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COMPLETE SEMIGROUPS OF BINARY RELATIONS DEFINED BY SEMILATTICES OF THE CLASS $Z-$ELEMENTARY $X-$SEMILATTICE OF UNIONS

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Abstract. In this paper we investigate idempotents of complete semigroups of binary relations defined by semilattices of the class $Z-$elementary $X-$semilattice of unions. For the case where X is a finite set we derive formulas by calculating the numbers of idempotents of the respective semigroup.

1. Introduction

Let X be an arbitrary nonempty set, D be an $X-$semilattice of unions, i.e. a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping from X into D. To each such a mapping f there corresponds a binary relation α_f on the set X that satisfies the condition

$$\alpha_f = \bigcup_{x\in X} (\{x\} \times f(x))$$

The set of all such α_f $(f : X \to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by an $X-$semilattice of unions D.

We denote by $\emptyset$ an empty binary relation or empty subset of the set X. The condition $(x, y) \in \alpha$ will be written in the form $x\alpha y$. Further let $x, y \in X$, $Y \subseteq X$, $\alpha \in B_X(D)$, $T \in D$, $\emptyset \neq D' \subseteq D$, $\breve{D} = \cup D = \bigcup_{Y\in D} Y$ and $t \in \breve{D}$. Then by symbols we denote the following sets:

$$y\alpha = \{x \in X \mid y\alpha x\}, \ Y\alpha = \bigcup_{y\in Y} y\alpha, \quad 2^X = \{Y \mid Y \subseteq X\}, \ X^* = 2^X \setminus \{\emptyset\}$$

$$V(D, \alpha) = \{Y\alpha \mid Y \in D\}, \ D'_T = \{T' \in D' \mid T \subseteq T'\}$$

$$\ddot{D}'_T = \{T' \in D' \mid T' \subseteq T\}, \ D'_t = \{Z' \in D' \mid t \in Z'\}, \ l(D', T) = \cup (D' \setminus D'_T).$$

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By symbol $\wedge(D, D')$ we mean an exact lower bound of the set D' in the semilattice D.

Definition 1.1. Let $\alpha \in B_X(D)$. If $\alpha \circ \alpha = \alpha$, then α is called an idempotent element of the semigroup $B_X(D)$.

Definition 1.2. We say that a complete X-semilattice of unions D is an XI-*semilattice of unions* if it satisfies the following two conditions:

a) $\wedge(D, D_t) \in D$ for any $t \in \breve{D}$;

b) $Z = \bigcup_{t \in Z} \wedge(D, D_t)$ for any nonempty Z element of D.

Definition 1.3. Let D be an arbitrary complete $X-$semilattice of unions, $\alpha \in B_X(D)$ and $Y_T^\alpha = \{x \in X \mid x\alpha = T\}$. If

$$V[\alpha] = \begin{cases} V(X^*, \alpha), & \text{if } \emptyset \notin D, \\ V(X^*, \alpha), & \text{if } \emptyset \in V(X^*, \alpha), \\ V(X^*, \alpha) \cup \{\emptyset\}, & \text{if } \emptyset \notin V(X^*, \alpha) \text{ and } \emptyset \in D, \end{cases}$$

then it is obvious that any binary relation α of a semigroup $B_X(D)$ can always be written in the form $\alpha = \bigcup_{T \in V[\alpha]} (Y_T^\alpha \times T)$. In the sequel, such a representation of a binary relation α will be called quasinormal.

Note that for a quasinormal representation of a binary relation α, not all sets Y_T^α $(T \in V[\alpha])$ can be different from the empty set. But for this representation the following conditions are always fulfilled:

a) $Y_T^\alpha \cap Y_{T'}^\alpha = \emptyset$, for any $T, T' \in D$ and $T \neq T'$;

b) $X = \bigcup_{T \in V[\alpha]} Y_T^\alpha$.

Lemma 1.4. [2, Equality 6.9] *Let $Y = \{y_1, y_2, \ldots, y_k\}$ and $D_j = \{T_1, T_2, \ldots, T_j\}$ be sets, where $k \geq 1$ and $j \geq 1$. Then the number $s(k, j)$ of all possible mappings of the set Y on any such subset of the set D_j' such that $T_j \in D_j'$ can be calculated by the formula $s(k, j) = j^k - (j-1)^k$.*

Lemma 1.5. *Let $D_j = \{T_1, T_2, \ldots, T_j\}$, X and Y be three such sets, that $\emptyset \neq Y \subseteq X$. If f is such mapping of the set X, in the set D_j, for which $f(y) = T_j$ for some $y \in Y$, then the number s of all those mappings f of the set X in the set D_j is equal to $s = j^{|X \setminus Y|} \cdot \left(j^{|Y|} - (j-1)^{|Y|}\right)$.*

Proof. Let f_1 be a mappings of the set $X \setminus Y$ in the set D_j, then the number of all such mappings is equal to $j^{|X \setminus Y|}$.

Now let f_2 be all mappings of the set Y in the set D_j, for which $f(y) = T_j$ for some $y \in Y$, then by Lemma 1.4 the number of all such mappings is equal to $j^{|Y|} - (j-1)^{|Y|}$.

We define the mapping f of the set X in the set D_j by

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in X \setminus Y \\ f_2(x) & \text{if } x \in Y \end{cases}$$

It is clear that the mapping f satisfies all the conditions of the given Lemma.

Thus the number s of all such maps is equal to all number of the pair (f_1, f_2). The number all such pair is equal to $s = j^{|X\setminus Y|} \cdot \left(j^{|Y|} - (j-1)^{|Y|}\right)$. □

The following Theorems are well known (see, [1, 2, 3, 4, 5, 6]).

Theorem 1.6. [2, Theorem 2.1] *A binary relation $\alpha \in B_X(D)$ is a right unit of this semigroup iff α is idempotent and $D = V(D, \alpha)$.*

Theorem 1.7. [2, Theorem 2.6] *Let D be a complete $X-$semilattice of unions. The semigroup $B_X(D)$ possesses a right unit iff D is an $XI-$semilattice of unions.*

Theorem 1.8. [1, Theorem 6.2.3], [5, Theorem 6] *Let D, $\Sigma(D)$, $E_X^{(r)}(Q)$ and I_D denote respectively the complete X-semilattice of unions, the set of all XI-subsemilattices of the semilattice D, the set of all right units of the semigroup $B_X(Q)$ and the set of all idempotents of the semigroup $B_X(D)$. Then for the sets $E_X^{(r)}(Q)$ and I_D the following statements are true:*

a) *If $\emptyset \in D$ and $\Sigma_{\emptyset}(D) = \{D' \in \Sigma(D) \mid \emptyset \in D'\}$ then*
   (1) *$E_X^{(r)}(Q) \cap E_X^{(r)}(Q') = \emptyset$ for any elements Q and Q' of the set $\Sigma_{\emptyset}(D)$ that satisfy the condition $Q \neq Q'$;*
   (2) *$I_D = \bigcup_{Q\in\Sigma_{\emptyset}(D)} E_X^{(r)}(Q)$;*
   (3) *The equality $|I_D| = \sum_{Q\in\Sigma_{\emptyset}(D)} \left|E_X^{(r)}(Q)\right|$ is fulfilled for the finite set X.*

b) *If $\emptyset \notin D$, then*
   (1) *$E_X^{(r)}(Q) \cap E_X^{(r)}(Q') = \emptyset$ for any elements Q and Q' of the set $\Sigma(D)$ that satisfy the condition $Q \neq Q'$;*
   (2) *$I_D = \bigcup_{Q\in\Sigma(D)} E_X^{(r)}(Q)$;*
   (3) *The equality $|I_D| = \sum_{Q\in\Sigma(D)} \left|E_X^{(r)}(Q)\right|$ is fulfilled for the finite set X.*

Theorem 1.9. [3] *Let $D = \left\{\breve{D}, Z_1, Z_2, \ldots, Z_{n-1}\right\}$ be some finite X-semilattice of unions and $C(D) = \{P_0, P_1, \ldots, P_{n-1}\}$ be the family of sets of pairwise non-intersecting subsets of the set X. If φ is a mapping of the semilattice D on the family of sets $C(D)$ which satisfies the condition $\varphi(\breve{D}) = P_0$ and $\varphi(Z_i) = P_i$ for any $i = 1, 2, \ldots, n-1$ and $\hat{D}_Z = D \setminus \{T \in D \mid Z \subseteq T\}$, then the following equalities are valid:*

$$\breve{D} = P_0 \cup P_1 \cup \cdots \cup P_{n-1},\ Z_i = P_0 \cup \bigcup_{T\in\hat{D}_{Z_i}} \varphi(T). \qquad (*)$$

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice D are represented in the form $(*)$, then among the parameters P_i $(i = 1, 2, \ldots, n-1)$ there exists a parameter that cannot be empty sets for D. Such sets P_i $(0 < i \leq n-1)$ are called basis sources, whereas sets P_i $(0 \leq j \leq n-1)$ which can be empty sets too are called completeness sources.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see, [[3]]).

Lemma 1.10. *Let* $D = \left\{\breve{D}, Z_1, Z_2, \ldots, Z_{n-1}\right\}$ *and* $C(D) = \{P_0, P_1, \ldots, P_{n-1}\}$ *be the finite semilattice of unions and the family of sets of pairwise nonintersecting subsets of the set* X*;* $\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2 & \ldots & Z_{n-1} \\ P_0 & P_1 & P_2 & \ldots & P_{n-1} \end{pmatrix}$ *is a mapping of the semilattice* D *on the family of sets* $C(D)$*. If* $\varphi(T) = P \in C(D)$ *for some* $\breve{D} \neq T \in D$*, then* $D_t = D \setminus \ddot{D}_T$ *for all* $t \in P$*.*

Proof. Let t and Z' be any elements of the set P ($P \neq P_0$) and of the semilattice D respectively. Then the equality $P \cap Z' = \emptyset$ (i.e., $Z' \notin D_t$ for any $t \in P$) is valid if and only if $T \notin \hat{D}_{Z'}$ (if $T \in \hat{D}_{Z'}$, then $\varphi(T) \subseteq Z'$ by definition of the formal equalities of the semilattice D). Since $\hat{D}_{Z'} = D \setminus \{T' \in D \mid Z' \subseteq T'\}$ by definition of the set $\hat{D}_{Z'}$. Thus the condition $T \notin \hat{D}_{Z'}$ hold iff $T \in \{T' \in D \mid Z' \subseteq T'\}$. So, $Z' \subseteq T$ and $Z' \in \ddot{D}_T$ by definition of the set $\ddot{D}_T$.

