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ON THE CONVOLUTION AND NEUTRIX CONVOLUTION OF THE FUNCTIONS $\sinh^{-1} x$ AND x^r

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ABSTRACT. The neutrix convolution $\sinh^{-1} x \circledast x^r$ is evaluated for $r = 0, 1, 2, \ldots$. Further results are also given.

1. Introduction

The functions $\sinh^{-1} x_+$ and $\sinh^{-1} x_-$ are defined by

$$\sinh^{-1} x_+ = H(x)\sinh^{-1} x, \qquad \sinh^{-1} x_- = H(-x)\sinh^{-1} x,$$

where H denotes Heaviside's function. Note that

$$\sinh^{-1} x = \sinh^{-1} x_+ + \sinh^{-1} x_-.$$

If f and g are locally summable functions then the classical definition for the convolution $f * g$ of f and g is as follows:

Definition 1. Let f and g be functions. Then the convolution $f * g$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt \tag{1}$$

for all points x for which the integral exists.

It follows easily from the definition that if the classical convolution $f * g$ of f and g exists, then $g * f$ exists and

$$f * g = g * f. \tag{2}$$

Further, if $(f * g)'$ and $f * g'$ (or $f' * g$) exist, then

$$(f * g)' = f * g' \quad (\text{or } f' * g). \tag{3}$$

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The classical definition of the convolution can be extended to define the convolution $f * g$ of two distributions f and g in $\mathcal{D}'$ with the following definition, see [9].

Definition 2. Let f and g be distributions in $\mathcal{D}'$. Then the convolution $f * g$ is defined by the equation

$$\langle (f*g)(x), \varphi(x)\rangle = \langle f(y), \langle g(x), \varphi(x+y)\rangle\rangle \tag{4}$$

for arbitrary φ in $\mathcal{D}'$, provided that f and g satisfy either of the following conditions:

(a) either f or g has bounded support,
(b) the supports of f and g are bounded on the same side.

It follows that if the convolution $f * g$ exists by this definition, then equations (2) and (3) are satisfied.

The following theorems were proved in [10].

Theorem 1. *The neutrix convolutions* $(\tan_+^{-1} x) \circledast x^{2r+1}$ *and* $(\tan_+^{-1} x) \circledast x^{2r}$ *exist and*

$$\begin{aligned}(\tan_+^{-1} x) \circledast x^{2r+1} &= \sum_{k=0}^{r} \binom{2r+1}{2k} \frac{(-1)^{k+1}}{(2k+1)^2} x^{2r-2k+1} \\ &\quad + \sum_{k=0}^{r} \binom{2r+1}{2k+1} \frac{(-1)^{k+1}\pi}{4(k+1)} x^{2r-2k}, \\ (\tan_+^{-1} x) \circledast x^{2r} &= \sum_{k=0}^{r} \binom{2r}{2k} \frac{(-1)^{k+1}}{(2k+1)^2} x^{2r-2k} \\ &\quad + \sum_{k=1}^{r} \binom{2r}{2k-1} \frac{(-1)^{k}\pi}{4k} x^{2r-2k+1},\end{aligned}$$

for $r = 0, 1, 2, \ldots$.

Theorem 2. *The neutrix convolutions* $x^{2r+1} \circledast \tan_+^{-1} x$ *and* $x^{2r} \circledast \tan_+^{-1} x$ *exist and*

$$x^{2r+1} \circledast \tan_+^{-1} x = \sum_{k=0}^{r} \binom{2r+1}{2k} x^{2r-2k+1} G_k(x) - \sum_{k=0}^{r} \binom{2r+1}{2k+1} x^{2r-2k} F_k(x),$$

$$x^{2r} \circledast \tan_+^{-1} x = \sum_{k=0}^{r} \binom{2r}{2k} x^{2r-2k} G_k(x) - \sum_{k=0}^{r-1} \binom{2r}{2k+1} x^{2r-2k-1} F_k(x),$$

for $r = 0, 1, 2, \ldots$.

The next theorem was proved in [6].

Theorem 3. *If $\lambda, \lambda + \mu < 0$ and $\mu \neq 0$, then the neutrix convolution $\mathrm{ei}_-(\lambda x) * e^{\mu x}$ exists and*

$$\mathrm{ei}_-(\lambda x) * e^{\mu x} = -\mu^{-1} \ln(1 + \mu/\lambda) e^{\mu x}.$$

The dilogarithm integral $\mathrm{Li}(x)$ see [6] is defined for by

$$\mathrm{Li}(x) = -\int_0^x \frac{\ln|1-t|}{t}\, dt$$

and the associated functions $\mathrm{Li}_+(x)$ and $\mathrm{Li}_-(x)$ are defined by

$$\mathrm{Li}_+(x) = H(x)\,\mathrm{Li}(x), \qquad \mathrm{Li}_-(x) = H(-x)\,\mathrm{Li}(x) = \mathrm{Li}(x) - \mathrm{Li}_+(x),$$

where $H(x)$ denotes Heaviside's function.

