^4f^2-A^2fE^2}{(f-E+A^2)^2}}+\frac{4\left(\frac{(f-E)^2}{f-E+A^2}\right)\left(A^2f^2+E^2\left(\frac{(f-E)^2}{f-E+A^2}\right)-AfE\left(\frac{A(E-f)}{f-E+A^2}\right)\right)}{\left(\frac{f^2E^2+E^4-2fE^3+A^2f^3+A^4f^2-A^2fE^2}{(f-E+A^2)^2}\right)^2}\\
&= \frac{-9A^2E^3+4A^4E^2+26f^2E^2+6E^4-20fE^3-16f^3E+8A^2f^3+4A^4f^2+4f^4+22A^2fE^2-21A^2f^2E-6A^4fE}{f^2E^2+E^4-2fE^3+A^2f^3+A^4f^2-A^2fE^2}
\end{aligned}$$

and

$$q=\frac{E^2\left(f-E+A^2\right)^2}{f^2E^2+E^4-2fE^3+A^2f^3+A^4f^2-A^2fE^2},$$

we obtain

$$p<1+q \Leftrightarrow \frac{(f-E)^2\left(f-E+A^2\right)\left(4f-4E+3A^2\right)}{f^2E^2+E^4-2fE^3+A^2f^3+A^4f^2-A^2fE^2} < 0,$$

which is true if

$$f+\frac{3}{4}A^2 < E < f+A^2,$$

because

$$\begin{aligned}
0 &< \left(\phi^2+f\right)\left(\psi^2+f\right) = f^2+f\left((\phi+\psi)^2-2\,(\phi\psi)\right)+\phi^2\psi^2\\
&= f^2+f\left(\left(\frac{A\,(E-f)}{f-E+A^2}\right)^2-2\left(\frac{(f-E)^2}{f-E+A^2}\right)\right)+\left(\frac{(f-E)^2}{f-E+A^2}\right)^2\\
&= \frac{f^2E^2+E^4-2fE^3+A^2f^3+A^4f^2-A^2fE^2}{\left(f-E+A^2\right)^2},
\end{aligned}$$

i.e.,

$$f^2E^2+E^4-2fE^3+A^2f^3+A^4f^2-A^2fE^2 = \left(\psi^2+f\right)\left(\phi^2+f\right)\left(f-E+A^2\right)^2. \quad \square$$

4. Global asymptotic stability

Notice that the function $f(u,v)$ is always increasing with respect to the second variable, and could be increasing or decreasing with respect to the first variable. The critical point of the function $f(u,v)$ in the first variable is $v = \dfrac{Af}{E}$.

So, if $v < \frac{Af}{E}$, the function $f(u,v)$ is increasing in the first variable, and if $v > \frac{Af}{E}$, the function $f(u,v)$ is decreasing in the first variable. Since

$$f\left(u, \frac{Af}{E}\right) = \frac{Au^2 + E\left(\frac{Af}{E}\right)}{u^2 + f} = A, \tag{18}$$

we distinguish the following three cases:

(1) $\dfrac{Af}{E} = A \Leftrightarrow E = f$,

(2) $\dfrac{Af}{E} > A \Leftrightarrow E < f$,

(3) $\dfrac{Af}{E} < A \Leftrightarrow E > f$.

4.1. Case $E = f$

This case corresponds to Case 4 in Table 1, where the zero equilibrium is nonhyperbolic of the resonance type $(-1,1)$ and the unique positive equilibrium solution is locally asymptotically stable.

If $E = f$, then Equation (1) becomes

$$x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + E}, \tag{19}$$

and we obtain the following global result.

Theorem 4.1. *If $E = f$, then the equilibrium point $\overline{x}_2 = \overline{x}_+ = A$ of Equation* (1) *is globally asymptotically stable in* $(0,\infty)$. *More precisely the following statements are true.*

(a) If $x_{-1} \geq A$ and $x_0 \geq A$, then $x_n \geq A$ for all $n > 0$ and $\lim\limits_{n\to\infty} x_n = A$.

(b) If $x_{-1} \leq A$ and $x_0 \leq A$, then $x_n \leq A$ for all $n > 0$ and $\lim\limits_{n\to\infty} x_n = A$.

(c) If either $x_{-1} < A < x_0$ or $x_0 < A < x_{-1}$, then $\{x_n\}_{n=-1}^{\infty}$ oscillates about the equilibrium $\overline{x}_2 = \overline{x}_+ = A$ with semicycles of length one and $\lim\limits_{n\to\infty} x_n = A$.

Proof. Assume that $(x_0, x_{-1}) \in (0,\infty) \times (0,\infty)$.

If $x_0 \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} A$, then $x_{2k} \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} A$, and if $x_{-1} \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} A$, then $x_{2k+1} \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} A$, for $k = 1, 2, \ldots$. This follows from

$$x_{n+1} - A = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + E} - A = E\frac{x_{n-1} - A}{x_n^2 + E}.$$

If $x_0 < A$ ($x_0 > A$), then

$$x_{2k} - x_{2k-2} = \frac{x_{2k-1}^2 (A - x_{2k-2})}{x_{2k-1}^2 + E} > 0 \ \ (< 0),$$

which means that the sequence $\{x_{2k}\}_{k=0}^{\infty}$ is increasing (decreasing) and is bounded from above (below) by A. Therefore, since there is a unique positive equilibrium solution and there is no a minimal period-two solution, we have

$$\lim_{k\to\infty} x_{2k} = A.$$

Similarly, if $x_{-1} < A$ ($x_{-1} > A$), then

$$x_{2k+1} - x_{2k-1} = \frac{x_{2k}^2 (A - x_{2k-1})}{x_{2k}^2 + E} > 0 \ \ (< 0),$$

which means that the sequence $\{x_{2k+1}\}_{k=0}^{\infty}$ is increasing (decreasing) and is bounded from above (below) by A. Therefore

$$\lim_{k\to\infty} x_{2k+1} = A.$$

It follows that $\lim\limits_{n\to\infty} x_n = A$ for all $(x_0, x_{-1}) \in (0,\infty) \times (0,\infty)$ (that is $\overline{x}_2 = \overline{x}_+ = A$ is a global attractor) and by Lemma 2.2 the equilibrium point $\overline{x}_2 = \overline{x}_+ = A$ is globally asymptotically stable in $(0,\infty)$. $\square$

Remark 4.1. Equation (19) is an example of a difference equation that has solutions which oscillate between two regions $(0,A] \times [A,\infty)$ and $(A,\infty) \times (0,A)$ where the function $f(u,v)$ is increasing and decreasing respectively in u so the theory of monotone maps is inapplicable. Notice that the function $f(u,v)$ is increasing in both variables in $(0,A]^2$ and is decreasing in first and increasing in the second variable in $(A,\infty)^2$.