Therefore, $\varphi(T) \cap Z' = \emptyset$ if and only if $Z' \in \ddot{D}_T$. Of this follows that the inclusion $\varphi(T) = P \subseteq Z'$ is true iff $D_t = D \setminus \ddot{D}_T$ for all $t \in \varphi(T) = P$. □

2. Results

Definition 2.1. Let D be complete $X-$semilattice of unions and Z be some fixed element of D. We say that a complete $X-$semilattice of unions D is $Z-$elementary if D satisfies the following conditions:

a) D is not a chain;
b) every subchain of the semilattice D is finite;
c) the set $D_Z = \{T \in D \mid Z \subseteq T\}$ is a chain with smallest element Z;
d) the condition $T \cup T' = Z$ holds for any incomparable elements T and T' of D.

Example 1. The diagrams 1, 2, 3, 4 of the Fig. 2.1 respectively are Z_1, $\breve{D}$, Z_1 and Z_2 elementary $X-$semilattices of unions:

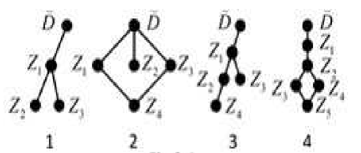

Fig. 2.1

Lemma 2.2. *If* D *is* $Z-$*elementary* $X-$*semilattice of unions, then* $D \setminus \{Z\}$ *is unique generated set of the semilattice* D*.*

Proof. The given Lemma immediately follows from the $Z-$elementary $X-$semilattice of unions. □

Lemma 2.3. *Let D be $Z-$elementary $X-$semilattice of unions. If subsemilattice D' of the semilattice D is not a chain, then D' is $Z-$elementary $X-$semilattice of unions.*

Proof. Let D be $Z-$elementary $X-$semilattice of unions. Suppose that the subsemilattice D' of the semilattice D is not a chain.

1) It is clear, that the length of any chain of the semilattice D' is finite since $D' \subseteq D$.

2) If $T \in D'_Z \setminus \{Z\}$, then $T \in D_Z$ since $T \in D' \subseteq D$, $Z \subset T$. We have $D'_Z \subseteq D_Z$. Therefore, it follows that D'_Z is a chain.

3) Further, let T and T' be such elements of the set D' such that $T \backslash T' \neq \emptyset$ and $T' \backslash T \neq \emptyset$ (i.e., the elements T and T' of D are incomparable). Then T,$T' \in D$, since $D' \subseteq D$. From this we have $T \cup T' = Z$ by the definition of the $Z-$elementary $X-$semilattice union D.

From the conditions (1), (2) and (3) it follows, that D' is $Z-$elementary $X-$semilattice of unions. □

Definition 2.4. Let C and C' be finite different chains of the set 2^X and $Z \in C \cap C'$. We say that the chains C and C' are $Z-$*compatible* if C and C' satisfy the following conditions:

a) $T \cup T' = Z$ for any $T \in C \backslash C'$ and $T' \in C' \backslash C$;

b) if $\overline{C}_Z = \{T \in C \mid Z \subseteq T\}$ and $\overline{C}'_Z = \{T' \in C' \mid Z \subseteq T'\}$, then $\overline{C}_Z = \overline{C}'_Z$ (see diagram 1 and 2 of the Fig.2.2).

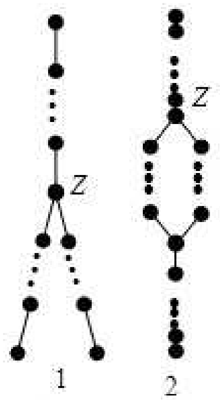

Fig. 2.2

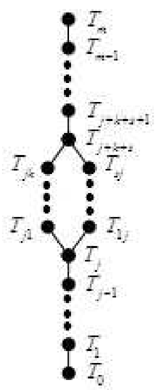

Fig. 2.3

Definition 2.5. The chain C of a $X-$semilattice of unions D is called *maximal*, if the inclusion $C \subseteq C'$ implies that $C = C'$ for any chain C' of the $X-$semilattice of unions D.

Theorem 2.6. *Suppose $X-$semilattice of unions D is not a chain. Then D is $Z-$elementary $X-$semilattice of unions iff any two maximal subchain of the $X-$semilattice of unions D are $Z-$compatible.*

Proof. Let D be $Z-$elementary $X-$semilattice of unions and C, C' be two different maximal subchains of the $X-$semilattice of unions D.

$a)$ Let $\overline{C}_Z = \{T \in C \mid Z \subseteq T\}$, $\overline{C}'_Z = \{T' \in C' \mid Z \subseteq T'\}$. By assumption the sets $\overline{C}_Z$, $\overline{C}'_Z$ and D_Z are maximal chains of the $X-$semilattice of union D with smallest element Z. Then $D_Z = \overline{C}_Z = \overline{C}'_Z$ since by definition of the $Z-$elementary $X-$semilattice of unions D the maximal subchains $D_Z, \overline{C}, \overline{C}'$ of the $X-$semilattice D, with the smallest element Z are by definition unique.

$b)$ Let $T \in C \backslash C'$ and $T' \in C' \backslash C$. Then $T \subset Z$, $T' \subset Z$ and $T \neq T'$. If $T \backslash T' \neq \emptyset$ and $T' \backslash T \neq \emptyset$ then $T \cup T' = Z$ by definition of the semilattice D. Therefore, the chains C and C' are $Z-$compatible.

1) So, we can assumed that $T \backslash T' = \emptyset$ and $T' \backslash T \neq \emptyset$. Further, let the element T' cover the element T'_1 in the chain C'. Then $T'_1 \backslash T \neq \emptyset$ or $T'_1 \backslash T = \emptyset$. If $T'_1 \backslash T \neq \emptyset$, then we have $T' \backslash T \supset T'_1 \backslash T \neq \emptyset$. But the inequality $T' \backslash T \neq \emptyset$ contradicts the equality $T \backslash T' = \emptyset$. So, $T'_1 \backslash T = \emptyset$.

2) Let $T'_1 \backslash T = \emptyset$. Then $T \supset T'_1$ and continue this process we obtain $T \supset T'_1 \supset T'_2 \supset \cdots \supset T'_q$ and $T'_q \backslash T = \emptyset$, where $T, T'_1, T'_2, \cdot, T'_q \in C'$ and element T'_i cover the element T'_{i+1} $(i = 1, 2, \ldots, q-1)$. But, this process must stop, since the chains C' is finite. So, there exists a natural number s, such that $T \supset T'_1 \supset T'_2 \supset \cdots \supset T'_s$ and $T'_s \backslash T \neq \emptyset$. We have $T' \backslash T \supset T'_1 \backslash T \supset T'_2 \backslash T \supset \cdots \supset T'_s \backslash T$, i.e., $T' \backslash T \neq \emptyset$. We have $T' \backslash T \neq \emptyset$, $T \backslash T' \neq \emptyset$ and $T' \cup T = Z$ by definition of the $Z-$elementary $X-$semilattice of unions D. So, $T \cup T' = Z$ for any $T \in C \backslash C'$ and $T' \in C' \backslash C$.

Therefore, the chains are compatible.

Let any two maximal subchains of the $X-$semilattice of unions D be $Z-$compatible. Then we have:

1) By supposition D is not a chain.

2) Every subchain of the semilattice D is finite, since all $Z-$comparable chains are finite.

3) By the definition of $Z-$comparable chains, the set $D_Z = \{T \in D \mid Z \subseteq T\}$ is a chain with smallest element Z;

4) If T and T' are any incomparable elements of D, then there exist two maximal different chains C and C' such that $T \in C \backslash C'$, $T' \in C' \backslash C$ and the chains C and C' are $Z-$compatible, then $T \cup T' = Z$. □

Let D be $Z-$elementary $X-$semilattice of unions and

$$Q = \{T_0, T_1, \ldots, T_{j-1}, T_j, T_{j+1}, \ldots, T_{k-1}, T_k, T_{k+1}, \ldots, T_{m-1}, T_m\}$$

be an $XI-$subsemilattice of the $Z-$elementary $X-$semilattice of unions D which satisfy the following conditions:

$$T_0 \subset T_1 \subset \cdots \subset T_{j-1} \subset T_j \subset T_{j+1} \subset \cdots \subset T_{k-2} \subset T_k \subset T_{k+1} \subset \cdots \subset T_{m-1} \subset T_m,$$
$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$
$$T_0 \subset T_1 \subset \cdots \subset T_{j-1} \subset T_j \subset T_{j+2} \subset \cdots \subset T_{k-1} \subset T_k \subset T_{k+1} \subset \cdots \subset T_{m-1} \subset T_m,$$
$$T_{jk} \backslash T_{sj} \neq \emptyset,\ T_{sj} \backslash T_{jk} \neq \emptyset,\ T_{jk} \cup T_{sj} = T_{j+k+s}.$$

(i.e., the number different elements covered by the element Z is two). Note that the diagram of the semilattice D is shown in Fig. 2.3.

