COMMON COUPLED FIXED POINT THEOREMS FOR TWO HYBRID PAIRS OF MAPPINGS SATISFYING AN IMPLICIT RELATION

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ABSTRACT. We establish two common coupled fixed point theorems for two hybrid pairs of mappings satisfying an implicit relation under weak commutativity and $w-$compatibility on a complete metric space, which is not partially ordered. We do not use the condition of continuity of any mapping for finding the coupled coincidence and common coupled fixed point. We improve, extend and generalize several known results.

1. INTRODUCTION AND PRELIMINARIES

Let (X, d) be a metric space and $CB(X)$ be the set of all nonempty closed bounded subsets of X. Let $D(x, A)$ denote the distance from x to $A \subset X$ and H denote the Hausdorff metric induced by d, that is,

$$D(x, \ A) = \inf_{a \in A} d(x, \ a),$$

$$H(A, \ B) = \max \left\{ \sup_{a \in A} D(a, \ B), \ \sup_{b \in B} D(b, \ A) \right\}, \ \text{for all } A, \ B \in CB(X).$$

The study of fixed points for multivalued contractions and non-expansive mappings using the Hausdorff metric was studied by many authors under different conditions. The theory of multivalued mappings has found application in control theory, convex optimization, differential inclusions and economics. There exists considerable literature about fixed point properties for two hybrid pairs of mappings, which have been studied by many authors including ([2], [11], [12], [13], [22], [28], [29], [36]).

Bhaskar and Lakshmikantham [8] introduced the concept of coupled fixed point for single-valued mappings and established some coupled fixed point

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results and found its application in the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ćirić [20] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces and extended the results established in [8]. Many authors focused and proved related remarkable results including ([3], [6], [10], [14], [16], [17], [18], [23], [25], [30], [37]).

Very recently, Samet, Karapinar, Aydi and Rajić [32] claimed that most of the coupled fixed point theorems in the setting of single-valued mappings on ordered metric spaces are consequences of well-known fixed point theorems.

The coupled fixed point theory for multivalued mappings was introduced by Abbas, Ćirić, Damjanović and Khan [1] and obtained coupled coincidence point and common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric space.

On the other hand, at present, coupled fixed point theorems for hybrid pair of mappings were studied by very few authors including ([1], [21]).

In [1], Abbas, Ćirić, Damjanović and Khan introduced the following:

Definition 1. Let X be a nonempty set, $F : X \times X \to 2^X$ (a collection of all nonempty subsets of X) and g be a self-mapping on X. An element $(x, y) \in X \times X$ is called

(1) A coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$.
(2) A coupled coincidence point of hybrid pair $\{F, g\}$ if $gx \in F(x, y)$ and $gy \in F(y, x)$.
(3) A common coupled fixed point of hybrid pair $\{F, g\}$ if $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$.

We denote the set of coupled coincidence points of mappings F and g by $C\{F, g\}$. Note that if $(x, y) \in C\{F, g\}$, then (y, x) is also in $C\{F, g\}$.

Definition 2. Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called $F-$weakly commuting at some point $(x, y) \in X \times X$ if $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$.

Definition 3. Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair $\{F, g\}$ is called $w-$compatible if $gF(x, y) \subseteq F(gx, gy)$ whenever $(x, y) \in C\{F, g\}$.

Lemma 1. [15]. *Let (X, d) be a metric space. Then, for each $a \in X$ and $B \in CB(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b \in B} d(a, b)$.*

Fixed point theorems satisfying an implicit relation for single-valued and multivalued mappings under different conditions have been studied by various authors including ([4], [5], [7], [9], [19], [24], [26], [27], [31], [33], [34], [35], [38]).

In this paper, we establish two common coupled fixed point theorems for two hybrid pairs of mappings satisfying an implicit relation under weak commutativity and $w-$compatibility respectively on a complete metric space, which is not partially ordered. We do not use the condition of continuity of any mapping involved therein for finding the coupled coincidence and common coupled fixed point. We improve, extend and generalize the results of Bhaskar and Lakshmikantham [8], Sedghi, Altun and Shobe [33] and many others results in the existing literature.

2. Implicit relation

Let R^+ be the set of all non-negative real numbers and let Ψ be the set of all continuous functions $\psi : (R^+)^9 \to R$ satisfying the following conditions:

$\psi_1 :$ $\psi(t_1, t_2, \ldots, t_9)$ is non-decreasing in t_1 and non-increasing in $t_2, t_3,$ $\ldots, t_9$.

$\psi_2 :$ There exists $0 < k < 1$ such that for every $u, v, p, q \in R^+$ such that

$$\psi(u, \ v, \ v, \ u, \ u + v, \ q, \ q, \ p, \ p + q) \le 0,$$

or

$$\psi(u, \ v, \ u, \ v, \ u + v, \ q, \ p, \ q, \ p + q) \le 0,$$

then $\max\{u, p\} \le k \max\{v, q\}$.

$\psi_3 :$ For all $u, v > 0$,

$$\psi(u, \ u, \ 0, \ 0, \ 2u, \ v, \ 0, \ 0, \ 2v) > 0.$$

Example 1. Let $\psi(t_1, t_2, \ldots, t_9) = t_1 - h \max\{t_2, t_3, t_4, \frac{t_5}{2}, t_6, t_7, t_8, \frac{t_9}{2}\}$ where $0 < h < 1$.