4.2. **Case** $E < f$

In this case, we have three qualitatively different situations: 1), 2) and 3) in Table 1.

Lemma 4.2. *Suppose that $E < f$. Then an invariant and attracting interval of Equation* (1) *is* $\left[0, \frac{Af}{E}\right]$ *and the function $f(u,v)$ is increasing in both arguments in this interval.*

Proof. Notice that the function $f(u,v)$ is nondecreasing with respect to both variables in $\left[0, \frac{Af}{E}\right]^2$. Also, we have

$$f : \left[0, \frac{Af}{E}\right]^2 \to \left[0, \frac{Af}{E}\right],$$

which implies that the interval $\left[0, \frac{Af}{E}\right]$ is an invariant interval, since

$$\max_{(x,y)\in\left[0,\frac{Af}{E}\right]^2} f(x,y) = f\left(\frac{Af}{E}, \frac{Af}{E}\right) \quad \text{and} \quad \min_{(x,y)\in\left[0,\frac{Af}{E}\right]^2} f(x,y) = f(0,0)$$

and $f\left(\frac{Af}{E}, \frac{Af}{E}\right) = A < \frac{Af}{E}$, $f(0,0) = 0$.

Now, we prove that the interval $\left[0, \frac{Af}{E}\right]$ is an attracting interval. If $x_{n-1} > \frac{Af}{E} > A$, then

$$x_{n+1} - x_{n-1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f} - x_{n-1} = \frac{x_n^2(A - x_{n-1}) + x_{n-1}(E - f)}{x_n^2 + f} < 0,$$

i.e., the sequences $\{x_{2k}\}_{k=0}^{\infty}$ and $\{x_{2k+1}\}_{k=0}^{\infty}$ are decreasing, which implies that there exist $k_0, l_0 \in \mathbb{N}$ such that $x_{2k} < \frac{Af}{E}$ for $k \geq k_0$ and $x_{2k+1} < \frac{Af}{E}$ for $k \geq l_0$. Otherwise $x_{2k} \geq \frac{Af}{E}$ and $x_{2k+1} \geq \frac{Af}{E}$ for all $k = -1, 0, 1, \ldots$, and $\lim_{n\to\infty} x_{2k} \geq \frac{Af}{E}$ and $\lim_{n\to\infty} x_{2k+1} \geq \frac{Af}{E}$, which is a contradiction because there is no minimal period-two solution of Equation (1). □

Theorem 4.3. *Assume that is* $0 < E < f - \frac{A^2}{4}$. *Then the unique equilibrium point of Equation* (1), $\bar{x} = 0$, *is globally asymptotically stable.*

Proof. From Lemma 4.2 we see that Equation (1) has only the zero equilibrium point in the invariant and attracting interval $\left[0, \frac{Af}{E}\right]$, and the function $f(u,v)$ is non-decreasing with respect to both variables in $\left[0, \frac{Af}{E}\right]^2$. From Theorem 1.4.8 in [14] and Lemma 2.1, we see that the equilibrium point $\bar{x} = 0$ is globally asymptotically stable. □

In the following analysis, we find conditions for local semi-stability of the positive equilibrium point $\bar{x}_2 = \frac{A}{2}$ of Equation (1), when $0 < E = f - \frac{A^2}{4}$, using center manifold theory.

Proposition 4.1. *Assume that* $0 < E = f - \frac{A^2}{4}$. *Then the nonhyperbolic equilibrium point* $\bar{x}_2 = \frac{A}{2}$ *of Equation* (1) *is semi-stable from above.*

Proof. To prove that $\bar{x}_2$ is semi-stable we will use center manifold theory. Equation (1) is of the form

$$x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + E + \frac{1}{4}A^2}. \tag{20}$$

By the change of variable $y_n = x_n - \frac{1}{2}A$, we obtain the following equation (for $\Omega = 2E + A^2$)

$$y_{n+1} = \frac{Ay_n^2 + A^2 y_n + 2Ey_{n-1}}{2y_n^2 + 2Ay_n + \Omega} \tag{21}$$

which has a zero equilibrium. By the substitution $y_{n-1} = u_n, y_n = v_n$, Equation (21) becomes the system

$$\left.\begin{aligned} u_{n+1} &= v_n, \\ v_{n+1} &= \frac{Av_n^2 + A^2 v_n + 2Eu_n}{2v_n^2 + 2Av_n + \Omega}. \end{aligned}\right\} \tag{22}$$

The Jacobian matrix J_0 at the zero equilibrium for (22) is

$$J_0 = \begin{bmatrix} 0 & 1 \\ \frac{2E}{\Omega} & \frac{A^2}{\Omega} \end{bmatrix}$$

and the corresponding characteristic equation has the form

$$\lambda^2 - \frac{A^2}{\Omega}\lambda - \frac{2E}{\Omega} = 0,$$

with

$$\lambda_1 = 1 \text{ and } \lambda_2 = -\frac{2E}{\Omega} \in (-1, 0).$$

System (22) can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = J_0 \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \gamma(u_n, v_n) \\ \zeta(u_n, v_n) \end{bmatrix} \tag{23}$$

where

$$\left.\begin{aligned} \gamma(u, v) &= 0, \\ \zeta(u, v) &= \frac{-v\left(2A^2v^2 + A\left(A^2 - 2E\right)v + 4AEu + 4Euv\right)}{\Omega\left(2v^2 + 2Av + \Omega\right)}. \end{aligned}\right\} \tag{24}$$