Further, let

$$C(Q) = \{P_i \mid i = 0, 1, \dots, j, j+k+s, j+k+s+1, \dots m-1, m\} \\ \cup \{P_{j1}, \dots, P_{jk}\} \cup \{P_{1j}, \dots, P_{sj}\}$$

be a family of sets, where every two elements are pairwise disjoint subsets of the set X,

$$\varphi = \begin{pmatrix} T_0 & T_1 & \cdots & T_j & T_{j1} & \cdots & T_{jk} & T_{1j} & \cdots & T_{sj} & T_{j+k+s} & T_{j+k+s+1} & \cdots & T_{m-1} & T_m \\ P_0 & P_1 & \cdots & P_j & P_{j1} & \cdots & P_{jk} & P_{1j} & \cdots & P_{sj} & P_{j+k+s} & P_{j+k+s+1} & \cdots & P_{m-1} & P_m \end{pmatrix}.$$

Then for the formal equalities of the semilattice Q we have :

$T_m = P_m \cup P_{m-1} \cup \cdots \cup P_{j+k+s+1} \cup P_{j+k+s} \cup P_{sj} \cup \cdots \cup P_{1j} \cup P_{jk} \cup \cdots \cup P_{j1}$
$\cup P_j \cup \cdots \cup P_1 \cup P_0$
$T_m = P_m \cup P_{m-2} \cup \cdots \cup P_{j+k+s+1} \cup P_{j+k+s} \cup P_{sj} \cup \cdots \cup P_{1j} \cup P_{jk} \cup \cdots \cup P_{j1}$
$\cup P_j \cup \cdots \cup P_1 \cup P_0$
$\vdots$
$T_{j+k+s+1} = P_m \cup P_{j+k+s} \cup P_{sj} \cup \cdots \cup P_{1j} \cup P_{jk} \cup \cdots \cup P_{j1} \cup P_j \cup \cdots \cup P_1 \cup P_0$
$T_{j+k+s} = P_m \cup P_{sj} \cup \cdots \cup P_{1j} \cup P_{jk} \cup \cdots \cup P_{j1} \cup P_j \cup \cdots \cup P_1 \cup P_0$
$T_{jk} = P_m \cup P_{sj} \cup \cdots \cup P_{1j} \cup P_{jk} \cup \cdots \cup P_{j1} \cup P_j \cup \cdots \cup P_1 \cup P_0$
$T_{sj} = P_m \cup P_{s-1j} \cup \cdots \cup P_{1j} \cup P_{jk} \cup \cdots \cup P_{j1} \cup P_j \cup \cdots \cup P_1 \cup P_0$
$\vdots$
$T_{j1} = P_m P_{sj} \cup \cdots \cup P_{1j} \cup P_{jk} \cup \cdots \cup P_{j1} \cup P_j \cup \cdots \cup P_1 \cup P_0$
$T_{1j} = P_m \cup P_{jk} \cup \cdots \cup P_{j1} \cup P_j \cup \cdots \cup P_1 \cup P_0$
$\vdots$
$T_1 = P_m \cup P_0$
$T_1 = P_m$

Here the elements $P_0, P_1, \dots, P_{j-1}, P_{j1}, \dots, P_{jk}, P_{1j}, \dots, P_{sj}, P_{j+k+s}, P_{j+k+s+1}, \dots, P_{m-1}$ are basis sources, the elements P_j, P_m are sources of completeness of the semilattice Q.

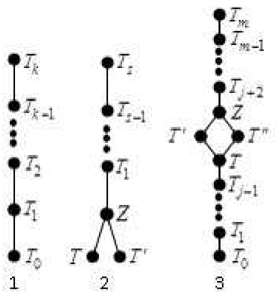

Fig. 2.4

Fig. 2.5

Lemma 2.7. *Let Q be a semilattice, whose diagram is shown in Fig.2.3. Then Q is a $XI-$semilattice of unions if and only if $k = s = 1$.*

Proof. Let $t \in \breve{Q}$, $Q_t = \{Z \in Q \mid t \in Z\}$ and $\wedge(Q, Q_t)$ be the exact lower bound of the set Q_t in Q. Then from the formal equalities and by Lemma 1.10 we have:

$$\begin{array}{ll} t \in P_m, & Q_t = Q \\ t \in P_{m-1}, & Q_t = Q \backslash \ddot{Q}_{T_{m-1}} \\ \cdots & \cdots \\ t \in P_{j+k+s+1}, & Q_t = Q \backslash \ddot{Q}_{T_{j+k+s+1}} \\ t \in P_{j+k+s}, & Q_t = Q \backslash \ddot{Q}_{T_{j+k+s}} \\ t \in P_{sj}, & Q_t = Q \backslash \ddot{Q}_{T_{sj}} \\ t \in P_{s-1j}, & Q_t = Q \backslash \ddot{Q}_{T_{s-1j}} \\ \cdots & \cdots \\ t \in P_{1j}, & Q_t = Q \backslash \ddot{Q}_{T_{1j}} \\ t \in P_{jk}, & Q_t = Q \backslash \ddot{Q}_{T_{jk}} \\ t \in P_{jk-1}, & Q_t = Q \backslash \ddot{Q}_{T_{jk-1}} \\ \cdots & \cdots \\ t \in P_{j1}, & Q_t = Q \backslash \ddot{Q}_{T_{j1}} \\ t \in P_j, & Q_t = Q \backslash \ddot{Q}_{T_j} \\ \cdots & \cdots \\ t \in P_1, & Q_t = Q \backslash \ddot{Q}_{T_1} \\ t \in P_0, & Q_t = Q \backslash \ddot{Q}_{T_0} \end{array} \qquad \wedge(Q, Q_t) = \begin{cases} T_0, & \text{if } t \in P_m, \\ T_m, & \text{if } t \in P_{m-1} \\ \cdots & \cdots \\ T_{j+k+s+2}, & \text{if } t \in P_{j+k+s+1}, \\ T_{j+k+s+1}, & \text{if } t \in P_{j+k+s}, \\ T_{j1}, & \text{if } t \in P_{sj}, \\ T_j, & \text{if } t \in P_{s-1j}, \\ \cdots & \cdots \\ T_j, & \text{if } t \in P_{1j}, \\ T_{1j}, & \text{if } t \in P_{jk}, \\ T_j, & \text{if } t \in P_{jk-1}, \\ \cdots & \cdots \\ T_j, & \text{if } t \in P_{j1}, \\ T_j, & \text{if } t \in P_j, \\ \cdots & \cdots \\ T_2, & \text{if } t \in P_1, \\ T_1, & \text{if } t \in P_0, \end{cases}$$

Thus we have obtained that $\wedge(Q, Q_t) \in Q$ for all $t \in T_m$. Let

$$\begin{aligned} Q^\wedge &= \{\wedge(Q, Q_t) \mid t \in T_m\} \\ &= \{T_0, T_1, T_2, \ldots, T_j, T_{j1}, \ldots, T_{1j}, T_{j+k+s+1}, \ldots, T_{m-1}, T_m\} \end{aligned}$$

and Q' be the semilattice of unions generated by the set $Q^\wedge$.

If $k \geq 2$ or $s \geq 2$, i.e., $T_{j2} \in Q$ or $T_{2j} \in Q$ then $T_{j2} \notin Q'$ or $T_{2j} \notin Q'$. So, if $k \geq 2$ or $s \geq 2$, then Q is not $XI-$semilattice of unions.

If $k = s = 1$, then $T_{j1} \cup T_{1j} = T_{j+2} \in Q'$, i.e., $Q' = Q$.

Therefore, Q is $XI-$semilattice of unions.

□

Theorem 2.8. *Let D be a $Z-$elementary $X-$semilattice of unions and Q be any $XI-$subsemilattice of the $X-$semilattice of unions D. Then for the $XI-$semilattice Q we have:*

a) *Q is a finite chain (see. diagram 1 of Fig. 2.4);*
b) *$Q = \{T, T', Z\} \cup Q'$, where T and T' are elements of the semilattice D such that $T \cap T' = \emptyset$ and $Q' = \{T_1, T_2, \ldots, T_s\} \subseteq D_Z \backslash \{Z\}$ (see. diagram 2 of Fig. 2.4);*
c) *$Q = Q' \cup \{T, T', T'', Z\} \cup Q''$, where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, T', T'' are incomparable elements of D; $Q' = \emptyset$ or $Q'' = \emptyset$, or Q', Q'' are subchains of the semilattice D satisfying the conditions $Q' \subseteq D_Z \backslash \{Z\}$ and $Q'' \subseteq \ddot{D}_T \backslash \{T\}$ (see. diagram 3 of Fig. 2.4).*

Proof. Let D be a $Z-$elementary $X-$semilattice of unions and Q be any $XI-$subsemilattice of the $X-$semilattice of unions D.

(a') If $Z \notin Q$ then by the definition of $Z-$elementary $X-$semilattice of unions it follows that Q is a finite $X-$chain.

Now, let $Z \in Q$ and T be the unique element of the semilattice Q which is covered by the element Z; If T_1 and T_2 are any incomparable elements of the semilattice Q satisfying the conditions $T_1 \subset T$ and $T_2 \subset T$, then by the definition $Z-$elementary $X-$semilattice of unions it follows that $Z = T_1 \cup T_2 \subseteq T$. The inclusion $Z \subseteq T$ contradicts the condition $T \subset Z$. So, we have T_1 and T_2 are comparable elements of the semilattice Q, i.e., $T_1 \subset T_2$ or $T_2 \subset T_1$. Therefore Q is a finite $X-$chain. The statement (a') is proved.

(b') Let T, T' and T'' be different elements of the semilattice Q which are covered by the element Z in the semilattice Q. Then

$$Z = T \cup T' = T \cup T'' = T' \cup T''$$

1) If $T \cap T' = \emptyset$, then $T = Z \backslash T'$, $T' = Z \backslash T$ and

$$T = Z \backslash T' = (T' \cup T'') \backslash T' \subseteq T'',\ T' = Z \backslash T = (T \cup T'') \backslash T \subseteq T''.$$

It follows, that $Z = T \cup T' \subseteq T'' \cup T'' = T''$, i.e., $T'' = Z$ since $T'' \subseteq Z$. But the equality $T'' = Z$ contradict, that T'' is an element which is covered by the element Z in the semilattice Q.

2) Now suppose that the intersection any two different elements which are covered by the element Z in the semilattice Q is not empty.

It is clear that $T \neq \emptyset$ and $T = \bigcup_{t \in T} \wedge (Q, Q_t)$, since Q is $XI-$semilattice of unions. From Lemma 2.3 it follows that Q is $Z-$elementary $X-$semilattice of unions. By the definition of the $Z-$elementary $X-$semilattice of unions D immediately follows that $D \backslash \{Z\}$ is unique generated set of the semilattice D. It follows that $T = \wedge (Q, Q_{t'})$ for some $t' \in T$. On the other hand, $t' \in T \subset Z = T' \cup T''$, i.e., $t' \in T'$ or $t' \in T''$. If $t' \in T'$, then we have $T' \in Q_{t'}$ and $T = \wedge (Q, Q_{t'}) \subset T'$. The inclusion $T \subset T' \subset Z$ contradicts the assumption that element T is covered by the element Z in the semilattice Q. This contradiction shows that number the elements which are covered by the element Z of the $XI-$semilattice Q are less or equal two.