(ψ_1) Obvious. (ψ_2) Let $\max\{u, p\} > 0$ and $\psi(u, v, v, u, u + v, q, q, p, p + q) = u - h \max\{u, v, p, q\} \le 0$. Thus $u \le h \max\{\max\{u, p\}, \max\{v, q\}\}$. Similarly $p \le h \max\{\max\{u, p\}, \max\{v, q\}\}$. Thus $\max\{u, p\} \le h \max\{\max\{u, p\}, \max\{v, q\}\}$. Now, if $\max\{u, p\} \ge \max\{v, q\}$, then $\max\{u, p\} \le h \max\{u, p\} < \max\{u, p\}$, which is a contradiction. Thus $\max\{u, p\} < \max\{v, q\}$ and $\max\{u, p\} \le h \max\{v, q\}$. Similarly, let $\max\{u, p\} > 0$ and $\psi(u, v, u, v, u+v, q, p, q, p+q) = u - h \max\{u, v, p, q\} \le 0$, then we have $\max\{u, p\} \le h \max\{v, q\}$. Thus (ψ_2) is satisfying with $k = h < 1$. If $\max\{u, p\} = 0$, then $\max\{u, p\} \le k \max\{v, q\}$. (ψ_3) $\psi(u, u, 0, 0, 2u, v, 0, 0, 2v) = u - h \max\{u, v\} = \max\{u - hu, u - hv\} = \max\{u(1 - h), u - hv\} > 0$.

Therefore $\psi \in \Psi$.

Example 2. Let $\psi(t_1, t_2, \ldots, t_9) = t_1 - \alpha \max\{t_2, t_3, t_4, t_6, t_7, t_8\} - \beta \max\{t_5, t_9\}$ where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$.

(ψ_1) Obvious. (ψ_2) Let $\max\{u, p\} > 0$ and $\psi(u, v, v, u, u+v, q, q, p, p+q) = u - \alpha \max\{v, u, q, p\} - \beta \max\{u+v, p+q\} \leq 0$, then $u \leq \alpha \max\{\max\{u, p\}, \max\{v, q\}\} + \beta[\max\{u, p\} + \max\{v, q\}]$, it follows that $u \leq \max\{(\alpha + \beta) \max\{u, p\} + \beta \max\{v, q\}, (\alpha + \beta) \max\{v, q\} + \beta \max\{u, p\}\}$. Similarly $p \leq \max\{(\alpha + \beta) \max\{u, p\} + \beta \max\{v, q\}, (\alpha + \beta) \max\{v, q\} + \beta \max\{u, p\}\}$. Thus $\max\{u, p\} \leq \max\{(\alpha + \beta) \max\{u, p\} + \beta \max\{v, q\}, (\alpha + \beta) \max\{v, q\} + \beta \max\{u, p\}\}$. Now, if $\max\{u, p\} \geq \max\{v, q\}$, then $\max\{u, p\} \leq (\alpha + 2\beta) \max\{u, p\} < \max\{u, p\}$, which is a contradiction. Thus $\max\{u, p\} < \max\{v, q\}$ and so $\max\{u, p\} \leq (\alpha + 2\beta) \max\{v, q\}$. Similarly, let $\max\{u, p\} > 0$ and $\psi(u, v, u, v, u+v, q, p, q, p+q) = u - \alpha \max\{v, u, q, p\} - \beta \max\{u+v, p+q\} \leq 0$, then we have $\max\{u, p\} \leq (\alpha + 2\beta) \max\{v, q\}$. Thus (ψ_2) is satisfying with $k = (\alpha + 2\beta) < 1$. If $\max\{u, p\} = 0$, then $\max\{u, p\} \leq k \max\{v, q\}$. (ψ_3) $\psi(u, u, 0, 0, 2u, v, 0, 0, 2v) = u - \alpha \max\{u, v\} - \beta \max\{2u, 2v\} = u - \alpha \max\{u, v\} - 2\beta \max\{u, v\} = u - (\alpha + 2\beta) \max\{u, v\} = \max\{u(1 - \alpha - 2\beta), u - (\alpha + 2\beta)v\} > 0$.

Therefore $\psi \in \Psi$.

Example 3. $\psi(t_1, t_2, \ldots, t_9) = t_1 - a \max\{t_2, t_6\} - b \max\{t_3 + t_4, t_7 + t_8\} - c \max\{t_5, t_9\}$, where $a, b, c \in [0, 1)$ and $a + 2b + 2c < 1$.