Let

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = P \begin{bmatrix} r_n \\ s_n \end{bmatrix} \tag{25}$$

where P is the matrix that diagonalizes J_0 defined by

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -\frac{2E}{\Omega} \end{bmatrix},$$

such that

$$P^{-1} = -\frac{\Omega}{\Omega + 2E} \begin{bmatrix} -\frac{2E}{\Omega} & -1 \\ -1 & 1 \end{bmatrix},$$

and

$$P^{-1} J_0 P = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{2E}{\Omega} \end{bmatrix}. \tag{26}$$

By (25) we have

$$u_n = r_n + s_n,$$
$$v_n = r_n - \frac{2E}{\Omega} s_n,$$

and by substitution in (24) we have

$$\gamma(u_n,v_n)=\overline{\gamma}(r_n,s_n),$$
$$\zeta(u_n,v_n)=\overline{\zeta}(r_n,s_n),$$

i.e.,

$$\overline{\gamma}(r_n,s_n)=0,$$
$$\overline{\zeta}(r_n,s_n)=\frac{(2Es_n-\Omega r_n)\left(2\Omega^3r_n^2+A\Omega^3r_n-4A^2E\Omega r_ns_n+2AE\left(6E+A^2\right)\Omega s_n-16E^3s_n^2\right)}{\Omega^2\left(2\Omega^2r_n^2+2A\Omega^2r_n-8E\Omega r_ns_n-4AE\Omega s_n+8E^2s_n^2+\Omega^3\right)}.$$

Thus, (23) can be written as

$$P\begin{bmatrix} r_{n+1}\\ s_{n+1}\end{bmatrix}=J_0P\begin{bmatrix} r_n\\ s_n\end{bmatrix}+\begin{bmatrix}\overline{\gamma}(r_n,s_n)\\ \overline{\zeta}(r_n,s_n)\end{bmatrix}$$

or equivalently

$$\begin{bmatrix} r_{n+1}\\ s_{n+1}\end{bmatrix}=P^{-1}J_0P\begin{bmatrix} r_n\\ s_n\end{bmatrix}+P^{-1}\begin{bmatrix}\overline{\gamma}(r_n,s_n)\\ \overline{\zeta}(r_n,s_n)\end{bmatrix},$$

and by using (26):

$$\begin{bmatrix} r_{n+1}\\ s_{n+1}\end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & -\frac{2E}{\Omega}\end{bmatrix}\begin{bmatrix} r_n\\ s_n\end{bmatrix}+P^{-1}\begin{bmatrix}\overline{\gamma}(r_n,s_n)\\ \overline{\zeta}(r_n,s_n)\end{bmatrix}. \tag{27}$$

So, the normal form of System (23) is:

$$\begin{bmatrix} r_{n+1}\\ s_{n+1}\end{bmatrix}=\begin{bmatrix} 1 & 0\\ 0 & -\frac{2E}{\Omega}\end{bmatrix}\begin{bmatrix} r_n\\ s_n\end{bmatrix}+\begin{bmatrix}\widehat{\gamma}(r_n,s_n)\\ \widehat{\zeta}(r_n,s_n)\end{bmatrix}, \tag{28}$$

where

$$\widehat{\gamma}(r_n,s_n)=-\widehat{\zeta}(r_n,s_n),$$

and

$$\widehat{\gamma}(r_n,s_n)=\frac{(2Es_n-\Omega r_n)\left(2\Omega^3r_n^2+A\Omega^3r_n-4A^2E\Omega r_ns_n+2AE\Omega(\Omega+4E)s_n-16E^3s_n^2\right)}{\Omega(\Omega+2E)\left(2\Omega^2r_n^2+2A\Omega^2r_n-8E\Omega r_ns_n-4AE\Omega s_n+8E^2s_n^2+\Omega^3\right)}.$$

Now, we let $s=\chi(r)=\Psi(r)+O\left(r^4\right)$, where $\Psi(r)=\alpha r^2+\beta r^3$, $\alpha,\beta\in\mathbb{R}$ is the center manifold, and where the map χ must satisfy the center manifold equation (for $\lambda_2=-\frac{2E}{\Omega}$):

$$\chi(r+\widehat{\gamma}(r,\chi(r)))-\lambda_2\chi(r)-\widehat{\zeta}(r,\chi(r))=0. \tag{29}$$

If we approximate $\widehat{\gamma}(r,s)$ by a Taylor polynomial as follows

$$\widehat{\gamma}(r,s)=\sum_{i=1}^{3}\frac{1}{i!}\left(r\frac{\partial}{\partial r}y(0,0)+s\frac{\partial}{\partial s}y(0,0)\right)^i+O_4, \tag{30}$$

we obtain

$$\widehat{\gamma}(r,\chi(r)) = r^2 \frac{-A\Omega^2 - 2r\Omega^2 + 2A^2 r\Omega - 8Ar\alpha E^2}{\Omega^2(\Omega + 2E)} + O\left(r^4\right),$$

and

$$\chi(r + \widehat{\gamma}(r,\chi(r))) = r^2\alpha + \left(\beta - \frac{2A\alpha}{\Omega + 2E}\right) r^3 + O\left(r^4\right).$$

Then from (29) we have the following system

$$2A\left(28\left(2E + A^2\right)E + 3A^4\right)\alpha + \left(2E + A^2\right)\left(4E + A^2\right)^2\beta = 4\left(2E + A^2\right)\left(E + 2A^2\right),$$
$$\alpha\left(4E + A^2\right)^2 = A\left(2E + A^2\right),$$

with the solution $(\alpha,\beta) = \left(\frac{A\left(2E+A^2\right)}{\left(4E+A^2\right)^2}, \frac{2\left(2E+A^2\right)\left(16E^2+4A^2E+A^4\right)}{\left(4E+A^2\right)^4}\right)$.