For the elements T and T' of the semilattice Q we consider two case.

3) If T and T' are minimal elements of the $X-$semilattice unions Q. $T \cap T' = \emptyset$ and $Q' = Q \backslash \{T, T', Z\}$, then $Q = \{T, T', Z\} \cup Q'$, where $Q' = \{T_1, T_2, \ldots, T_s\} \subseteq D_Z \backslash \{Z\}$ and Q' is a chain by definition of $Z-$elementary $X-$semilattice of unions and Q. The statement (b') is proved.

$c')$ Now suppose that the elements T' and T'' covered by the element Z in the semilattice Q are not minimal elements of the semilattice Q, i.e., $T \subset T'$ and $T \subset T''$ for some $T \in Q$. Then by Lemma 2.7 we have the element T covered by the elements T' and T'' in the semilattice Q. It is clear, that the set $\{T, T', T'', Z\}$ is a $X-$subsemilattice of the semilattice Q.

Further, let $Q' = \{Z' \in Q \mid Z \subset Z'\}$ and $Q'' = Q \setminus (Q' \cup \{T, T', T'', Z\})$. Then we have

$$Q = Q' \cup \{T, T', T'', Z\} \cup Q''.$$

It is clear that $Q' \subseteq D_Z \setminus \{Z\}$ and is a subchain of the chain D_Z.

Now, let Z'' be any element of the set Q''. Then $Z'' \in Q$, $Z'' \notin Q' \cup \{T, T', T'', Z\}$ and $Z'' \subset T'$ or $Z'' \subset T''$ since T' and T'' are maximal elements of the set $Q \setminus (Q' \cup \{Z\})$. If Z'' and T are incomparable elements of the semilattice Q then $Z = T \cup Z'' \subseteq T'$ by the definition of $Z-$elementary $X-$semilattice unions and by the conditions $T \subset T'$ and $Z'' \subset T'$. But the inclusion $Z \subseteq T'$ contradicts the conditions $T' \subset Z$. So, Z'' and T are comparable elements of the semilattice Q. From this follows that $Z'' \subset T$.

In the case $Z'' \subset T''$ we can similarly prove that $Z'' \subset T$.

Further let Z''_1 and Z''_2 are any incomparable elements of the set Q'' satisfying the conditions $Z''_1 \subset T$ and $Z''_2 \subset T$. Then by the definition $Z-$elementary $X-$semilattice of unions it follows that $Z = Z''_1 \cup Z''_2 \subseteq T$. The inclusion $Z \subseteq T$ contradicts the condition $T \subset Z$. So, we have Z''_1 and Z''_2 are comparable elements of the set Q'', i.e., $Z''_1 \subset Z''_2$ or $Z''_2 \subset Z''_1$. Therefore Q'' is a finite $X-$chain for which $\ddot{Q} \subseteq \ddot{D}_T \setminus \{T\}$. The statement (c') is proved. □

Definition 2.9. Let $C(D)$ denote the set all chains of the $Z-$elementary $X-$semilattice unions D. $N(D) = \{|C| \mid C \in C(D)\}$, $h(D)$ be the largest natural number of the set $N(D)$,

$$C_k(D) = \{C \in C(D) \mid |C| = k\} \ (1 \le k \le h(D)),$$
$$I^*_{C_k(D)} = \{\alpha \in B_X(D) \mid \alpha \circ \alpha = \alpha, \ V(D, \alpha) \subseteq C_k(D)\},$$
$$I_{C(D)} = \{\alpha \in B_X(D) \mid \alpha \circ \alpha = \alpha, \ V(D, \alpha) \subseteq C(D)\}.$$

It is easy to see, that: $C(D) = C_1(D) \cup C_2(D) \cup \cdots \cup C_{h(D)}(D)$.

Theorem 2.10. *Let $Q = \{T_0, T_1, \ldots, T_m\}$ be a subsemilattice of the semilattice D such that $T_0 \subset T_1 \subset \cdots \subset T_m$ (see Fig. 2.5). Then a binary relation α of the semigroup $B_X(D)$ that has a quasinormal representation of the form $\alpha = \bigcup_{i=0}^{m} (Y_i^\alpha \times T_i)$ is a right unit of the semigroup $B_X(Q)$ iff $Q = V(D, \alpha)$ and $Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_p^\alpha \supseteq T_p$, $Y_q^\alpha \cap T_q \neq \emptyset$ for any $p = 1, 2, \ldots, m-1$ and $q = 1, 2, \ldots, m$.*

Proof. Let $Q = \{T_0, T_1, \ldots, T_m\}$ be a subsemilattice of the semilattice D such that $T_0 \subset T_1 \subset \cdots \subset T_m$. Then the given Theorem immediately follows from the Theorem 1.6 and Corollary 3 of [5]. (see, also, Corollary 13.1.2 of [1]) . □

Theorem 2.11. *Let $Q = \{T_0, T_1, \ldots, T_m\}$ be a subsemilattice of the semilattice D such that $T_0 \subset T_1 \subset \cdots \subset T_m$. If $E_X^{(r)}(Q)$ is the set of all right units of the semigroup $B_X(Q)$, then*

$$E_X^{(r)}(Q) = \left(2^{|T_1 \setminus T_0|} - 1\right)\left(3^{|T_2 \setminus T_1|} - 2^{|T_2 \setminus T_1|}\right) \cdots$$
$$\left((m+1)^{|T_m \setminus T_{m-1}|} - m^{|T_m \setminus T_{m-1}|}\right)(m+1)^{|X \setminus T_m|}$$

(see, Theorem 6.5 of [2] or Corollary 13.1.5 of [1]).

Definition 2.12. Let $\mu = \{(T,T') \mid T,T' \in D,\ T \cap T' = \emptyset\} \neq \emptyset$, $Q(T,T',Q') = \{T,T',Z\} \cup Q'$ where $(T,T') \in \mu$, $Q' \subseteq D_Z \setminus \{Z\}$,

$$C'(D) = \{Q(T,T',Q') \mid (T,T') \in \mu,\ Q' \subseteq D_Z \setminus \{Z\}\}$$

and

$$C'_s(D) = \{Q(T,T',Q') \in C'(D) \mid |Q'| = s\} \ \left(0 \leq s \leq 2^{|D_z \setminus \{Z\}|}\right),$$
$$I^*_{C'_s(D)} = \{\alpha \in B_X(D) \mid \alpha \circ \alpha = \alpha,\ V(D,\alpha) \subseteq C'_s(D)\},$$
$$I^*_{C'(D)} = \{\alpha \in B_X(D) \mid \alpha \circ \alpha = \alpha,\ V(D,\alpha) \subseteq C'(D)\}.$$

It is easy to see, that $C'(D) = C'_0(D) \cup C'_1(D) \cup \cdots \cup C_{2^{|D_Z \setminus \{Z\}|}}(D)$.

Theorem 2.13. *Let $Q = \{T_1, T_2, \ldots, T_m\}$ $(m \geq 3)$ be a subsemilattice of the semilattice D such that $T_1, T_2 \notin \{\emptyset\}$, $T_1 \cap T_2 = \emptyset$, $T_1 \cup T_2 = T_3$, $T_3 \subset T_4 \subset \cdots \subset T_m$. Then the semigroup $B_X(Q)$ has right unit iff $T_1 \cap T_2 = \emptyset$ (see* [6], *Theorem* 1*).*

Theorem 2.14. *Let $Q = \{T_1, T_2, \ldots, T_m\}$ $(m \geq 3)$ be a subsemilattice of the semilattice D such that $T_1, T_2 \notin \{\emptyset\}$, $T_1 \cap T_2 = \emptyset$, $T_1 \cup T_2 = T_3$, $T_3 \subset T_4 \subset \cdots \subset T_m$, (see Fig.* 2.6*). Then a binary relation α of the semigroup $B_X(Q)$ that has a quasinormal representation of the form $\alpha = \bigcup_{i=0}^{m} (Y_i^\alpha \times T_i)$ is a right unit of the semigroup $B_X(Q)$ iff $Q = V(D,\alpha)$ and $Y_1^\alpha \supseteq T_1$, $Y_2^\alpha \supseteq T_2$, $Y_1^\alpha \cup Y_2^\alpha \cup \cdots \cup Y_k^\alpha \supseteq T_k$ and $Y_q^\alpha \cap T_q \neq \emptyset$ for any $k = 4, 5, \ldots, m-1$ and $q = 4, 5, \ldots, m-1$ (see Corollary* 13.2.3 *of* [1]*).*

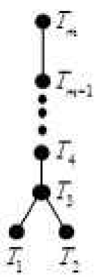

Fig. 2.6

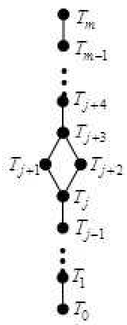

Fig. 2.7

Theorem 2.15. *Let $Q = \{T_1, T_2, \ldots, T_m\}$ $(m \geq 3)$ be a subsemilattice of the semilattice D such that $T_1, T_2 \notin \{\emptyset\}$, $T_1 \cap T_2 = \emptyset$, $T_1 \cup T_2 = T_3$, $T_3 \subset T_4 \subset \cdots \subset T_m$. If $E_X^{(r)}(Q)$ is the set of all right units of the semigroup $B_X(Q)$, then*

$$E_X^{(r)}(Q) = \left(4^{|T_4 \setminus T_3|} - 3^{|T_4 \setminus T_3|}\right)\left(5^{|T_5 \setminus T_4|} - 4^{|T_5 \setminus T_4|}\right) \cdots \left(m^{|T_m \setminus T_{m-1}|} - (m-1)^{|T_m \setminus T_{m-1}|}\right) m^{|X \setminus T_m|}$$

(see Corollary 13.2.1 *of* [1]*).*

Definition 2.16. Let

$$v=\{(T,T',T'')\mid T,T',T''\in D,\ T\subset T',T\subset T'',\ T'\backslash T''\neq\emptyset,T''\backslash T'\neq\emptyset\}\neq\emptyset,$$
$$Q(T,T',T'',Q',Q'')=Q'\cup\{T,T',T'',Z\}\cup Q''$$

and $C''(D)$ be set of all $Q(T,T',T'',Q',Q'')$, where $(T,T',T'')\in v$, $Q'=\emptyset$ or $Q''=\emptyset$, or Q', Q'' are subchains of the semilattice D satisfying the conditions $Q'\subseteq D_Z\backslash\{Z\}$ and $\ddot{Q}\subseteq \ddot{D}_T\backslash\{T\}$.