(ψ_1) Obvious. (ψ_2) Let $\max\{u, p\} > 0$ and $\psi(u, v, v, u, u+v, q, q, p, p+q) = u - a \max\{v, q\} - b \max\{u+v, p+q\} - c \max\{u+v, p+q\} \leq 0$, then $u \leq a \max\{v, q\} + b[\max\{u, p\} + \max\{v, q\}] + c[\max\{u, p\} + \max\{v, q\}]$. Similarly $p \leq a \max\{v, q\} + b[\max\{u, p\} + \max\{v, q\}] + c[\max\{u, p\} + \max\{v, q\}]$. Thus $\max\{u, p\} \leq a \max\{v, q\} + b[\max\{u, p\} + \max\{v, q\}] + c[\max\{u, p\} + \max\{v, q\}]$. Now, if $\max\{u, p\} \geq \max\{v, q\}$, then $\max\{u, p\} \leq (a + 2b + 2c) \max\{u, p\} < \max\{u, p\}$, which is a contradiction. Thus $\max\{u, p\} < \max\{v, q\}$ and $\max\{u, p\} \leq (a + 2b + 2c) \max\{v, q\}$. Similarly, let $\max\{u, p\} > 0$ and $\psi(u, v, u, v, u+v, q, p, q, p+q) = u - a \max\{v, q\} - b \max\{u+v, p+q\} - c \max\{u+v, p+q\} \leq 0$, then $\max\{u, p\} < (a + 2b + 2c) \max\{v, q\}$. Thus (ψ_2) is satisfying with $k = a + 2b + 2c < 1$. If $\max\{u, p\} = 0$, then $\max\{u, p\} \leq k \max\{v, q\}$. (ψ_3) $\psi(u, u, 0, 0, 2u, v, 0, 0, 2v) = u - a \max\{u, v\} - c \max\{2u, 2v\} = u - a \max\{u, v\} - 2c \max\{u, v\} = u - (a + 2c) \max\{u, v\} = \max\{u(1 - a - 2c), u - (a + 2c)v\} > 0$.

Therefore $\psi \in \Psi$.

3. Main results

Theorem 1. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $f, g : X \to X$ be mappings satisfying*

(1.1) $F(X \times X) \subseteq g(X)$, $G(X \times X) \subseteq f(X)$,

(1.2) *for all* x, y, u, $v \in X$ *and* $\psi \in \Psi$,

$$\psi \begin{pmatrix} H(F(x, \ y), \ G(u, \ v)), \\ d(fx, \ gu), \ D(fx, \ F(x, \ y)), \ D(gu, \ G(u, \ v)), \\ D(fx, \ G(u, \ v)) + D(gu, \ F(x, \ y)), \\ d(fy, \ gv), \ D(fy, \ F(y, \ x)), \ D(gv, \ G(v, \ u)), \\ D(fy, \ G(v, \ u)) + D(gv, \ F(y, \ x)) \end{pmatrix} \leq 0,$$

(1.3) $f(X)$ *and* $g(X)$ *are closed subsets of* X, *then*

(a) F *and* f *have a coupled coincidence point,*

(b) G *and* g *have a coupled coincidence point,*

(c) F *and* f *have a common coupled fixed point, if* f *is* $F-$*weakly commuting at* (x, y) *and* $f^2x = fx$ *and* $f^2y = fy$ *for* $(x, y) \in C\{F, f\}$,

(d) G *and* g *have a common coupled fixed point, if* g *is* $G-$*weakly commuting at* $(\widetilde{x}, \widetilde{y})$ *and* $g^2\widetilde{x} = g\widetilde{x}$ *and* $g^2\widetilde{y} = g\widetilde{y}$ *for* $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$,

(e) F, G, f, g *have common coupled fixed point provided that both* (c) *and* (d) *are true.*

Proof. Let x_0, $y_0 \in X$ be arbitrary. Choose $u_1 = gx_1 \in F(x_0, y_0)$ and $v_1 = gy_1 \in F(y_0, x_0)$, as $F(X \times X) \subseteq g(X)$. Since $F, G : X \times X \to CB(X)$, therefore by Lemma 1, there exist $u_2 \in G(x_1, y_1)$ and $v_2 \in G(y_1, x_1)$ such that

$$d(u_1, \ u_2) \leq H(F(x_0, \ y_0), \ G(x_1, \ y_1)),$$
$$d(v_1, \ v_2) \leq H(F(y_0, \ x_0), \ G(y_1, \ x_1)).$$

Since $G(X \times X) \subseteq f(X)$, there exist $x_2, y_2 \in X$ such that $u_2 = fx_2 \in G(x_1, y_1)$ and $v_2 = fy_2 \in G(y_1, x_1)$. Then we choose $u_3 \in F(x_2, y_2)$ and $v_3 \in F(y_2, x_2)$ such that

$$d(u_2, \ u_3) \leq H(G(x_1, \ y_1), \ F(x_2, \ y_2)),$$
$$d(v_2, \ v_3) \leq H(G(y_1, \ x_1), \ F(y_2, \ x_2)).$$

Continuing this process, we obtain sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \geq 0$, we have

$$u_{2n} = fx_{2n} \in G(x_{2n-1}, \ y_{2n-1}), \ u_{2n+1} = gx_{2n+1} \in F(x_{2n}, \ y_{2n}),$$
$$v_{2n} = fy_{2n} \in G(y_{2n-1}, \ x_{2n-1}), \ v_{2n+1} = gy_{2n+1} \in F(y_{2n}, \ x_{2n}),$$

and

$$d(u_{2n-1}, \ u_{2n}) \leq H(F(x_{2n-2}, \ y_{2n-2}), \ G(x_{2n-1}, \ y_{2n-1})),$$
$$d(u_{2n}, \ u_{2n+1}) \leq H(G(x_{2n-1}, \ y_{2n-1}), \ F(x_{2n}, \ y_{2n})),$$
$$d(v_{2n-1}, \ v_{2n}) \leq H(F(y_{2n-2}, \ x_{2n-2}), \ G(y_{2n-1}, \ x_{2n-1})),$$
$$d(v_{2n}, \ v_{2n+1}) \leq H(G(y_{2n-1}, \ x_{2n-1}), \ F(y_{2n}, \ x_{2n})).$$