The following theorem was proved in [6].

Theorem 4. *The neutrix convolution $\mathrm{Li}_+(x) \circledast x^r$ exists and*

$$\mathrm{Li}_+(x) \circledast x^r = \frac{1}{r+1} \sum_{i=0}^{r} \binom{r+1}{i} \frac{(-1)^{r-i}}{(r-i+1)^2}\, x^i$$

for $r = 0, 1, 2, \ldots$. In particular

$$\mathrm{Li}_+(x) \circledast H(x) = 1,$$
$$\mathrm{Li}_+(x) \circledast x_+ = x - \frac{1}{8}.$$

2. Main results

We need the following lemmas to prove our results on the convolution and neutrix convolution.

Lemma 1.

$$\sinh^{2r} x = 2^{1-2r} \sum_{k=1}^{r} \binom{2r}{r-k} (-1)^{r-k} \cosh(2kx) + (-1)^r 2^{-2r} \binom{2r}{r}, \tag{5}$$

$$\sinh^{2r-1} x = 2^{2-2r} \sum_{k=1}^{r} \binom{2r-1}{r-k} (-1)^{r-k} \sinh(2k-1)x, \tag{6}$$

for $r = 1, 2, \ldots$.

Proof. We have

$$\begin{aligned}\sinh^{2r} x &= 2^{-2r}(e^x - e^{-x})^{2r} = 2^{-2r}\sum_{k=0}^{2r}\binom{2r}{k}(-1)^k e^{(2r-2k)x}\\ &= 2^{-2r}\sum_{k=0}^{r-1}\binom{2r}{k}(-1)^k(e^{(2r-2k)x} + e^{-(2r-2k)x}) + (-1)^r 2^{-2r}\binom{2r}{r}\\ &= 2^{1-2r}\sum_{k=1}^{r}\binom{2r}{r-k}(-1)^{r-k}\cosh(2kx) + (-1)^r 2^{-2r}\binom{2r}{r},\end{aligned}$$

proving equation (5).

Similarly, we have

$$\begin{aligned}\sinh^{2r-1} x &= 2^{1-2r}(e^x - e^{-x})^{2r-1} = 2^{1-2r}\sum_{k=0}^{2r-1}\binom{2r-1}{k}(-1)^k e^{(2r-2k-1)x}\\ &= 2^{1-2r}\sum_{k=0}^{r-1}\binom{2r-1}{k}(-1)^k(e^{(2r-2k-1)x} - e^{-(2r-2k-1)x})\\ &= 2^{2-2r}\sum_{k=1}^{r}\binom{2r-1}{r-k}(-1)^{r-k}\sinh(2k-1)x,\end{aligned}$$

proving equation (6).

For shortness, we will write

$$\sinh^r x = \sum_{k=0}^{r}[a_{r,k}\cosh(kx) + b_{r,k}\sinh(kx)], \tag{7}$$

for $r = 1, 2, \dots,$ where

$$\begin{aligned} a_{2r,2k-1} &= 0; \quad k = 1, 2, \dots, r,\\ a_{2r-1,k} &= 0; \quad k = 0, 1, 2, \dots, 2r-1,\\ b_{2r-1,2k} &= 0; \quad k = 0, 1, 2, \dots, r-1,\\ b_{2r,k} &= 0; \quad k = 1, 2, \dots, 2r\end{aligned}$$

so that

$$\sinh^{2r} x = \sum_{k=0}^{r} a_{2r,2k}\cosh(2kx), \tag{8}$$

$$\sinh^{2r-1} x = \sum_{k=1}^{r} a_{2r-1,2k-1}\sinh(2k-1)x, \tag{9}$$

for $r = 1, 2, \dots$ and $k = 1, 2, \dots, r$. □

Lemma 2.

$$\sinh(2rx) = \sum_{k=1}^{r}\sum_{j=0}^{k-1}\binom{2r}{2k-1}\binom{k-1}{j}(-1)^{k+j+1}\cosh^{2r-2k+2j+1}x\sinh x, \tag{10}$$

$$\sinh(2r-1)x = \sum_{k=1}^{r}\sum_{j=0}^{k-1}\binom{2r-1}{2k-1}\binom{k-1}{j}(-1)^{k+j+1}\cosh^{2r-2k+2j}x\sinh x, \tag{11}$$

for $r = 1, 2, \ldots$.