Further, let

$$C''_{sk}\left(T,T',T'',D\right)=\{Q\left(T,T',T'',Q',Q''\right)\mid |Q'|=s,|Q''|=k\}\quad((T,T',T'')\in v)\,,$$
$$I^*_{C''_{sk}(T,T',T'',D)}=\{\alpha\in B_X(D)\mid \alpha\circ\alpha=\alpha,\ V(D,\alpha)\subseteq C''_{sk}\left(T,T',T'',D\right)\}\,,$$
$$I_{C''(D)}=\{\alpha\in B_X(D)\mid \alpha\circ\alpha=\alpha,\ V(D,\alpha)\subseteq C''\left(D\right)\}\,,$$

where $0\le s\le 2^{|D_Z\backslash\{Z\}|}$ and $0\le k\le 2^{\left|\ddot{D}_Z\backslash\{Z\}\right|}$.

It is easy to see, that $C''\left(D\right)=\bigcup\limits_{(T,T',T'')\in v}C''_{sk}\left(T,T',T'',D\right)$.

Theorem 2.17. *Let $Q=\{T_0,T_1,\ldots,T_m\}$ $(m\ge 3)$ be a semilattice and j be a fixed natural number such that $0\le j\le m-3$ and*

$$T_0\subset T_1\subset\cdots\subset T_j\subset T_{j+1}\subset T_{j+3}\subset\cdots\subset T_m,$$
$$T_0\subset T_1\subset\cdots\subset T_j\subset T_{j+2}\subset T_{j+3}\subset\cdots\subset T_m,$$
$$T_{j+1}\backslash T_{j+2}\neq\emptyset,\ T_{j+2}\backslash T_{j+1}\neq\emptyset,\ T_{j+1}\cup T_{j+2}=T_{j+3}$$

(see Fig. 2.7). A binary representation α of the semigroup $B_X(Q)$, which has a quasinormal representation of the form $\alpha=\bigcup_{i=0}^{m}\left(Y_i^\alpha\times T_i\right)$ such that $Q=V(D,\alpha)$, is an idempotent element of the semigroup $B_X(D)$ iff

$$Y_0^\alpha\cup Y_1^\alpha\cup\cdots\cup Y_j^\alpha\supseteq T_{j+1}\cap T_{j+2},$$
$$Y_0^\alpha\cup Y_1^\alpha\cup\cdots\cup Y_j^\alpha\cup Y_{j+2}^\alpha\supseteq T_{j+2},$$
$$Y_0^\alpha\cup Y_1^\alpha\cup\cdots\cup Y_p^\alpha\supseteq T_p,\ Y_p^\alpha\cap T_p\neq\emptyset$$

for any $p=0,1,2,\ldots,m-1$, $q=1,2,\ldots,m$ $(p\neq j+2,\ q\neq j+3)$ (see Corollary 13.3.1 *of* [1]*).*

Theorem 2.18. *Let $Q=\{T_0,T_1,\ldots,T_j,\ldots,T_m\}$ $(m\ge 3)$ be a semilattice and j be a fixed natural number such that $0\le j\le m-3$ and*

$$T_0\subset T_1\subset\cdots\subset T_j\subset T_{j+1}\subset T_{j+3}\subset\cdots\subset T_m,$$
$$T_0\subset T_1\subset\cdots\subset T_j\subset T_{j+2}\subset T_{j+3}\subset\cdots\subset T_m,$$
$$T_{j+1}\backslash T_{j+2}\neq\emptyset,\ T_{j+2}\backslash T_{j+1}\neq\emptyset,\ T_{j+1}\cup T_{j+2}=T_{j+3}.$$

If $E_X^{(r)}(Q)$ is the set of all right units of the semigroup $B_X(Q)$, then the following statements are true:

a) $$\left|E_X^{(r)}(Q)\right|=\left(2^{|T_1\backslash T_2|}-1\right)\left(2^{|T_2\backslash T_1|}-1\right)\left(5^{|T_4\backslash T_3|}-4^{|T_4\backslash T_3|}\right)\cdots\left((m+1)^{|T_m\backslash T_{m-1}|}-m^{|T_m\backslash T_{m-1}|}\right)(m+1)^{|X\backslash T_m|},$$

If $j=0$ (i.e., $T_j=T_0$);

b) $$\left|E_X^{(r)}(Q)\right| = \left(2^{|T_1\setminus T_0|}-1\right)\left((j+1)^{|T_i\setminus T_{i-1}|}-j^{|T_i\setminus T_{i-1}|}\right)$$
$$(j+1)^{|T_{i+1}\cap T_{i+2}\setminus T_i|}\left((j+2)^{|T_{j+1}\setminus T_{j+2}|}-(j+1)^{|T_{j+1}\setminus T_{j+2}|}\right)$$
$$\left((j+2)^{|T_{j+2}\setminus T_{j+1}|}-(j+1)^{|T_{j+2}\setminus T_{j+1}|}\right)$$
$$\left((j+5)^{|T_{j+4}\setminus T_{j+3}|}-(j+4)^{|T_{j+4}\setminus T_{j+3}|}\right)$$
$$\cdots\left((m+1)^{|T_m\setminus T_{m-1}|}-m^{|T_m\setminus T_{m-1}|}\right)(m+1)^{|X\setminus T_m|},$$
if $1\le j\le m-3$ $(T_j\ne T_0)$ *(see Corollary* 13.3.3 *of* [1]*).*

Theorem 2.19. *If D is $Z-$elementary $X-$semilattice of unions, then the following equalities are true:*

$$\left|I_{C(D)}\right| = \left|I^*_{C_1(D)}\right| + \left|I^*_{C_2(D)}\right| + \cdots + \left|I^*_{C_k(D)}\right|,$$
$$\left|I_{C'(D)}\right| = \left|I^*_{C'_0(D)}\right| + \left|I^*_{C'_1(D)}\right| + \cdots + \left|I^*_{C'_{2^{|DE_Z\setminus\{Z\}|}}(D)}\right|,$$
$$\left|I_{C''(D)}\right| = \sum_{(T,T',T'')\in v}\left|I^*_{C'_{sk}(T,T',T'',D)}\right|.$$

Proof. The given Theorem immediately follows from the Theorem 1.8. □

Theorem 2.20. *Let D be $Z-$elementary $X-$semilattice of unions and $\alpha\in B_X(D)$. Binary relation α is an idempotent relation of the semigroup $B_X(D)$ iff binary relation α satisfies only one condition of the following conditions:*

a) $\alpha=(X\times T)$, *where* $T\in D$;
b) $\alpha=(Y_0^\alpha\times T_0)\cup(Y_1^\alpha\times T_1)\cup\cdots\cup(Y_k^\alpha\times T_k)$, *where* $T_0, T_1,\cdots,T_k\in D$, $T_0\subset T_1\subset\cdots\subset T_k$, $2\le k\le h(D)$, $Y_1^\alpha,\ldots,Y_{k-1}^\alpha,Y_k^\alpha\notin\{\emptyset\}$ *and satisfies the conditions:* $Y_1^\alpha\cup Y_2^\alpha\cup\cdots\cup Y_p^\alpha\supseteq T_p$, $Y_q^\alpha\cap T_q\ne\emptyset$ *for any* $p=0,1,\ldots,k-1$ *and* $q=1,2,\ldots,k$;
c) $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')\cup(Y_Z^\alpha\times Z)$, *where* $T,T'\in D$, $T_1\cap T_2=\emptyset$, $Y_T^\alpha,Y_{T'}^\alpha\notin\{\emptyset\}$ *and satisfies the conditions:* $Y_T^\alpha\supseteq T$, $Y_{T'}^\alpha\supseteq T'$;
d) $\alpha=(Y_1^\alpha\times T_1)\cup(Y_2^\alpha\times T_2)\cup\cdots\cup(Y_s^\alpha\times T_s)$, *where* $T_1,T_2,\ldots,T_s\in D$, $T_1=T$, $T_2=T'$, $T_3=Z$, $4\le s\le 2^{|D_Z\setminus\{Z\}|}$, $T_1\cap T_2=\emptyset$, Y_1^α,Y_2^α, Y_4^α, $Y_5^\alpha,\ldots,Y_s^\alpha\notin\{\emptyset\}$ *and satisfies the conditions:* $Y_1^\alpha\supseteq T_1$, $Y_2^\alpha\supseteq T_2$, $Y_1^\alpha\cup Y_2^\alpha\cup Y_3^\alpha\cup\cdots\cup Y_p^\alpha\supseteq T_p$ *and* $Y_q^\alpha\cap T_q\ne\emptyset$ *for any* $p=4,5,\ldots,s-1$ *and* $q=4,5,\ldots,s$;
e) $\alpha=(Y_0^\alpha\times T_0)\cup(Y_1^\alpha\times T_1)\cup\cdots\cup\left(Y_{j-1}^\alpha\times T_{j-1}\right)\cup\left(Y_j^\alpha\times T_j\right)\cup\left(Y_{j+1}^\alpha\times T_{j+1}\right)$ $\cup\left(Y_{j+2}^\alpha\times T_{j+2}\right)\cup\left(Y_{j+3}^\alpha\times T_{j+3}\right)\cup\cdots\cup\left(Y_{m-1}^\alpha\times T_{m-1}\right)\cup(Y_m^\alpha\times T_m)$, *where* $T_0,\ldots,T_{j-1},T,T',T'',Z,T_{j+3},\ldots,T_{m-1},T_m\in D$, $T_j=T$, $T_{j+1}=T'$, $T_{j+2}=T''$, $T_{j+3}=Z$, $Y_0^\alpha,Y_1^\alpha,\ldots,Y_{j-1}^\alpha$, $Y_j^\alpha,Y_{j+1}^\alpha,Y_{j+2}^\alpha,Y_{j+4}^\alpha,\ldots,Y_m^\alpha\notin\{\emptyset\}$ *and satisfies the conditions:*

$$Y_0^\alpha\cup Y_1^\alpha\cup\cdots\cup Y_j^\alpha\supseteq T_{j+1}\cap T_{j+2},$$
$$Y_0^\alpha\cup Y_1^\alpha\cup\cdots\cup Y_j^\alpha\cup Y_{j+2}^\alpha\supseteq T_{j+2},$$
$$Y_0^\alpha\cup Y_1^\alpha\cup\cdots\cup Y_p^\alpha\supseteq T_p^\alpha,\ Y_q^\alpha\cap T_q\ne\emptyset$$

for any $p = 0, 1, \ldots, m-1$, $q = 1, 2, \ldots, m$ $(p \neq j+2,\ q \neq j+3)$ *(see Corollary* 13.3.1 *of* [1]*).*