Then by condition (1.2), we get

$$\psi \begin{pmatrix} H(F(x_{2n},\ y_{2n}),\ G(x_{2n-1},\ y_{2n-1})), \\ d(fx_{2n},\ gx_{2n-1}),\ D(fx_{2n},\ F(x_{2n},\ y_{2n})), \\ D(gx_{2n-1},\ G(x_{2n-1},\ y_{2n-1})), \\ D(fx_{2n},\ G(x_{2n-1},\ y_{2n-1})) + D(gx_{2n-1},\ F(x_{2n},\ y_{2n})), \\ d(fy_{2n},\ gy_{2n-1}),\ D(fy_{2n},\ F(y_{2n},\ x_{2n})), \\ D(gy_{2n-1},\ G(y_{2n-1},\ x_{2n-1})), \\ D(fy_{2n},\ G(y_{2n-1},\ x_{2n-1})) + D(gy_{2n-1},\ F(y_{2n},\ x_{2n})) \end{pmatrix} \leq 0.$$

Using (ψ_1), we get

$$\psi \begin{pmatrix} d(u_{2n+1},\ u_{2n}), \\ d(u_{2n},\ u_{2n-1}),\ d(u_{2n},\ u_{2n+1}),\ d(u_{2n-1},\ u_{2n}), \\ 0 + d(u_{2n-1},\ u_{2n+1}), \\ d(v_{2n},\ v_{2n-1}),\ d(v_{2n},\ v_{2n+1}),\ d(v_{2n-1},\ v_{2n}), \\ 0 + d(v_{2n-1},\ v_{2n+1}) \end{pmatrix} \leq 0,$$

which implies that

$$\psi \begin{pmatrix} d(u_{2n+1},\ u_{2n}), \\ d(u_{2n},\ u_{2n-1}),\ d(u_{2n},\ u_{2n+1}),\ d(u_{2n-1},\ u_{2n}), \\ d(u_{2n-1},\ u_{2n}) + d(u_{2n},\ u_{2n+1}), \\ d(v_{2n},\ v_{2n-1}),\ d(v_{2n},\ v_{2n+1}),\ d(v_{2n-1},\ v_{2n}), \\ d(v_{2n-1},\ v_{2n}) + d(v_{2n},\ v_{2n+1}) \end{pmatrix} \leq 0.$$

By (ψ_2), we get

$$\max\{d(u_{2n+1},\ u_{2n}),\ d(v_{2n+1},\ v_{2n})\}$$
$$\leq k \max\{d(u_{2n},\ u_{2n-1}),\ d(v_{2n},\ v_{2n-1})\}.$$

Similarly, we can obtain

$$\max\{d(u_{2n},\ u_{2n-1}),\ d(v_{2n},\ v_{2n-1})\}$$
$$\leq k \max\{d(u_{2n-1},\ u_{2n-2}),\ d(v_{2n-1},\ v_{2n-2})\}.$$

Thus, we have for all $n \in \mathbb{N}$,

$$\max\{d(u_n,\ u_{n+1}),\ d(v_n,\ v_{n+1})\} \leq k \max\{d(u_{n-1},\ u_n),\ d(v_{n-1},\ v_n)\}$$
$$\leq k^n \max\{d(u_0,\ u_1),\ d(v_0,\ v_1)\}$$
$$\leq k^n \delta.$$

Thus

$$\max\{d(u_n,\ u_{n+1}),\ d(v_n,\ v_{n+1})\} \leq k^n \delta, \qquad (1.4)$$

where

$$\delta = \max\{d(u_0,\ u_1),\ d(v_0,\ v_1)\}.$$

Thus, for $m, n \in \mathbb{N}$ with $m > n$, by triangle inequality and (1.4), we get

$$\max\{d(u_n, u_{m+n}), d(v_n, v_{m+n})\} \leq \max\{d(u_n, u_{n+1}), d(v_n, v_{n+1})\}$$
$$+ \max\{d(u_{n+1}, u_{n+2}), d(v_{n+1}, v_{n+2})\}$$
$$+ \cdots + \max\{d(u_{m+n-1}, u_{m+n}), d(v_{m+n-1}, v_{m+n})\}$$
$$\leq k^n\delta + k^{n+1}\delta + \cdots + k^{n+m-1}\delta$$
$$\leq \sum_{i=n}^{n+m-1} k^i\delta,$$

which shows that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in X. Since X is complete, there exist $u, v \in X$ such that

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = u, \qquad (1.5)$$
$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} fy_{2n} = \lim_{n\to\infty} gy_{2n+1} = v.$$