Let $s = \chi(r) = \Psi(r) + O\left(r^4\right)$, where

$$\Psi(r) = \frac{A\left(2E + A^2\right)}{\left(4E + A^2\right)^2} r^2 + \frac{2\left(2E + A^2\right)\left(16E^2 + 4A^2E + A^4\right)}{\left(4E + A^2\right)^4} r^3.$$

In view of Theorem 5.9 of [6] the study of the stability of the zero equilibrium of Equation (21), that is the positive nonhyperbolic equilibrium $\overline{x}_2 = \frac{A}{2}$ of Equation (20), reduces to the stability of the following equation

$$r_{n+1} = r_n + \widehat{\gamma}(r_n, s_n) = G(r_n), \tag{31}$$

where

$$G(r) = r + \widehat{\gamma}(r,\Psi(r)) = -\frac{r\left(4E\left(8E + A^2\right)r^2 + A\left(4E + A^2\right)^2 r - \left(4E + A^2\right)^3\right)}{\left(A^2 + 4E\right)^3}.$$

Since $\dfrac{d}{dr}G(0) = 1$ and

$$\frac{d^2}{dr^2}G(0) = -\frac{2A}{A^2 + 4E} < 0,$$

from Theorem 1.6 of [15], the zero equilibrium of (31) is an unstable fixed point, that is semi-stable from above. Therefore, from Theorem 5.9 of [6], the zero equilibrium of Equation (21), that is the positive nonhyperbolic equilibrium $\overline{x}_2 = \frac{A}{2}$ of Equation (20) is semi-stable from above. □

The next result shows global behavior of Equation (1) when $0 < E = f - \frac{A^2}{4}$.

Theorem 4.4. *Assume that $0 < E = f - \frac{A^2}{4}$. Then Equation (1) has two equilibrium points: $\overline{x}_1 = 0$ which is locally asymptotically stable and $\overline{x}_2 = \frac{A}{2}$ which is nonhyperbolic of the stable type, more precisely $\overline{x}_2$ is semi-stable. There exists a set $\mathcal{C} \subset Q_2(\overline{x}_2,\overline{x}_2) \cup Q_4(\overline{x}_2,\overline{x}_2)$ with endpoints on the axes and $\mathcal{W}^s((\overline{x}_2,\overline{x}_2)) = \mathcal{C}$ is an invariant subset of the basin of attraction of $(\overline{x}_2,\overline{x}_2)$. Also, $\mathcal{C}$ is a graph of a* *strictly decreasing continuous function of the first variable on an interval and separates* $\mathcal{R} = [0,+\infty)^2$ *into two connected and invariant components* $\mathcal{W}_-((\overline{x}_2,\overline{x}_2))$ *and* $\mathcal{W}_+((\overline{x}_2,\overline{x}_2))$*, where*

$$\mathcal{W}_-((\overline{x}_2,\overline{x}_2)) := \{(x,y) \in \mathcal{R} \setminus \mathcal{C} : \exists (x',y') \in \mathcal{C} \text{ with } (x,y) \preceq_{ne} (x',y')\}$$

and

$$\mathcal{W}_+((\overline{x}_2,\overline{x}_2)) := \{(x,y) \in \mathcal{R} \setminus \mathcal{C} : \exists (x',y') \in \mathcal{C} \text{ with } (x',y') \preceq_{ne} (x,y)\},$$

such that:

(i) if $(x_{-1},x_0) \in \mathcal{W}_+((\overline{x}_2,\overline{x}_2))$*, then* $\lim_{n\to\infty} x_n = \frac{A}{2}$*;*

(ii) if $(x_{-1},x_0) \in \mathcal{W}_-((\overline{x}_2,\overline{x}_2))$*, then* $\lim_{n\to\infty} x_n = 0$*.*

Proof. Since

$$x_{n+1} - \frac{A}{2} = \frac{\frac{A}{2}\left(x_n - \frac{A}{2}\right)\left(x_n + \frac{A}{2}\right) + E\left(x_{n-1} - \frac{A}{2}\right)}{x_n^2 + E + \frac{A^2}{4}},$$

this means that $\left[0,\frac{A}{2}\right]$ and $\left[\frac{A}{2},+\infty\right)$ are invariant intervals, i.e., if $(x_{-1},x_0) \in \left[0,\frac{A}{2}\right]^2$ (or $\left[\frac{A}{2},+\infty\right)^2$), then $x_n \in \left[0,\frac{A}{2}\right]$ (or $\left[\frac{A}{2},+\infty\right)$) for all $n = 1,2,\ldots$.

By using Lemma 4.2, Proposition 4.1 and the theory of monotone maps in the plane, more precisely, the theory of cooperative maps, since the corresponding map T^2 from (17) is a cooperative map, the conclusion of the theorem follows. In other words, the version of Theorem 1.2 for a function increasing in both variables applies. □

Theorem 4.5. *Assume that* $0 < f - \frac{A^2}{4} < E < f$*. Then Equation* (1) *has three equilibrium points:* $\overline{x}_1 = 0$ *and* $\overline{x}_3 = \overline{x}_+$ *which are locally asymptotically stable and* $\overline{x}_2 = \overline{x}_-$ *which is a saddle point. There exists a set* $\mathcal{C} \subset \mathcal{Q}_2(\overline{x}_2,\overline{x}_2) \cup \mathcal{Q}_4(\overline{x}_2,\overline{x}_2)$ *with endpoints on the axes and* $\mathcal{W}^s((\overline{x}_2,\overline{x}_2)) = \mathcal{C}$ *is an invariant subset of the basin of attraction of* $(\overline{x}_2,\overline{x}_2)$*. Also,* $\mathcal{C}$ *is a graph of a strictly decreasing continuous function of the first variable on an interval and separates* $\mathcal{R} = [0,+\infty)^2$ *into two connected and invariant components* $\mathcal{W}_-((\overline{x}_2,\overline{x}_2))$ *and* $\mathcal{W}_+((\overline{x}_2,\overline{x}_2))$ *such that:*

(i) if $(x_{-1},x_0) \in \mathcal{W}_+((\overline{x}_2,\overline{x}_2))$*, then* $\lim_{n\to\infty} x_n = \overline{x}_+$*;*

(ii) if $(x_{-1},x_0) \in \mathcal{W}_-((\overline{x}_2,\overline{x}_2))$*, then* $\lim_{n\to\infty} x_n = 0$*.*

Proof. It follows from Lemmas 2.2 and 4.2 and the theory of monotone maps in the plane, more precisely theory of cooperative maps, since the corresponding map T^2 is a cooperative map. □

4.3. **Case** $E > f$

In this case, we have four qualitatively different situations: 5), 6), 7) and 8) in Table 1.