Proof. Using de Moivre's Theorem, we have

$$\cos(2rx) + i\sin(2rx) \quad = \quad (\cos x + i\sin x)^{2r}.$$

Equating the imaginary parts, we have

$$\begin{aligned}\sin(2rx) &= \sum_{k=1}^{r}\binom{2r}{2k-1}(-1)^{k+1}\cos^{2r-2k+1}x\sin^{2k-1}x\\ &= \sum_{k=1}^{r}\binom{2r}{2k-1}(-1)^{k+1}\cos^{2r-2k+1}x(1-\cos^2 x)^{k-1}\sin x\\ &= \sum_{k=1}^{r}\binom{2r}{2k-1}(-1)^{k+1}\cos^{2r-2k+1}x\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\cos^{2j}x\sin x.\end{aligned} \tag{12}$$

Replacing x by ix in equation (12), we get equation (10).

Similarly, we have

$$\begin{aligned}\sin(2r-1)x &= \sum_{k=1}^{r}\binom{2r-1}{2k-1}(-1)^{k+1}\cos^{2r-2k}x\sin^{2k-1}x\\ &= \sum_{k=1}^{r}\binom{2r-1}{2k-1}(-1)^{k+1}\cos^{2r-2k}x(1-\cos^2 x)^{k-1}\sin x\\ &= \sum_{k=1}^{r}\binom{2r-1}{2k-1}(-1)^{k+j+1}\cos^{2r-2k}x\sum_{j=0}^{k-1}\binom{k-1}{j}\cos^{2j}x\sin x.\end{aligned} \tag{13}$$

Replacing x by ix in equation (13), we get equation (11).

For shortness, we will write

$$\sinh(rx) = \sum_{k=1}^{r} c_{r,k}\cosh^k x\sinh x, \tag{14}$$

for $r = 1, 2, \ldots$ and $k = 1, 2, \ldots, r$, where $c_{2r,2k} = c_{2r-1,2k-1} = 0$, for $k = 1, 2, \ldots, r$, so that

$$\sinh(2rx) = \sum_{k=1}^{r} c_{2r,2k+1} \cosh^{2r-2k+1} x \sinh x, \tag{15}$$

$$\sinh(2r-1)x = \sum_{k=1}^{r} c_{2r-1,2k} \cosh^{2r-2k} x \sinh x, \tag{16}$$

for $r = 1, 2, \ldots$ and $k = 1, 2, \ldots, r$. □

Lemma 3.

$$\int \sinh^r x\, dx = \sum_{k=1}^{r} \sum_{i=1}^{k} \frac{a_{r,k} b_{k,i}}{k} \cosh^i x \sinh x, \tag{17}$$

for $r = 1, 2, \ldots$.

Proof. Using equations (7) and (14), we have

$$\begin{aligned} \int \sinh^r x\, dx &= \sum_{k=1}^{r} \int [a_{r,k} \cosh(kx) + b_{r,k} \sinh(kx)]\, dx \\ &= \sum_{k=1}^{r} \frac{a_{r,k} \sinh(kx) + b_{r,k} \cosh(kx)}{k} \\ &= \sum_{k=1}^{r} \sum_{i=1}^{k} \frac{a_{r,k} b_{k,i}}{k} \cosh^i x \sinh x, \end{aligned}$$

proving equation (17). □

Theorem 5. *The convolution* $\sinh^{-1} x_+ * x_+^r$ *exists and*

$$\begin{aligned} \sinh^{-1} x_+ * x_+^r = \sum_{k=0}^{r} \binom{r}{k} \Bigg[& \frac{(-1)^k x_+^{r+1} \sinh^{-1} x}{k+1} \\ & - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j}}{i(k+1)} x_+^{r-k+1} (x^2+1)^{j/2} \Bigg], \end{aligned} \tag{18}$$

for $r = 0, 1, 2, \ldots$.