Since $f(X)$ and $g(X)$ are closed subsets of X, then there exist $x, y, \widetilde{x}, \widetilde{y} \in X$,

$$u = fx = g\widetilde{x} \text{ and } v = fy = g\widetilde{y}. \qquad (1.6)$$

Now, since $fx_{2n} \in G(x_{2n-1}, y_{2n-1})$ and $fy_{2n} \in G(y_{2n-1}, x_{2n-1})$, therefore by using condition (1.2), we get

$$\psi\begin{pmatrix} H(F(x, y), G(x_{2n-1}, y_{2n-1})), \\ d(fx, gx_{2n-1}), D(fx, F(x, y)), D(gx_{2n-1}, G(x_{2n-1}, y_{2n-1})), \\ D(fx, G(x_{2n-1}, y_{2n-1})) + D(gx_{2n-1}, F(x, y)), \\ d(fy, gy_{2n-1}), D(fy, F(y, x)), D(gy_{2n-1}, G(y_{2n-1}, x_{2n-1})), \\ D(fy, G(y_{2n-1}, x_{2n-1})) + D(gy_{2n-1}, F(y, x)) \end{pmatrix} \leq 0,$$

which implies, by (ψ_1), that

$$\psi\begin{pmatrix} D(F(x, y), fx_{2n}), \\ d(fx, gx_{2n-1}), D(fx, F(x, y)), d(gx_{2n-1}, fx_{2n}), \\ d(fx, fx_{2n}) + D(gx_{2n-1}, F(x, y)), \\ d(fy, gy_{2n-1}), D(fy, F(y, x)), d(gy_{2n-1}, fy_{2n}), \\ d(fy, fy_{2n}) + D(gy_{2n-1}, F(y, x)) \end{pmatrix} \leq 0.$$

Letting $n \to \infty$ in the above inequality, by using the continuity of ψ, (1.5) and (1.6), we obtain

$$\psi\begin{pmatrix} D(F(x, y), fx), \\ 0, D(fx, F(x, y)), 0, 0 + D(fx, F(x, y)), \\ 0, D(fy, F(y, x)), 0, 0 + D(fy, F(y, x)) \end{pmatrix} \leq 0.$$

Thus, by (ψ_2), we obtain

$$D(fx, F(x, y)) = 0 \text{ and } D(fy, F(y, x)) = 0,$$

which implies that

$$fx \in F(x, \ y) \text{ and } fy \in F(y, \ x),$$

that is, $(x, \ y)$ is a coupled coincidence point of F and f. This proves (a). Again, since $gx_{2n+1} \in F(x_{2n}, \ y_{2n})$ and $gy_{2n+1} \in F(y_{2n}, \ x_{2n})$, therefore by using condition (1.2), we get

$$\psi \begin{pmatrix} H(F(x_{2n}, \ y_{2n}), \ G(\widetilde{x}, \ \widetilde{y})), \\ d(fx_{2n}, \ g\widetilde{x}), \ D(fx_{2n}, \ F(x_{2n}, \ y_{2n})), \ D(g\widetilde{x}, \ G(\widetilde{x}, \ \widetilde{y})), \\ D(fx_{2n}, \ G(\widetilde{x}, \ \widetilde{y})) + D(g\widetilde{x}, \ F(x_{2n}, \ y_{2n})), \\ d(fy_{2n}, \ g\widetilde{y}), \ D(fy_{2n}, \ F(y_{2n}, \ x_{2n})), \ D(g\widetilde{y}, \ G(\widetilde{y}, \ \widetilde{x})), \\ D(fy_{2n}, \ G(\widetilde{y}, \ \widetilde{x})) + D(g\widetilde{y}, \ F(y_{2n}, \ x_{2n})) \end{pmatrix} \leq 0,$$

which implies, by (ψ_1), that

$$\psi \begin{pmatrix} D(gx_{2n+1}, \ G(\widetilde{x}, \ \widetilde{y})), \\ d(fx_{2n}, \ g\widetilde{x}), \ d(fx_{2n}, \ gx_{2n+1}), \ D(g\widetilde{x}, \ G(\widetilde{x}, \ \widetilde{y})), \\ D(fx_{2n}, \ G(\widetilde{x}, \ \widetilde{y})) + d(g\widetilde{x}, \ gx_{2n+1}), \\ d(fy_{2n}, \ g\widetilde{y}), \ d(fy_{2n}, \ gy_{2n+1}), \ D(g\widetilde{y}, \ G(\widetilde{y}, \ \widetilde{x})), \\ D(fy_{2n}, \ G(\widetilde{y}, \ \widetilde{x})) + d(g\widetilde{y}, \ gy_{2n+1}) \end{pmatrix} \leq 0.$$

Letting $n \to \infty$ in the above inequality, by using the continuity of ψ, (1.5) and (1.6), we obtain

$$\psi \begin{pmatrix} D(g\widetilde{x}, \ G(\widetilde{x}, \ \widetilde{y})), \\ 0, \ 0, \ D(g\widetilde{x}, \ G(\widetilde{x}, \ \widetilde{y})), \ D(g\widetilde{x}, \ G(\widetilde{x}, \ \widetilde{y})) + 0, \\ 0, \ 0, \ D(g\widetilde{y}, \ G(\widetilde{y}, \ \widetilde{x})), \ D(g\widetilde{y}, \ G(\widetilde{y}, \ \widetilde{x})) + 0 \end{pmatrix} \leq 0.$$