Lemma 4.6. *If $f < E < f + A^2$ then an invariant and attracting interval of Equation* (1) *is* $\left[A, \frac{A^3}{f+A^2-E}\right]$ *and the function* $f(u,v)$ *is decreasing in the first and increasing in the second argument in this interval.*

Proof. If $v \geq A > \frac{Af}{E}$, then the function $f(u,v)$ is decreasing in the first variable and increasing in the second variable, which implies that

$$x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f} > \frac{Ax_n^2 + E\frac{Af}{E}}{x_n^2 + f} = A.$$

Therefore, if there exists an invariant interval of Equation (1), then it is of the form $[A, U]$. Since, the function f is decreasing in the first variable and increasing in the second variable in $[A,U]^2$, we obtain that

$$U \geq \max_{(x,y)\in[A,U]^2} f(x,y) = f(A,U) \quad \text{and} \quad \min_{(x,y)\in[A,U]^2} f(x,y) = f(U,A) \geq A,$$

i.e.,

$$U \geq f(A,U) \Leftrightarrow U \geq \frac{A^3 + EU}{A^2 + f} \Leftrightarrow \left(U \geq \frac{A^3}{f + A^2 - E} \text{ and } E < f + A^2\right),$$

from which we can set $U = \frac{A^3}{f+A^2-E}$ when $E < f + A^2$. This means that $\left[A, \frac{A^3}{f+A^2-E}\right]$ is an invariant interval of Equation (1) when $E < f + A^2$.

On the other hand, if $x_{n-1} < \frac{Af}{E} < A$, we obtain

$$x_{n+1} - x_{n-1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f} - x_{n-1} = \frac{x_n^2 (A - x_{n-1}) + x_{n-1}(E - f)}{x_n^2 + f} > 0,$$

i.e., the sequences $\{x_{2k}\}_{k=0}^{\infty}$ and $\{x_{2k+1}\}_{k=0}^{\infty}$ are increasing, which implies that there exist $k_0, l_0 \in \mathbb{N}$ such that $x_{2k} > \frac{Af}{E}$ for $k \geq k_0$ and $x_{2k+1} > \frac{Af}{E}$ for $k \geq l_0$. Otherwise $x_{2k} \leq \frac{Af}{E}$ and $x_{2k+1} \leq \frac{Af}{E}$ for all $k = -1, 0, 1, \ldots$, and $\lim_{n\to\infty} x_{2k} \geq \frac{Af}{E}$ and $\lim_{n\to\infty} x_{2k+1} \geq \frac{Af}{E}$, which is a contradiction because the zero equilibrium is a repeller and there is no a minimal period-two solution of Equation (1) in $\left[0, \frac{Af}{E}\right]$. This means that $\left[A, \frac{A^3}{f+A^2-E}\right]$ is also an attracting interval for Equation (1) when $E < f + A^2$. □

Theorem 4.7. *Assume that $f < E < f + \frac{3}{4}A^2$. Then Equation* (1) *has two equilibrium points: $\bar{x}_1 = 0$ which is a repeller and $\bar{x}_2 = \bar{x}_+$ which is locally asymptotically stable. Also, the positive equilibrium point $\bar{x}_2 = \bar{x}_+$ is globally asymptotically stable in* $(0, +\infty)$.

Proof. The function $f(u,v)$ is decreasing in the first variable and increasing in the second variable in $\left[A, \frac{A^3}{f+A^2-E}\right]^2$ and there is no a minimal period-two solution. Since $\left[A, \frac{A^3}{f+A^2-E}\right]$ is an invariant and an attracting interval of Equation (1), using

Theorem 1.4.6 in [14] and Lemma 2.2, we see that the positive equilibrium point $\overline{x}_2 = \overline{x}_+$ is globally asymptotically stable in $(0,+\infty)$. □

Now, by using center manifold theory we find conditions for the stability of the positive equilibrium point $\overline{x}_2 = \overline{x}_+ = \frac{3}{2}A$ of Equation (1), when $E = f + \frac{3}{4}A^2$.

Proposition 4.2. *Assume that $E = f + \frac{3}{4}A^2$. Then the positive equilibrium point $= \overline{x}_+ = \frac{3}{2}A$ of Equation* (1) *is locally asymptotically stable.*

Proof. To prove that $\overline{x}_2$ is a local sink we will use center manifold theory. Since $E = f + \frac{3}{4}A^2$, Equation (1) is of the form

$$x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + E - \frac{3}{4}A^2}. \tag{32}$$

By the change of variable $y_n = x_n - \frac{3}{2}A$, we get the equation (for $\Phi = 3A^2 + 2E$)

$$y_{n+1} = \frac{-Ay_n^2 - 3A^2 y_n + 2Ey_{n-1}}{2y_n^2 + 6Ay_n + \Phi}, \tag{33}$$

which has a zero equilibrium. By the substitution $y_{n-1} = u_n$, $y_n = v_n$, Equation (33) becomes the system

$$\left.\begin{array}{c} u_{n+1} = v_n, \\ v_{n+1} = \frac{-Av_n^2 - 3A^2 v_n + 2Eu_n}{2v_n^2 + 6Av_n + \Phi}. \end{array}\right\} \tag{34}$$

The Jacobian matrix J_0 at the zero equilibrium for (34) is

$$J_0 = \begin{bmatrix} 0 & 1 \\ \frac{2E}{\Phi} & \frac{-3A^2}{\Phi} \end{bmatrix}$$

and the corresponding characteristic equation is

$$\lambda^2 + \frac{3A^2}{\Phi}\lambda - \frac{2E}{\Phi} = 0,$$

with

$$\lambda_1 = -1, \lambda_2 = \frac{2E}{\Phi} \in (0,1).$$

System (34) can be written as

$$\begin{bmatrix} u_{n+1} \\ v_{n+1} \end{bmatrix} = J_0 \begin{bmatrix} u_n \\ v_n \end{bmatrix} + \begin{bmatrix} \gamma(u_n, v_n) \\ \zeta(u_n, v_n) \end{bmatrix} \tag{35}$$

where

$$\left.\begin{array}{c} \gamma(u,v) = 0, \\ \zeta(u,v) = v\dfrac{18A^3 v + 6A^2 v^2 - Av\Phi - 12AuE - 4uvE}{\Phi(\Phi + 6Av + 2v^2)}. \end{array}\right\} \tag{36}$$