Proof. The given Theorem immediately follows from the Theorem 2.10, 2.14 and 2.17. □

Theorem 2.21. *Let D and I_D be any $Z-$elementary $X-$semilattice of unions and all idempotent elements of the $Z-$elementary $X-$semilattice of unions respectively. Then the following conditions are true.*

a) $|I_D| = \left|I_{C(D)}\right|$, *if* $\mu = \emptyset$ *and* $\nu = \emptyset$*;*
b) $|I_D| = \left|I_{C(D)}\right| + \left|I_{C'(D)}\right|$ *if* $\mu \neq \emptyset$ *and* $\nu = \emptyset$*;*
c) $|I_D| = \left|I_{C(D)}\right| + \left|I_{C'(D)}\right|$ *if* $\mu = \emptyset$ *and* $\nu \neq \emptyset$*;*
d) $|I_D| = \left|I_{C(D)}\right| + \left|I_{C'(D)}\right| + \left|I_{C'(D)}\right|$ *if* $\mu \neq \emptyset$ *and* $\nu \neq \emptyset$*.*

Proof. The given Theorem immediately follows from the Theorem 2.19. □

Theorem 2.22. *If D is any $Z-$elementary $X-$semilattice of unions, then for any idempotent binary relation ε from the semigroup $B_X(D)$ the order of maximal subgroup $G_X(D,\varepsilon)$ is not greater than two.*

Proof. Let D be any $Z-$elementary $X-$semilattice of unions and $\varepsilon \circ \varepsilon = \varepsilon$. As is known (see [1]) the group $G_X(D,\varepsilon)$ is anti-isomorphic to the group of all complete automorphisms of the semilattice $V(D,\varepsilon)$. In this case the number of all complete automorphisms of the semilattice $V(D,\varepsilon)$ is not greater than two. Therefore the order of maximal subgroup $G_X(D,\varepsilon)$ is not greater than two. □

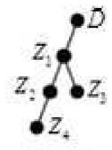

Fig. 2.8

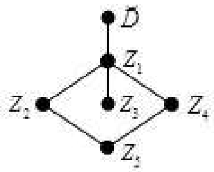

Fig. 2.9

Example 2. Let $D = \left\{Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$ be Z_1-elementary $X-$semilattice of unions satisfying the conditions

$$\begin{array}{l} Z_3 \subset Z_2 \subset Z_1 \subset \breve{D},\ Z_3 \subset Z_1 \subset \breve{D},\ Z_4 \backslash Z_3 \neq \emptyset \\ Z_3 \backslash Z_4 \neq \emptyset,\ Z_3 \backslash Z_2 \neq \emptyset,\ Z_2 \backslash Z_3 \neq \emptyset \\ Z_4 \cup Z_3 = Z_1,\ Z_3 \cup Z_2 = Z_1. \end{array} \tag{2.1}$$

The semilattice satisfying the conditions (2.1) is shown in Fig. 2.8.

Let $C(D) = \{P_0, P_1, P_2, P_3, P_4\}$ be a family sets, where P_0, P_1, P_2, P_3, P_4 are pairwise disjoint subsets of the set X and

$$\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2, Z_3 & Z_4 \\ P_0 & P_1 & P_2 & P_3 & P_4 \end{pmatrix}$$

is a mapping of the semilattice D onto the family sets $C(D)$. Then for the formal equalities of the semilattice D we have a form:

$$\begin{aligned}
\breve{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \\
Z_1 &= P_0 \cup P_2 \cup P_3 \cup P_4 \\
Z_2 &= P_0 \cup P_3 \cup P_4 \\
Z_3 &= P_0 \cup P_2 \cup P_4 \\
Z_4 &= P_0 \cup P_3.
\end{aligned}$$

Here the elements P_1, P_2, P_3, P_4 are basic sources; the elements P_0 are sources of completeness of the Z_1-elementary $X-$semilattice of unions D.

Further, we have $Z_4 \cap Z_3 = (P_0 \cup P_3) \cap (P_0 \cup P_2 \cup P_4) = P_0$.

(1) If $Z_4 \cap Z_3 \neq \emptyset$ $(P_0 \neq \emptyset)$, then $h(D) = 4$, $\mu = \nu = \emptyset$

$$\begin{aligned}
&C_1(D) = \left\{ \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \left\{\breve{D}\right\} \right\} \\
&C_2(D) = \left\{ \begin{array}{l} \{Z_4, Z_2\}, \{Z_4, Z_1\}, \left\{Z_4, \breve{D}\right\}, \{Z_3, Z_1\}, \left\{Z_3, \breve{D}\right\}, \{Z_2, Z_1\}, \\ \left\{Z_2, \breve{D}\right\} \left\{Z_1, \breve{D}\right\} \end{array} \right\} \\
&C_3(D) = \left\{ \left\{Z_2, Z_1, \breve{D}\right\}, \left\{Z_4, Z_1, \breve{D}\right\}, \left\{Z_4, Z_2, \breve{D}\right\}, \{Z_4, Z_2, Z_1\}, \left\{Z_3, Z_1, \breve{D}\right\} \right\} \\
&C_4(D) = \left\{ \left\{Z_4, Z_2, Z_1, \breve{D}\right\} \right\} \\
&C(D) = C_1(D) \cup C_2(D) \cup C_3(D) \cup C_4(D)
\end{aligned}$$

and $\left|I_{C(D)}\right| = \left|I_{C_1(D)}\right| + \left|I_{C_2(D)}\right| + \left|I_{C_3(D)}\right| + \left|I_{C_4(D)}\right|$, where

$$\begin{aligned}
&\left|I^*_{C_1(D)}\right| = 5; \\
&\left|I^*_{C_2(D)}\right| = \left(2^{|Z_2 \setminus Z_4|} - 1\right) 2^{|X \setminus Z_2|} + \left(2^{|Z_1 \setminus Z_4|} + 2^{|Z_1 \setminus Z_3|} + 2^{|Z_1 \setminus Z_2|} - 3\right) 2^{|X \setminus Z_1|} \\
&+ \left(2^{|\breve{D} \setminus Z_4|} + 2^{|\breve{D} \setminus Z_3|} + 2^{|\breve{D} \setminus Z_2|} + 2^{|\breve{D} \setminus Z_1|} - 4\right) 2^{|X \setminus \breve{D}|}; \\
&\left|I^*_{C_3(D)}\right| = \left(2^{|Z_1 \setminus Z_2|} - 1\right) \left(3^{\left|\breve{D} \setminus Z_1\right|} - 2^{\left|\breve{D} \setminus Z_1\right|}\right) 3^{\left|X \setminus \breve{D}\right|} \\
&+ \left(2^{|Z_1 \setminus Z_4|} - 1\right) \left(3^{\left|\breve{D} \setminus Z_1\right|} - 2^{\left|\breve{D} \setminus Z_1\right|}\right) 3^{\left|X \setminus \breve{D}\right|} + \left(2^{|Z_2 \setminus Z_4|} - 1\right) \left(3^{\left|\breve{D} \setminus Z_2\right|} - 2^{\left|\breve{D} \setminus Z_2\right|}\right) 3^{\left|X \setminus \breve{D}\right|} \\
&+ \left(2^{|Z_2 \setminus Z_4|} - 1\right) \left(3^{|Z_1 \setminus Z_2|} - 2^{|Z_1 \setminus Z_2|}\right) 3^{|X \setminus Z_1|} + \left(2^{|Z_1 \setminus Z_3|} - 1\right) \left(3^{\left|\breve{D} \setminus Z_1\right|} - 2^{\left|\breve{D} \setminus Z_1\right|}\right) 3^{\left|X \setminus \breve{D}\right|} \\
&\left|I^*_{C_4(D)}\right| = \left(2^{|Z_2 \setminus Z_4|} - 1\right) \left(3^{|Z_1 \setminus Z_2|} - 2^{|Z_1 \setminus Z_2|}\right) \left(3^{\left|\breve{D} \setminus Z_1\right|} - 2^{\left|\breve{D} \setminus Z_1\right|}\right) 4^{\left|X \setminus \breve{D}\right|}
\end{aligned}$$

(see Theorem 2.4).

If $X = \{1, 2, 3, 4, 5\}$, $D = \{\{3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}$ then $\left|I^*_{C_1(D)}\right| = 5$, $\left|I^*_{C_2(D)}\right| = 28$, $\left|I^*_{C_3(D)}\right| = 13$, $\left|I^*_{C_4(D)}\right| = 1$, $\left|I_{C(D)}\right| = 47$.