Thus, by (ψ_2), we obtain

$$D(g\widetilde{x}, \ G(\widetilde{x}, \ \widetilde{y})) = 0 \text{ and } D(g\widetilde{y}, \ G(\widetilde{y}, \ \widetilde{x})) = 0,$$

which implies that

$$g\widetilde{x} \in G(\widetilde{x}, \ \widetilde{y}) \text{ and } g\widetilde{y} \in G(\widetilde{y}, \ \widetilde{x}).$$

that is, $(\widetilde{x}, \ \widetilde{y})$ is a coupled coincidence point of G and g. This proves (b).

Furthermore, from condition (c), we have f is $F-$weakly commuting at (x, y), that is, $f^2x \in F(fx, fy)$, $f^2y \in F(fy, fx)$ and $f^2x = fx$, $f^2y = fy$. Thus $fx = f^2x \in F(fx, fy)$ and $fy = f^2y \in F(fy, fx)$, that is, $u = fu \in F(u, v)$ and $v = fv \in F(v, u)$. This proves (c). A similar argument proves (d). Then (e) holds immediately. $\qquad\square$

Put $f = g$ in the Theorem 1, we get the following result:

Corollary 2. *Let $(X, \ d)$ be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying*
(2.1) $F(X \times X) \subseteq g(X)$, $G(X \times X) \subseteq g(X)$,

(2.2) *for all x, y, u, $v \in X$ and $\psi \in \Psi$,*

$$\psi \begin{pmatrix} H(F(x,\ y),\ G(u,\ v)), \\ d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ D(gx,\ G(u,\ v)) + D(gu,\ F(x,\ y)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ D(gy,\ G(v,\ u)) + D(gv,\ F(y,\ x)) \end{pmatrix} \leq 0.$$

(2.3) *$g(X)$ is a closed subset of X, then*

(a) F and g have a coupled coincidence point,

(b) G and g have a coupled coincidence point,

(c) F and g have a common coupled fixed point, if g is $F-$weakly commuting at (x, y) and $g^2 x = gx$ and $g^2 y = gy$ for $(x, y) \in C\{F, g\}$,

(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x}, \widetilde{y})$ and $g^2 \widetilde{x} = g\widetilde{x}$ and $g^2 \widetilde{y} = g\widetilde{y}$ for $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$,

(e) F, G, g have common coupled fixed point provided that both (c) and (d) are true.

Put $F = G$ and $f = g$ in the Theorem 1, we get the following result:

Corollary 3. *Let (X, d) be a complete metric space. Assume $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying*

(3.1) $F(X \times X) \subseteq g(X)$,

(3.2) *for all x, y, u, $v \in X$ and $\psi \in \Psi$,*

$$\psi \begin{pmatrix} H(F(x,\ y),\ F(u,\ v)), \\ d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ D(gx,\ F(u,\ v)) + D(gu,\ F(x,\ y)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ D(gy,\ F(v,\ u)) + D(gv,\ F(y,\ x)) \end{pmatrix} \leq 0.$$

If (2.3) holds, then

(a) F and g have a coupled coincidence point,

(b) F and g have a common coupled fixed point, if g is $F-$weakly commuting at (x, y) and $g^2 x = gx$ and $g^2 y = gy$ for $(x, y) \in C\{F, g\}$.

Examples 1-3 and Theorem 1 imply the following:

Corollary 4. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $f, g : X \to X$ be mappings satisfying (1.1) and*

(4.1) *for all x, y, u, $v \in X$, where $0 < h < 1$,*

$$H(F(x,\ y),\ G(u,\ v))$$
$$\leq\ h \max \left\{ \begin{array}{c} d(fx,\ gu),\ D(fx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ d(fy,\ gv),\ D(fy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ \frac{D(fx,\ G(u,\ v)) + D(gu,\ F(x,\ y))}{2},\ \frac{D(fy,\ G(v,\ u)) + D(gv,\ F(y,\ x))}{2} \end{array} \right\},$$

or for all x, y, u, $v \in X$, where α, $\beta \geq 0$ and $\alpha + 2\beta < 1$,

$$H(F(x, y), G(u, v))$$

$$\leq \alpha \max \left\{ \begin{array}{l} d(fx, gu), \ D(fx, F(x, y)), \ D(gu, G(u, v)), \\ d(fy, gv), \ D(fy, F(y, x)), \ D(gv, G(v, u)) \end{array} \right\}$$

$$+ \beta \max \left\{ \begin{array}{l} D(fx, G(u, v)) + D(gu, F(x, y)), \\ D(fy, G(v, u)) + D(gv, F(y, x)) \end{array} \right\},$$

or for all x, y, u, $v \in X$, where a, b, $c \in [0, 1)$ and $a + 2b + 2c < 1$,

$$H(F(x, y), G(u, v))$$

$$\leq a \max \{d(fx, gu), \ d(fy, gv)\}$$

$$+ b \max \left\{ \begin{array}{l} D(fx, F(x, y)) + D(gu, G(u, v)), \\ D(fy, F(y, x)) + D(gv, G(v, u)) \end{array} \right\}$$

$$+ c \max \left\{ \begin{array}{l} D(fx, G(u, v)) + D(gu, F(x, y)), \\ D(fy, G(v, u)) + D(gv, F(y, x)) \end{array} \right\}.$$