Let

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = P \begin{bmatrix} r_n \\ s_n \end{bmatrix}, \tag{37}$$

where P is the matrix that diagonalizes J_0 defined by

$$P=\left[\begin{array}{cc}1 & 1\\ -1 & \frac{2E}{\Phi}\end{array}\right].$$

Then

$$P^{-1}=\frac{\Phi}{\Phi+2E}\left[\begin{array}{cc}\frac{2E}{\Phi} & -1\\ 1 & 1\end{array}\right],$$

and

$$P^{-1}J_0P=\left[\begin{array}{cc}-1 & 0\\ 0 & \frac{2E}{\Phi}\end{array}\right]. \tag{38}$$

The normal form of system (35), obtained in a similar manner as in the proof of Proposition 4.1, is

$$\left[\begin{array}{c}r_{n+1}\\ s_{n+1}\end{array}\right]=\left[\begin{array}{cc}-1 & 0\\ 0 & \frac{2E}{\Phi}\end{array}\right]\left[\begin{array}{c}r_n\\ s_n\end{array}\right]+\left[\begin{array}{c}\widehat{\gamma}(r_n,s_n)\\ \widehat{\zeta}(r_n,s_n)\end{array}\right], \tag{39}$$

where

$$\widehat{\gamma}(r_n,s_n)=-\widehat{\zeta}(r_n,s_n),$$

and

$$\widehat{\gamma}(r_n,s_n)=-\frac{\Phi r_n-2Es_n}{\Phi(\Phi+2E)}\frac{-2\Phi^3r_n^2+16E^3s_n^2+5A\Phi^3r_n+2A\Phi E(\Phi+12E)s_n+12A^2\Phi Er_ns_n}{2\Phi^2r_n^2+8E^2s_n^2-6A\Phi^2r_n+12A\Phi Es_n-8\Phi Er_ns_n+\Phi^3}.$$

Now, let $s=\chi(r)=\Psi(r)+O\left(r^4\right)$, where $\Psi(r)=\alpha r^2+\beta r^3$, $\alpha,\beta\in\mathbb{R}$ is the center manifold, and where map χ must satisfy the center manifold equation (for $\lambda_2=\frac{2E}{\Phi}$)

$$\chi\left(-r+\widehat{\gamma}(r,\chi(r))\right)-\lambda_2\chi(r)-\widehat{\zeta}(r,\chi(r))=0. \tag{40}$$

By (30), we have that

$$\widehat{\gamma}(r,\chi(r))=-\frac{5A\Phi^2r^2+2\left(\Phi\left(-\Phi+15A^2\right)-4A\alpha E(\Phi-3E)\right)r^3}{\Phi^2(\Phi+2E)}+O\left(r^4\right),$$

and

$$\chi\left(-r+\widehat{\gamma}(r,\chi(r))\right)=\frac{\alpha(\Phi+2E)r^2+(10A\alpha-(\Phi+2E)\beta)r^3}{\Phi+2E}+O\left(r^4\right).$$

Then from (40) we have the system

$$\begin{aligned}2A(6E-5\Phi)(\Phi+2E)\alpha+\Phi(\Phi+2E)^2\beta&=2\Phi\left(\Phi-15A^2\right),\\ \alpha\left(\Phi^2-4E^2\right)&=5A\Phi,\end{aligned}$$

with the solution $(\alpha,\beta)=\left(\frac{5}{3}\frac{2E+3A^2}{A(4E+3A^2)},\frac{26}{3}\frac{2E+3A^2}{(4E+3A^2)^2}\right)$.

Let $s=\chi(r)=\Psi(r)+O\left(r^4\right)$, where

$$\Psi(r)=\frac{5}{3}\frac{2E+3A^2}{A(4E+3A^2)}r^2+\frac{26}{3}\frac{2E+3A^2}{(4E+3A^2)^2}r^3.$$

Now, according to Theorem 5.9 of [6] the study of the stability of the zero equilibrium of Equation (33), that is the positive nonhyperbolic equilibrium $\overline{x}_2 = \frac{3}{2}A$ of Equation (32), reduces to the study of stability of the following equation

$$r_{n+1} = -r_n + \widehat{\gamma}(r_n, s_n) = G(r_n), \tag{41}$$

where

$$G(r) = -r + \widehat{\gamma}(r, \Psi(r)) = -\frac{r\left(4\left(-E+18A^2\right)r^2 + 15A\left(4E+3A^2\right)r + 3\left(4E+3A^2\right)^2\right)}{3\left(3A^2+4E\right)^2}.$$

Since $\frac{d}{dr}G(0) = -1$, $\frac{d^2}{dr^2}G(0) = -\frac{10A}{3A^2+4E}$ and $\frac{d^3}{dr^3}G(0) = -\frac{8\left(18A^2-E\right)}{\left(3A^2+4E\right)^2}$, then the corresponding Schwarzian is of the form

$$SG(0) = -\frac{d^3}{dr^3}G(0) - \frac{3}{2}\left(\frac{d^2}{dr^2}G(0)\right)^2 = -\frac{2}{3A^2+4E} < 0,$$

and from Theorem 1.6 of [15], the zero equilibrium of (41) is a sink. Therefore, from Theorem 5.9 of [6], the zero equilibrium of Equation (33), that is the positive nonhyperbolic equilibrium $\overline{x}_2 = \frac{3}{2}A$ of Equation (32) is a sink. □

The next result shows global behavior of Equation (1) when is $E = f + \frac{3}{4}A^2$.