(2) If $Z_4 \cap Z_3 = \emptyset$ $(P_0 = \emptyset)$, then $\mu = \{\{Z_4, Z_3\}\}$, $\nu = \emptyset$, $h(D) = 4$, $s = 0, 1$ and

$$\begin{aligned}
&C_1(D) = \left\{ \{Z_4,\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \left\{\breve{D}\right\} \right\} \\
&C_2(D) = \left\{ \begin{array}{l} \{Z_4, Z_2\}, \{Z_4, Z_1\}, \left\{Z_4, \breve{D}\right\}, \{Z_3, Z_1\}, \left\{Z_3, \breve{D}\right\}, \{Z_2, Z_1\}, \\ \left\{Z_2, \breve{D}\right\} \left\{Z_1, \breve{D}\right\} \end{array} \right\},
\end{aligned}$$

$C_3(D) = \left\{\left\{Z_2, Z_1, \breve{D}\right\}, \left\{Z_4, Z_1, \breve{D}\right\}, \left\{Z_4, Z_2, \breve{D}\right\}, \{Z_4, Z_2, Z_1\}, \left\{Z_3, Z_1, \breve{D}\right\}\right\}$
$C_4(D) = \left\{\left\{Z_4, Z_2, Z_1, \breve{D}\right\}\right\}$
$C(D) = C_1(D) \cup C_2(D) \cup C_3(D) \cup C_4(D)$
$C_0'(D) = \{\{Z_4, Z_3, Z_1\}\}$
$C_1'(D) = \left\{\left\{Z_4, Z_3, Z_1, \breve{D}\right\}\right\}.$

and $|I_D| = \left|I_{C_1(D)}\right| + \left|I_{C_2(D)}\right| + \left|I_{C_3(D)}\right| + \left|I_{C_4(D)}\right| + \left|I_{C_0'(D)}\right| + \left|I_{C_1'(D)}\right|$, where

$$\left|I^*_{C_1(D)}\right| = 5;$$
$$\left|I^*_{C_2(D)}\right| = \left(2^{|Z_2\setminus Z_4|} - 1\right) 2^{|X\setminus Z_2|} + \left(2^{|Z_1\setminus Z_4|} + 2^{|Z_1\setminus Z_3|} + 2^{|Z_1\setminus Z_2|} - 3\right) 2^{|X\setminus Z_1|} + \left(2^{\left|\breve{D}\setminus Z_4\right|} + 2^{\left|\breve{D}\setminus Z_3\right|} + 2^{\left|\breve{D}\setminus Z_2\right|} + 2^{\left|\breve{D}\setminus Z_1\right|} - 4\right) 2^{\left|X\setminus \breve{D}\right|};$$
$$\left|I^*_{C_3(D)}\right| = \left(2^{|Z_1\setminus Z_2|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 3^{\left|X\setminus \breve{D}\right|} + \left(2^{|Z_1\setminus Z_4|} - 1\right) + \left(2^{|Z_1\setminus Z_4|} - 1\right) \left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 3^{\left|X\setminus \breve{D}\right|} + \left(2^{|Z_2\setminus Z_4|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_2\right|} - 2^{\left|\breve{D}\setminus Z_2\right|}\right) 3^{\left|X\setminus \breve{D}\right|} + \left(2^{|Z_2\setminus Z_4|} - 1\right)\left(3^{|Z_1\setminus Z_2|} - 2^{|Z_1\setminus Z_2|}\right) 3^{|X\setminus Z_1|} + \left(2^{|Z_1\setminus Z_3|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_2\right|}\right) 3^{\left|X\setminus \breve{D}\right|}$$
$$\left|I^*_{C_4(D)}\right| = \left(2^{|Z_2\setminus Z_4|} - 1\right)\left(3^{|Z_1\setminus Z_2|} - 2^{|Z_1\setminus Z_2|}\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 4^{\left|X\setminus \breve{D}\right|}$$
$$\left|I^*_{C_0'(D)}\right| = 3^{|X\setminus Z_1|},$$
$$\left|I^*_{C_1'(D)}\right| = \left(4^{\left|\breve{D}\setminus Z_1\right|} - 3^{\left|\breve{D}\setminus Z_1\right|}\right) 4^{\left|X\setminus \breve{D}\right|}$$

(see Theorems 2.11 and 2.15).

If $X = \{1, 2, 3, 4\}$, $D = \{\{3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$ then $\left|I^*_{C_1(D)}\right| = 5$, $\left|I^*_{C_2(D)}\right| = 28$, $\left|I^*_{C_3(D)}\right| = 13$, $\left|I^*_{C_4(D)}\right| = 1$, $\left|I^*_{C_0'(D)}\right| = 3$, $\left|I^*_{C_1'(D)}\right| = 1$, $|I_D| = 51$.

Example 3. Let $D = \left\{Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$ be Z_1-elementary $X-$semilattice of unions satisfying the conditions

$$\begin{array}{l} Z_5 \subset Z_2 \subset Z_1 \subset \breve{D},\ Z_5 \subset Z_4 \subset Z_1 \subset \breve{D},\ Z_3 \subset Z_1 \subset \breve{D}\ Z_4\setminus Z_3 \neq \emptyset \\ Z_3\setminus Z_4 \neq \emptyset,\ Z_4\setminus Z_2 \neq \emptyset,\ Z_2\setminus Z_4 \neq \emptyset,\ Z_3\setminus Z_2 \neq \emptyset,\ Z_2\setminus Z_3 \neq \emptyset \\ Z_4 \cup Z_3 = Z_4 \cup Z_2 = Z_3 \cup Z_2 = Z_5 \cup Z_3 = Z_1. \end{array} \quad (2.2)$$

The semilattice satisfying the conditions (2.2) is shown in Fig. 2.9.

Let $C(D) = \{P_0, P_1, P_2, P_3, P_4, P_5\}$ be a family sets, where $P_0, P_1, P_2, P_3, P_4, P_5$ are pairwise disjoint subsets of the set X and

$$\varphi = \begin{pmatrix} \breve{D} & Z_1 & Z_2, Z_3 & Z_4 & Z_5 \\ P_0 & P_1 & P_2 \ P_3 & P_4 & P_5 \end{pmatrix}$$

be a mapping of the semilattice D onto the family sets $C(D)$. Then for the formal equalities of the semilattice D we have a form:

$$\begin{array}{l} \breve{D} = P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \\ Z_1 = P_0 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \\ Z_2 = P_0 \cup P_3 \cup P_4 \cup P_5 \end{array}$$

$$Z_3 = P_0 \cup P_2 \cup P_4 \cup P_5$$
$$Z_4 = P_0 \cup P_2 \cup P_3 \cup P_5$$
$$Z_5 = P_0 \cup P_3.$$

Here the elements P_1, P_2, P_3, P_4 are basic sources; the elements P_0, P_5 are sources of completeness of the Z_1-elementary $X-$semilattice of unions D.

Further, we have $Z_5 \cap Z_3 = (P_0 \cup P_3) \cap (P_0 \cup P_2 \cup P_4 \cup P_5) = P_0$.

(1) If $Z_5 \cap Z_3 \neq \emptyset$ $(P_0 \neq \emptyset)$, then $\mu = \emptyset$, $\nu = \{(Z_5, Z_4, Z_2)\}$, $h(D) = 4$, $s = 0, 1$,

$$C_1(D) = \left\{\{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \left\{\breve{D}\right\}\right\},$$
$$C_2(D) = \left\{\begin{array}{l} \{Z_5, Z_4\}, \{Z_5, Z_2\}, \{Z_5, Z_1\}, \left\{Z_5, \breve{D}\right\}, \{Z_4, Z_1\}, \left\{Z_4, \breve{D}\right\}, \{Z_3, Z_1\}, \\ \left\{Z_3, \breve{D}\right\}, \{Z_2, Z_1\}, \left\{Z_2, \breve{D}\right\} \left\{Z_1, \breve{D}\right\} \end{array}\right\},$$
$$C_3(D) = \left\{\begin{array}{l} \{Z_5, Z_4, Z_1\}, \left\{Z_5, Z_4, \breve{D}\right\}, \{Z_5, Z_2, Z_1\}, \left\{Z_5, Z_2, \breve{D}\right\}, \left\{Z_5, Z_1, \breve{D}\right\}, \\ \left\{Z_4, Z_1, \breve{D}\right\}, \left\{Z_3, Z_1, \breve{D}\right\}, \left\{Z_2, Z_1, \breve{D}\right\} \end{array}\right\},$$
$$C_4(D) = \left\{\left\{Z_5, Z_4, Z_1, \breve{D}\right\}, \left\{Z_5, Z_2, Z_1, \breve{D}\right\}\right\}$$
$$C''_0(D) = \{\{Z_5, Z_4, Z_2, Z_1\}\}$$
$$C''_1(D) = \left\{\left\{Z_5, Z_4, Z_2, Z_1, \breve{D}\right\}\right\}$$
$$C(D) = C_1(D) \cup C_2(D) \cup C_3(D) \cup C_4(D),\ C''(D) = C''_0(D) \cup C''_1(D).$$

and $|I_D| = \left|I_{C_1(D)}\right| + \left|I_{C_2(D)}\right| + \left|I_{C_3(D)}\right| + \left|I_{C_4(D)}\right| + \left|I_{C''_0(D)}\right| + \left|I_{C''_1(D)}\right|$, where