If (1.3) *holds, then*

(a) *F and f have a coupled coincidence point,*

(b) *G and g have a coupled coincidence point,*

(c) *F and f have a common coupled fixed point, if f is $F-$weakly commuting at (x, y) and $f^2 x = fx$ and $f^2 y = fy$ for $(x, y) \in C\{F, f\}$,*

(d) *G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x}, \widetilde{y})$ and $g^2 \widetilde{x} = g\widetilde{x}$ and $g^2 \widetilde{y} = g\widetilde{y}$ for $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$,*

(e) *F, G, f, g have common coupled fixed point provided that both (c) and (d) are true.*

Examples 1-3 and Corollary 2 imply the following:

Corollary 5. *Let (X, d) be a complete metric space. Assume F, $G : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying* (2.1) *and*

(5.1) *for all x, y, u, $v \in X$, where $0 < h < 1$,*

$$H(F(x, y), G(u, v))$$

$$\leq h \max \left\{ \begin{array}{l} d(gx, gu), \ D(gx, F(x, y)), \ D(gu, G(u, v)), \\ d(gy, gv), \ D(gy, F(y, x)), \ D(gv, G(v, u)), \\ \frac{D(gx, G(u, v)) + D(gu, F(x, y))}{2}, \ \frac{D(gy, G(v, u)) + D(gv, F(y, x))}{2} \end{array} \right\},$$

or for all x, y, u, $v \in X$, where α, $\beta \geq 0$ and $\alpha + 2\beta < 1$,

$$H(F(x, y), G(u, v))$$

$$\leq \alpha \max \left\{ \begin{array}{l} d(gx, gu), \ D(gx, F(x, y)), \ D(gu, G(u, v)), \\ d(gy, gv), \ D(gy, F(y, x)), \ D(gv, G(v, u)) \end{array} \right\}$$

$$+ \beta \max \left\{ \begin{array}{l} D(gx, G(u, v)) + D(gu, F(x, y)), \\ D(gy, G(v, u)) + D(gv, F(y, x)) \end{array} \right\},$$

or for all x, y, u, v ∈ X, where a, b, c ∈ [0, 1) and a + 2b + 2c < 1,

$$H(F(x, \ y), \ G(u, \ v))$$

$$\leq \ a \max \{ d(gx, \ gu), \ d(gy, \ gv) \}$$

$$+ b \max \left\{ \begin{array}{c} D(gx, \ F(x, \ y)) + D(gu, \ G(u, \ v)), \\ D(gy, \ F(y, \ x)) + D(gv, \ G(v, \ u)) \end{array} \right\}$$

$$+ c \max \left\{ \begin{array}{c} D(gx, \ G(u, \ v)) + D(gu, \ F(x, \ y)), \\ D(gy, \ G(v, \ u)) + D(gv, \ F(y, \ x)) \end{array} \right\}.$$

If (2.3) holds, then

(a) *F and g have a coupled coincidence point,*

(b) *G and g have a coupled coincidence point,*

(c) *F and g have a common coupled fixed point, if g is F−weakly commuting at (x, y) and $g^2 x = gx$ and $g^2 y = gy$ for $(x, \ y) \in C\{F, \ g\}$,*

(d) *G and g have a common coupled fixed point, if g is G−weakly commuting at $(\widetilde{x}, \ \widetilde{y})$ and $g^2 \widetilde{x} = g\widetilde{x}$ and $g^2 \widetilde{y} = g\widetilde{y}$ for $(\widetilde{x}, \ \widetilde{y}) \in C\{G, \ g\}$,*

(e) *F, G, g have common coupled fixed point provided that both (c) and (d) are true.*

Examples 1-3 and Corollary 3 imply the following:

Corollary 6. *Let (X, d) be a complete metric space. Assume $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying (3.1) and*

(6.1) *for all x, y, u, v ∈ X, where 0 < h < 1,*

$$H(F(x, \ y), \ F(u, \ v))$$

$$\leq \ h \max \left\{ \begin{array}{c} d(gx, \ gu), \ D(gx, \ F(x, \ y)), \ D(gu, \ F(u, \ v)), \\ d(gy, \ gv), \ D(gy, \ F(y, \ x)), \ D(gv, \ F(v, \ u)), \\ \frac{D(gx, \ F(u, \ v)) + D(gu, \ F(x, \ y))}{2}, \ \frac{D(gy, \ F(v, \ u)) + D(gv, \ F(y, \ x))}{2} \end{array} \right\},$$

or for all x, y, u, v ∈ X, where α, β ≥ 0 and α + 2β < 1,

$$H(F(x, \ y), \ F(u, \ v))$$

$$\leq \ \alpha \max \left\{ \begin{array}{c} d(gx, \ gu), \ D(gx, \ F(x, \ y)), \ D(gu, \ F(u, \ v)), \\ d(gy, \ gv), \ D(gy, \ F(y, \ x)), \ D(gv, \ F(v, \ u)) \end{array} \right\}$$