Theorem 4.8. *If $E = f + \frac{3}{4}A^2$, then the positive equilibrium point $\overline{x}_2 = \overline{x}_+ = \frac{3}{2}A$ is globally asymptotically stable in $(0, +\infty)$.*

Proof. The proof is the same as the proof of Theorem 4.7 since the equilibrium point $\overline{x}_2$ is stable (see Proposition 4.2). □

Theorem 4.9. *Assume that $f + \frac{3}{4}A^2 < E < f + A^2$. Then Equation (1) has two equilibrium points: : $\overline{x}_1 = 0$, which is a repeller and $\overline{x}_2 = \overline{x}_+$, which is a saddle point, and has the unique minimal period-two solution $\{\ldots\phi, \psi, \phi, \psi, \ldots\}$, which is locally asymptotically stable, where ϕ and ψ are of the form (10). There exists a set $\mathcal{C} \subset \mathcal{Q}_1(\overline{x}_2, \overline{x}_2) \cup \mathcal{Q}_3(\overline{x}_2, \overline{x}_2)$ and $\mathcal{W}^s((\overline{x}_2, \overline{x}_2)) = \mathcal{C}$ is the basin of attraction of $(\overline{x}_2, \overline{x}_2)$. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval and separates $\mathcal{R}_1 = [0, \infty)^2 \setminus \{(0,0)\}$ into two connected and invariant parts, $\mathcal{W}_-^{(1)}((\overline{x}_2, \overline{x}_2))$ and $\mathcal{W}_+^{(1)}((\overline{x}_2, \overline{x}_2))$, where*

$$\begin{aligned}
\mathcal{W}_-^{(1)}((\overline{x}_2, \overline{x}_2)) \quad &: \quad = \{(x,y) \in \mathcal{R}_1 \setminus \mathcal{C} : \exists (x', y') \in \mathcal{C} \text{ with } (x,y) \preceq_{se} (x', y')\} \\
\mathcal{W}_+^{(1)}((\overline{x}_2, \overline{x}_2)) \quad &: \quad = \{(x,y) \in \mathcal{R}_1 \setminus \mathcal{C} : \exists (x', y') \in \mathcal{C} \text{ with } (x', y') \preceq_{se} (x,y)\},
\end{aligned}$$

such that:

(i) if $(x_{-1}, x_0) \in \mathcal{W}_+^{(1)}((\overline{x}_2, \overline{x}_2))$, then $\lim_{n\to\infty} x_{2n} = \phi$ and $\lim_{n\to\infty} x_{2n+1} = \psi$;

(ii) if $(x_{-1}, x_0) \in \mathcal{W}_-^{(1)}((\overline{x}_2, \overline{x}_2))$, then $\lim_{n\to\infty} x_{2n} = \psi$ and $\lim_{n\to\infty} x_{2n+1} = \phi$.

Proof. It is follows from Lemma 4.6 and the theory of monotone maps in the plane, more precisely, competitive maps, since the corresponding map T^2 is a competitive map, see [7, 16]. In other words Theorem 1.3 applies. □

Lemma 4.10. *Suppose that $E \geq f + A^2$. Then an invariant and attracting interval of Equation (1) is $[A, +\infty)$.*

Proof. If $x_{n-1} > A$ $(> \frac{Af}{E})$, then

$$x_{n+1} = \frac{Ax_n^2 + Ex_{n-1}}{x_n^2 + f} > \frac{Ax_n^2 + EA}{x_n^2 + f} > \frac{Ax_n^2 + E\frac{Af}{E}}{x_n^2 + f} = A > \frac{Af}{E},$$

so that

$$x_{n-1}, x_n \in [A, +\infty) \Rightarrow x_{n+1} \in [A, +\infty),$$

i.e., $[A, +\infty)$ is an invariant interval for Equation (1). The proof of the fact that $[A, +\infty)$ is also an attracting interval of Equation (1) is the same as in Lemma 4.6. □

Theorem 4.11. *If $E \geq f + A^2$, then Equation (1) has two equilibrium points: $\overline{x}_1 = 0$, which is a repeller and $\overline{x}_2 = \overline{x}_+$, which is a saddle point. There exists a set $\mathcal{C} \subset \mathcal{Q}_1(\overline{x}_2, \overline{x}_2) \cup \mathcal{Q}_3(\overline{x}_2, \overline{x}_2)$ and $\mathcal{W}^s((\overline{x}_2, \overline{x}_2)) = \mathcal{C}$ is the basin of attraction of $(\overline{x}_2, \overline{x}_2)$. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval and separates $\mathcal{R}_1 = [0, \infty)^2 \setminus \{(0,0)\}$ into two connected and invariant parts, $\mathcal{W}_-^{(1)}((\overline{x}_2, \overline{x}_2))$ and $\mathcal{W}_+^{(1)}((\overline{x}_2, \overline{x}_2))$, such that:*

(i) if $(x_{-1}, x_0) \in \mathcal{W}_+^{(1)}((\overline{x}_2, \overline{x}_2))$, then $\lim_{n\to\infty} x_{2n+1} = +\infty$ and $\lim_{n\to\infty} x_{2n} = A$;

(ii) if $(x_{-1}, x_0) \in \mathcal{W}_-^{(1)}((\overline{x}_2, \overline{x}_2))$, then $\lim_{n\to\infty} x_{2n+1} = A$ and $\lim_{n\to\infty} x_{2n} = +\infty$.

Proof. By using Lemma 4.10 it follows that every solution of Equation (1) eventually enters the interval $[A, +\infty)$. Since then the function $f(x_n, x_{n-1})$ is decreasing in the first variable and increasing in the second variable in $[A, +\infty)$, we can apply Theorem 1.1 and the theory of monotone maps, the corresponding map T^2 is a competitive. More precisely Theorem 1.3 applies. It is easy to see that if $\lim_{n\to\infty} x_{2n+1} = +\infty$, then $\lim_{n\to\infty} x_{2n} = l < \infty$, which implies

$$l = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} \frac{A + E\frac{x_{2n}}{x_{2n+1}^2}}{1 + \frac{f}{x_{2n+1}^2}} = A.$$

Also, if $\lim_{n\to\infty} x_{2n} = +\infty$ then $\lim_{n\to\infty} x_{2n+1} = A$. □

Acknowledgment. The authors are grateful to one of the referees for numerous corrections and improvements.

This work is supported in part by the Fundamental Research Funds of Bosnia and Herzegovina FMON number 01-6211-1-IV/19.

REFERENCES

[1] A. M. Amleh, E. Camouzis, and G. Ladas, *On the Dynamics of a Rational Difference Equation, Part 1.* Inter. J. Difference Equ., 3 (1)(2008), 1–35.