$$\left|I^*_{C_1(D)}\right| = 6;$$
$$\left|I^*_{C_2(D)}\right| = \left(2^{|Z_1\setminus Z_5|} + 2^{|Z_1\setminus Z_4|} + 2^{|Z_1\setminus Z_3|} + 2^{|Z_1\setminus Z_2|} - 4\right) 2^{|X\setminus Z_1|} + \left(2^{|Z_2\setminus Z_5|} - 1\right) 2^{|X\setminus Z_1|}$$
$$+ \left(2^{|Z_4\setminus Z_5|} - 1\right) 2^{|X\setminus Z_4|} + \left(2^{\left|\breve{D}\setminus Z_5\right|} + 2^{\left|\breve{D}\setminus Z_4\right|} + 2^{\left|\breve{D}\setminus Z_3\right|} + 2^{\left|\breve{D}\setminus Z_2\right|} + 2^{\left|\breve{D}\setminus Z_1\right|} - 5\right) 2^{\left|X\setminus\breve{D}\right|};$$
$$\left|I^*_{C_3(D)}\right| = \left(2^{|Z_4\setminus Z_5|} - 1\right)\left(3^{|Z_1\setminus Z_4|} - 2^{|Z_1\setminus Z_4|}\right) 3^{|X\setminus Z_1|}$$
$$+ \left(2^{|Z_2\setminus Z_5|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_2\right|} - 2^{\left|\breve{D}\setminus Z_2\right|}\right) 3^{\left|X\setminus\breve{D}\right|} + \left(2^{|Z_2\setminus Z_5|} - 1\right)\left(3^{|Z_1\setminus Z_2|} - 2^{|Z_1\setminus Z_2|}\right) 3^{|X\setminus Z_1|}$$
$$+ \left(2^{|Z_2\setminus Z_5|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_2\right|} - 2^{\left|\breve{D}\setminus Z_2\right|}\right) 3^{\left|X\setminus\breve{D}\right|} + \left(2^{|Z_1\setminus Z_5|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 3^{\left|X\setminus\breve{D}\right|}$$
$$+ \left(2^{|Z_1\setminus Z_4|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 3^{\left|X\setminus\breve{D}\right|} + \left(2^{|Z_1\setminus Z_3|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 3^{\left|X\setminus\breve{D}\right|}$$
$$+ \left(2^{|Z_1\setminus Z_2|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 3^{\left|X\setminus\breve{D}\right|};$$
$$\left|I^*_{C_4(D)}\right| = \left(2^{|Z_4\setminus Z_5|} - 1\right)\left(3^{|Z_1\setminus Z_4|} - 2^{|Z_1\setminus Z_4|}\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 4^{\left|X\setminus\breve{D}\right|}$$
$$+ \left(2^{|Z_2\setminus Z_5|} - 1\right)\left(3^{|Z_1\setminus Z_2|} - 2^{|Z_1\setminus Z_2|}\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 4^{\left|X\setminus\breve{D}\right|}$$
$$\left|I^*_{C''_0(D)}\right| = \left(2^{|Z_2\setminus Z_4|} - 1\right)\left(2^{|Z_4\setminus Z_2|} - 1\right) 4^{|X\setminus Z_1|};$$
$$\left|I^*_{C''_1(D)}\right| = \left(2^{|Z_2\setminus Z_4|} - 1\right)\left(2^{|Z_4\setminus Z_2|} - 1\right)\left(5^{\left|\breve{D}\setminus Z_1\right|} - 4^{\left|\breve{D}\setminus Z_1\right|}\right) 5^{\left|X\setminus\breve{D}\right|}$$

(see Theorems 2.11 and 2.18).

If $X = \{1, 2, 3, 4, 5, 6\}$,

$$D = \{\{3, 6\}, \{2, 3, 5, 6\}, \{2, 4, 5, 6\}, \{3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}$$

then $\left|I^*_{C_1(D)}\right| = 6$, $\left|I^*_{C_2(D)}\right| = 69$, $\left|I^*_{C_3(D)}\right| = 58$, $\left|I^*_{C_4(D)}\right| = 6$, $\left|I^*_{C''_0(D)}\right| = 4$, $\left|I^*_{C''_1(D)}\right| = 1$, $|I_D| = 144$.

(2) If $Z_5 \cap Z_3 = \emptyset$ $(P_0 = \emptyset)$, then $\mu = \{(Z_5, Z_3)\}$, $\nu = \{(Z_5, Z_2, Z_4)\}$, $h(D) = 4$, $s = 0, 1$,

$$\begin{aligned}
&C_1(D) = \left\{\{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \left\{\breve{D}\right\}\right\},\\
&C_2(D) = \left\{\begin{array}{l} \{Z_5, Z_2\}, \{Z_5, Z_4\}, \{Z_5, Z_1\}, \left\{Z_5, \breve{D}\right\}, \{Z_4, Z_1\}, \left\{Z_4, \breve{D}\right\}, \{Z_3, Z_1\}, \\ \left\{Z_3, \breve{D}\right\}, \{Z_2, Z_1\}, \left\{Z_2, \breve{D}\right\} \left\{Z_1, \breve{D}\right\} \end{array}\right\},\\
&C_3(D) = \left\{\begin{array}{l} \{Z_5, Z_4, Z_1\}, \left\{Z_5, Z_4, \breve{D}\right\}, \{Z_5, Z_2, Z_1\}, \left\{Z_5, Z_2, \breve{D}\right\}, \left\{Z_5, Z_1, \breve{D}\right\}, \\ \left\{Z_4, Z_1, \breve{D}\right\}, \left\{Z_3, Z_1, \breve{D}\right\}, \left\{Z_2, Z_1, \breve{D}\right\} \end{array}\right\},\\
&C_4(D) = \left\{\left\{Z_5, Z_4, Z_1, \breve{D}\right\}, \left\{Z_5, Z_2, Z_1, \breve{D}\right\}\right\},\\
&C_0''(D) = \{\{Z_5, Z_3, Z_1\}\},\ C_1''(D) = \left\{\left\{Z_5, Z_3, Z_1, \breve{D}\right\}\right\},\\
&C_0''(D) = \{\{Z_5, Z_4, Z_2, Z_1\}\},\ C_1''(D) = \left\{\left\{Z_5, Z_4, Z_2, Z_1, \breve{D}\right\}\right\},\\
&C(D) = C_1(D) \cup C_2(D) \cup C_3(D) \cup C_4(D),\\
&C'(D) = C_0'(D) \cup C_1'(D),\ \ C''(D) = C_0''(D) \cup C_1''(D)
\end{aligned}$$

and $|I_D| = \left|I_{C_1(D)}\right| + \left|I_{C_2(D)}\right| + \left|I_{C_3(D)}\right| + \left|I_{C_4(D)}\right| + \left|I_{C_0'(D)}\right| + \left|I_{C_1'(D)}\right| + \left|I_{C_0''(D)}\right| + \left|I_{C_1''(D)}\right|$, where

$$\begin{aligned}
&\left|I^*_{C_1(D)}\right| = 6;\\
&\left|I^*_{C_2(D)}\right| = \left(2^{|Z_1\setminus Z_5|} + 2^{|Z_1\setminus Z_4|} + 2^{|Z_1\setminus Z_3|} + 2^{|Z_1\setminus Z_2|} - 4\right) 2^{|X\setminus Z_1|} + \left(2^{|Z_2\setminus Z_5|} - 1\right) 2^{|X\setminus Z_1|}\\
&+\left(2^{|Z_4\setminus Z_5|} - 1\right) 2^{|X\setminus Z_4|} + \left(2^{\left|\breve{D}\setminus Z_5\right|} + 2^{\left|\breve{D}\setminus Z_4\right|} + 2^{\left|\breve{D}\setminus Z_3\right|} + 2^{\left|\breve{D}\setminus Z_2\right|} + 2^{\left|\breve{D}\setminus Z_1\right|} - 5\right) 2^{\left|X\setminus \breve{D}\right|};\\
&\left|I^*_{C_3(D)}\right| = \left(2^{|Z_4\setminus Z_5|} - 1\right)\left(3^{|Z_1\setminus Z_4|} - 2^{|Z_1\setminus Z_4|}\right) 3^{|X\setminus Z_1|}\\
&+\left(2^{|Z_2\setminus Z_5|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_2\right|} - 2^{\left|\breve{D}\setminus Z_2\right|}\right) 3^{\left|X\setminus \breve{D}\right|} + \left(2^{|Z_2\setminus Z_5|} - 1\right)\left(3^{|Z_1\setminus Z_2|} - 2^{|Z_1\setminus Z_2|}\right) 3^{|X\setminus Z_1|}\\
&+\left(2^{|Z_2\setminus Z_5|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_2\right|} - 2^{\left|\breve{D}\setminus Z_2\right|}\right) 3^{\left|X\setminus \breve{D}\right|} + \left(2^{|Z_1\setminus Z_5|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 3^{\left|X\setminus \breve{D}\right|}\\
&+\left(2^{|Z_1\setminus Z_4|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 3^{\left|X\setminus \breve{D}\right|} + \left(2^{|Z_1\setminus Z_3|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 3^{\left|X\setminus \breve{D}\right|}\\
&+\left(2^{|Z_1\setminus Z_2|} - 1\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 3^{\left|X\setminus \breve{D}\right|};\\
&\left|I^*_{C_4(D)}\right| = \left(2^{|Z_4\setminus Z_5|} - 1\right)\left(3^{|Z_1\setminus Z_4|} - 2^{|Z_1\setminus Z_4|}\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 4^{\left|X\setminus \breve{D}\right|}\\
&+\left(2^{|Z_2\setminus Z_5|} - 1\right)\left(3^{|Z_1\setminus Z_2|} - 2^{|Z_1\setminus Z_2|}\right)\left(3^{\left|\breve{D}\setminus Z_1\right|} - 2^{\left|\breve{D}\setminus Z_1\right|}\right) 4^{\left|X\setminus \breve{D}\right|}\\
&\left|I^*_{C_0'(D)}\right| = 3^{|X\setminus Z_1|},\ \left|I^*_{C_1'(D)}\right| = \left(4^{\left|\breve{D}\setminus Z_1\right|} - 3^{\left|\breve{D}\setminus Z_1\right|}\right) 4^{\left|X\setminus \breve{D}\right|}\\
&\left|I^*_{C_0''(D)}\right| = \left(2^{|Z_2\setminus Z_4|} - 1\right)\left(2^{|Z_4\setminus Z_2|} - 1\right) 4^{|X\setminus Z_1|},\\
&\left|I^*_{C_1''(D)}\right| = \left(2^{|Z_2\setminus Z_4|} - 1\right)\left(2^{|Z_4\setminus Z_2|} - 1\right)\left(5^{\left|\breve{D}\setminus Z_1\right|} - 4^{\left|\breve{D}\setminus Z_1\right|}\right) 5^{\left|X\setminus \breve{D}\right|}
\end{aligned}$$

(see Theorems 2.11, 2.15 and 2.18).

If $X = \{1, 2, 3, 4, 5\}$,

$$D = \{\{3\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\}$$

then $\left|I^*_{C_1(D)}\right| = 6$, $\left|I^*_{C_2(D)}\right| = 69$, $\left|I^*_{C_3(D)}\right| = 58$, $\left|I^*_{C_4(D)}\right| = 6$, $\left|I^*_{C_0'(D)}\right| = 3$, $\left|I^*_{C_1'(D)}\right| = 1$, $\left|I^*_{C_0''(D)}\right| = 4$, $\left|I^*_{C_1''(D)}\right| = 1$, $|I_D| = 148$.

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