Proof. It is obvious that $\sinh^{-1} x_+ * x_+^r = 0$ if $x < 0$. When $x > 0$, we have

$$\begin{aligned} \sinh^{-1} x_+ * x_+^r &= \int_0^x \sinh^{-1} t (x-t)^r\, dt \\ &= \sum_{k=0}^{r} \binom{r}{k} x^{r-k} \int_0^x (-t)^k \sinh^{-1} t\, dt. \end{aligned} \tag{19}$$

Making the substitution $t = \sinh u$, we get

$$\begin{aligned}\int_0^x t^k \sinh^{-1} t\, dt &= \int_0^{\sinh^{-1} x} u \sinh^k u \cosh u\, du \\ &= \frac{x^{k+1}\sinh^{-1} x}{k+1} - \int_0^{\sinh^{-1} x} \frac{\sinh^{k+1} u}{k+1} du \\ &= \frac{x^{k+1}\sinh^{-1} x}{k+1} - \sum_{i=1}^{k+1}\sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j}}{i(k+1)} x(x^2+1)^{j/2}, \qquad (20)\end{aligned}$$

on using equation (17). Equation (18) now follows from equations (19) and (20). □

Replacing x by $-x$ in equation (18), we get

Corollary 1. *The convolution* $\sinh^{-1} x_- * x_-^r$ *exists and*

$$\sinh^{-1} x_- * x_-^r = \sum_{k=0}^{r} \binom{r}{k} \Bigg[\frac{(-1)^k x_-^{r+1} \sinh^{-1} x}{k+1} - \sum_{i=1}^{k+1}\sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j}}{i(k+1)} x_-^{r-k+1} (x^2+1)^{j/2} \Bigg], \quad (21)$$

for $r = 0, 1, 2, \ldots$.

The definition of the convolution is rather restrictive and so the non-commutative neutrix convolution was introduced in [2]. In order to define the neutrix convolution we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function τ_n is then defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n \end{cases}$$

for $n = 1, 2, \ldots$.

The following definition was given in [2].

Definition 3. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the *neutrix convolution* $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit h exists in the sense

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'', the real numbers, with negligible functions being finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0, r = 1, 2, \ldots)$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, if

$$\lim_{n\to\infty}\langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, we say that the *convolution* $f * g$ exists and equals h.

Note that in this definition the convolution $f_n * g$ is as defined in Gel'fand and Shilov's sense, the distribution f_n having compact support. Note also that because of the lack of symmetry in the definition of $f \circledast g$, the neutrix convolution is in general non-commutative.

The following theorem was proved in [2], showing that the neutrix convolution is a generalization of the convolution.

Theorem 6. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution $f \circledast g$ exists and*

$$f \circledast g = f * g.$$

We now prove the following theorem.

Theorem 7. *The neutrix convolution* $\sinh^{-1} x_+ \circledast x^r$ *exists and*

$$\sinh^{-1} x_+ \circledast x^r = \sum_{k=0}^{r}\binom{r}{k}(-1)^k x^{r-k}\left[c_{k+1} - \sum_{i=1}^{k+1}\sum_{j=1}^{i}\frac{a_{k+1,i}b_{i,j}d_j}{i(k+1)}\right], \tag{22}$$

for $r = 0, 1, 2, \ldots,$ *where*

$$c_k = \begin{cases} 0, & k \text{ odd}, \\ -\binom{-1/2}{k/2}\dfrac{1}{k}, & k \text{ even}, \end{cases} \qquad d_k = \begin{cases} 0, & j \text{ even}, \\ \binom{1/2}{(j+1)/2}, & j \text{ odd}. \end{cases}$$

Proof. Putting $[\sinh^{-1} x_+]_n = \sinh^{-1} x_+ \tau_n(x)$, we have

$$\begin{aligned}
[\sinh^{-1} x_+]_n * x^r &= \int_0^n \sinh^{-1} t(x-t)^r\,dt + \int_n^{n+n^{-n}} \sinh^{-1} t(x-t)^r \tau_n(t)\,dt \\
&= \sum_{k=0}^{r} \binom{r}{k} (-1)^k x^{r-k} \int_0^n t^k \sinh^{-1} t\,dt \\
&\qquad + \int_0^{n+n^{-n}} \sinh^{-1} t(x-t)^r \tau_n(t)\,dt \\
&= I_1 + I_2. \qquad (23)
\end{aligned}$$

Replacing x by n in equation (20), we get

$$\int_0^n t^k \sinh^{-1} t\,dt = \frac{n^{k+1}\sinh^{-1} n}{k+1} - \sum_{i=1}^{k+1}\sum_{j=1}^{i} \frac{a_{k+1,i}b_{i,j}}{i(k+1)} n(n^2+1)^{j/2}. \qquad (24)$$

Now,

$$[\sinh^{-1} x]' = (x^2+1)^{-1/2} = x^{-1}\sum_{i=0}^{\infty} \binom{-1/2}{i} x^{-2i}$$

and so

$$\sinh^{-1} x = \ln x - \sum_{i=1}^{\infty} \binom{-1/2}{i} \frac{x^{-2i}}{2i} + \text{ const.} \qquad (25)$$

Hence, for $k = 0, 1, 2, \dots$, we have

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} n^k \sinh^{-1} n &= \begin{cases} 0, & k \text{ odd}, \\ -\binom{-1/2}{k/2}\dfrac{1}{k}, & k \text{ even} \end{cases} \\
&= c_k, \qquad (26)
\end{aligned}$$

for short.