[2] A. Bilgin, M. R. S. Kulenović, and E. Pilav, *Basins of Attraction of Period-Two Solutions of Monotone Difference Equations*, Adv. Difference Equ., (2016), 2016: 74, DOI 10.1186/s13662-016-0801-y.

[3] A. Bilgin and M. R. S. Kulenović, Global Asymptotic Stability for Discrete Single Species Population Models, *Discrete Dyn. Nat. Soc.*, Vol. 2017, (2017), 15 p.

[4] A. Brett and M. R. S. Kulenović, *Basins of Attraction of Equilibrium Points of Monotone Difference Equations.* Sarajevo J. Math., 5 (18) (2009), 211–233.

[5] E. Camouzis and G. Ladas, *When does local asymptotic stability imply global attractivity in rational equations?*, J. Difference Equ. Appl., 12 (2006), 863–885.

[6] S. Elaydi, *Discrete Chaos*, Chapman & Hall/CRC, Boca Raton, London, New York, 2000.

[7] M. Garić-Demirović, M. R. S. Kulenović and M. Nurkanović, *Basins of Attraction of Equilibrium Points of Second Order Difference Equations*, Applied Math. Letters, 25 (2012), 2110–2115.

[8] M. Garić-Demirović, M. R. S. Kulenović and M. Nurkanović, *Basins of Attraction of Certain Homogeneous Second Order Quadratic Fractional Difference Equation*, Journal of Concrete and Applicable Mathematics, 13 (1-2)(2015), 35–50.

[9] S. Jašarević Hrustić, M.R.S. Kulenović and M. Nurkanović, *Global Dynamics and Bifurcations of Certain Second Order Rational Difference Equation with Quadratic Terms*, Qualitative Theory of Dynamical Systems, 15 (1) (2016), 283–307, DOI 10.1007/s12346-015-0148-x.

[10] S. Jašarević Hrustić, M.R.S. Kulenović and M. Nurkanović, *Global Dynamics of Certain Second Order Fractional Difference Equations with Quadratic Terms*, Discrete Dynamics in Nature and Society, Vol. 2016 (2016), Article ID 3716042, 14 pages, http://dx.doi.org/10.1155/2016/3716042.

[11] S. Jašarević Hrustić, M.R.S. Kulenović, Z. Nurkanović and E. Pilav, *Birkhoff normal forms, KAM theory and symmetries for certain second order rational difference equation with quadratic term*, Int. J. Difference Equ., 10 (2015), 181–199.

[12] S. Kalabušić, M. Nurkanović and Z. Nurkanović, *Global Dynamics of Certain Mix Monotone Difference Equations*, Mathematics **2018**, 6(1), 10; https://doi.org/10.3390/math6010010

[13] Y. Kostrov and Z. Kudlak, *On a second-order rational difference equation with a quadratic term*, International Journal of Difference Equations, 11 (2) (2016), 179-202.

[14] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman and Hall/CRC, Boca Raton, London, 2001.

[15] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica,* Chapman and Hall/CRC, Boca Raton, London, 2002.

[16] M. R. S. Kulenović and O. Merino, *Global Bifurcations for Competitive Systems in the Plane*, Discrete Contin. Dyn. Syst. Ser. B 12(2009), 133–149.

[17] M. R. S. Kulenović and O. Merino, *Invariant manifolds for planar competitive and cooperative maps*, Journal of Difference Equations and Applications, 24 (6) (2018), 898-915.

[18] M.R.S. Kulenović, O. Merino and M. Nurkanović, *Global dynamics of certain competitive system in the plane*, Journal of Difference Equations and Applications, 18 (12) (2012), 1951-1966.

[19] M.R.S. Kulenović, S. Moranjkić, M. Nurkanović and Z. Nurkanović, *Global Asymptotic Stability and Naimark_Sacker Bifurcation of Certain Mix Monotone Difference Equation*, Discrete Dynamics in Nature and Society, Vol. 2018, Article ID 7052935, 22 pages, https://doi.org/10.1155/2018/7052935.

[20] M. R. S. Kulenović and M. Nurkanović, *Asymptotic Behavior of a Competitive System of Linear Fractional Difference Equations*, Advances in Difference Equations, Vol. 2006 (2006), Article ID 19756,1-13.

[21] S. Moranjkić and Z. Nurkanović, *Local and Global Dynamics of Certain Second-Order Rational Difference Equations Containing Quadratic Terms*, Advances in Dynamical Systems and Applications, ISSN 0973-5321, 12 (2) (2017), 123-157.

[22] C. Mylona, N. Psarros, G. Papaschinopoulos and C. Schinas, *Stability of the Non-Hyperbolic Zero Equilibrium of Two Close-to-Symmetric Systems of Difference Equations with Exponential Terms*, Symmetry **2018**, 10, 188; doi:10.3390/sym10060188.

[23] N. Psarros, G. Papaschinopoulos and C.J. Schinas, *Semistability of two systems of differens equations using centre manifold theory*, Math. Meth. Appl. Sci. **2016**; DOI:10.1002/mma.3904.

[24] H.L. Smith, *Planar competitive and cooperative difference equations*, Journal of Difference Equations and Applications, 3 (5-6) (1998),335-357.

[25] G.G. Thomson, ”A proposal for a treshold stock size and maximum fishing mortality rate”, in *Risk Evaluation and Biological Reference Points for Fisheries Management, Canad. Spec. Publ. Fish. Aquat. Sci*, S.J. Smith, J.J. Hunt, and D. Rivard, Eds., vol.120, pp. 303-320, NCR Research Press, Ottawa Canada, 1993.

(Received: July 24, 2019)
(Revised: December 12, 2019)

Mustafa R.S. Kulenović
University of Rhode Island
Department of Mathematics
Kingston, RI 02881, USA
e-mail: *kulenm@math.uri.edu*
and
Mehmed and Zehra Nurkanović
University of Tuzla
Department of Mathematics
Tuzla, Univerzitetska 4,
Bosnia and Herzegovina
e-mail: *mehmed.nurkanovic@untz.ba*
e-mail: *zehra.nurkanovic@untz.ba*