Further,

$$(n^2+1)^{j/2} = n^j \sum_{i=0}^{\infty} \binom{j/2}{i} n^{-2i}$$

and so for $j = 1, 2, \dots$, we have

$$\begin{aligned}
\operatorname*{N-lim}_{n\to\infty} n(n^2+1)^{j/2} &= \begin{cases} 0, & j \text{ even}, \\ \binom{j/2}{(j+1)/2}, & j \text{ odd} \end{cases} \\
&= d_j, \qquad (27)
\end{aligned}$$

for short.

It now follows from equations (24) to (26) that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = \sum_{k=0}^{r} \binom{r}{k} (-1)^k x^{r-k} \left[c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right]. \tag{28}$$

Next, it is easily seen that $I_2 = O(n^{-n})$ and so

$$\lim_{n\to\infty} I_2 = 0. \tag{29}$$

Equation (22) now follows from equations (23), (28) and (29). □

Replacing x by $-x$ in equation (22), we get

Corollary 2. *The neutrix convolution* $\sinh^{-1} x_- \circledast x^r$ *exists and*

$$\sinh^{-1} x_- \circledast x^r = -\sum_{k=0}^{r} \binom{r}{k} x^{r-k} \left[c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right], \tag{30}$$

for $r = 0, 1, 2, \ldots$.

Corollary 3. *The neutrix convolution* $\sinh^{-1} x \circledast x^r$ *exists and*

$$\sinh^{-1} x \circledast x^r = \sum_{k=0}^{r} \binom{r}{k} [(-1)^k - 1] x^{r-k} \left[c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right], \tag{31}$$

for $r = 0, 1, 2, \ldots$.

Proof. We have

$$\sinh^{-1} x \circledast x^r = \sinh^{-1} x_+ \circledast x^r + \sinh^{-1} x_- \circledast x^r$$

and then equation (31) follows from equations (22) and (30). □

Corollary 4. *The neutrix convolution* $\sinh^{-1} x_+ \circledast x_-^r$ *exists and*

$$\begin{aligned} \sinh^{-1} x_+ \circledast x_-^r = {} & \sum_{k=0}^{r} \binom{r}{k} (-1)^{r+k} x^{r-k} \left[c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right] \\ & - \sum_{k=0}^{r} \binom{r}{k} \left[\frac{(-1)^{r+k} x_+^{r+1} \sinh^{-1} x}{k+1} \right. \\ & \left. - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j}}{i(k+1)} x_+^{r-k+1} (x^2+1)^{j/2} \right], \end{aligned} \tag{32}$$

for $r = 0, 1, 2, \ldots$.

Proof. We have

$$
\begin{aligned}
(-1)^r \sinh^{-1} x_+ \circledast x_-^r &= \sinh^{-1} x_+ \circledast x^r - \sinh^{-1} x_+ * x_+^r \\
&= \sum_{k=0}^{r} \binom{r}{k} (-1)^k x^{r-k} \left[c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right] \\
&\quad - \sum_{k=0}^{r} \binom{r}{k} \left[\frac{(-1)^k x_+^{r+1} \sinh^{-1} x}{k+1} \right. \\
&\quad \left. - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j}}{i(k+1)} x_+^{r-k+1} (x^2+1)^{j/2} \right]
\end{aligned}
$$

on using equations (18 and (22) and equation (32) follows. □

Replacing x by $-x$ in equation (32), we get

Corollary 5. *The neutrix convolution* $\sinh^{-1} x_- \circledast x_+^r$ *exists and*

$$
\begin{aligned}
\sinh^{-1} x_- \circledast x_+^r &= \sum_{k=0}^{r} \binom{r}{k} x^{r-k} \left[c_{k+1} - \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j} d_j}{i(k+1)} \right] \\
&\quad - \sum_{k=0}^{r} \binom{r}{k} \left[\frac{(-1)^{r+k} x_-^{r+1} \sinh^{-1} x}{k+1} \right. \\
&\quad \left. + \sum_{i=1}^{k+1} \sum_{j=1}^{i} \frac{a_{k+1,i} b_{i,j}}{i(k+1)} (-1)^{r+j+k} x_-^{r-k+1} (x^2+1)^{j/2} \right], \quad (33)
\end{aligned}
$$

for $r = 0, 1, 2, \ldots$.

For further related results, see [4], [5], [7] and [8].

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