$$+ \beta \max \left\{ \begin{array}{c} D(gx, \ F(u, \ v)) + D(gu, \ F(x, \ y)), \\ D(gy, \ F(v, \ u)) + D(gv, \ F(y, \ x)) \end{array} \right\},$$

or for all x, y, u, v ∈ X, where a, b, c ∈ [0, 1) and a + 2b + 2c < 1,

$$H(F(x, \ y), \ F(u, \ v))$$

$$\leq \ a \max \{ d(gx, \ gu), \ d(gy, \ gv) \}$$

$$+ b \max \left\{ \begin{array}{c} D(gx, \ F(x, \ y)) + D(gu, \ F(u, \ v)), \\ D(gy, \ F(y, \ x)) + D(gv, \ F(v, \ u)) \end{array} \right\}$$

$$+c \max \left\{ \begin{array}{c} D(gx, \ F(u, \ v)) + D(gu, \ F(x, \ y)), \\ D(gy, \ F(v, \ u)) + D(gv, \ F(y, \ x)) \end{array} \right\}.$$

If (2.3) holds, then

(a) F and g have a coupled coincidence point,

(c) F and g have a common coupled fixed point, if g is $F-$weakly commuting at (x, y) and $g^2 x = gx$ and $g^2 y = gy$ for $(x, y) \in C\{F, g\}$.

Theorem 7. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $f, g : X \to X$ be mappings satisfying (1.1), (1.2) and*

(7.1) $\{F, f\}$ *and* $\{G, g\}$ *are* $w-$*compatible,*

(7.2) $f(X)$ *or* $g(X)$ *is a closed subset of* X,

then F, G, f, g have a common coupled fixed point.

Proof. We can prove like Theorem 1 that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in X. Since X is complete, there exist $u, v \in X$ satisfying (1.5). Suppose that $f(X)$ is a closed subset of X, then there exist $x, y \in X$, we have

$$u = fx \text{ and } v = fy. \tag{7.3}$$

As in Theorem 1, we can prove that

$$fx \in F(x, \ y) \text{ and } fy \in F(y, \ x), \tag{7.4}$$

that is, (x, y) is a coupled coincidence point of F and f. Hence $(x, y) \in C\{F, f\}$. From $w-$compatibility of $\{F, f\}$, we have $fF(x, y) \subseteq F(fx, fy)$, hence $f^2 x \in F(fx, fy)$ and $f^2 y \in F(fy, fx)$, that is, $fu \in F(u, v)$ and $fv \in F(v, u)$. Now, by condition (1.2), we get

$$\psi \left(\begin{array}{c} H(F(u, \ v), \ G(x_{2n-1}, \ y_{2n-1})), \\ d(fu, \ gx_{2n-1}), \ D(fu, \ F(u, \ v)), \ D(gx_{2n-1}, \ G(x_{2n-1}, \ y_{2n-1})), \\ D(fu, \ G(x_{2n-1}, \ y_{2n-1})) + D(gx_{2n-1}, \ F(u, \ v)), \\ d(fv, \ gy_{2n-1}), \ D(fv, \ F(v, \ u)), \ D(gy_{2n-1}, \ G(y_{2n-1}, \ x_{2n-1})), \\ D(fv, \ G(y_{2n-1}, \ x_{2n-1})) + D(gy_{2n-1}, \ F(v, \ u)) \end{array} \right) \leq 0.$$

From (ψ_1) and by triangle inequality, we have

$$\psi \left(\begin{array}{c} d(fu, \ u_{2n}), \\ d(fu, \ u_{2n-1}), \ 0, \ d(u_{2n-1}, \ u_{2n}), \ d(fu, \ u_{2n}) + d(u_{2n-1}, \ fu), \\ d(fv, \ v_{2n-1}), \ 0, \ d(v_{2n-1}, \ v_{2n}), \ d(fv, \ v_{2n}) + d(v_{2n-1}, \ fv) \end{array} \right) \leq 0.$$

Letting $n \to \infty$ in the above inequality, we get

$$\psi \left(\begin{array}{c} d(fu, \ u), \\ d(fu, \ u), \ 0, \ 0, \ 2d(fu, \ u), \\ d(fv, \ v), \ 0, \ 0, \ 2d(fv, \ v) \end{array} \right) \leq 0.$$

Hence, by (ψ_3), we have $d(fu, u) = d(fv, v) = 0$. Thus

$$u = fu \in F(u, \ v) \text{ and } v = fv \in F(v, \ u).$$

Since $F(X \times X) \subseteq g(X)$, then there exist $\widetilde{x}, \widetilde{y} \in X$ such that $g\widetilde{x} = u = fu \in F(u, v)$ and $g\widetilde{y} = v = fv \in F(v, u)$. Again, by condition (1.2), we get

$$\psi \begin{pmatrix} H(F(u, v), G(\widetilde{x}, \widetilde{y})), \\ d(fu, g\widetilde{x}), D(fu, F(u, v)), D(g\widetilde{x}, G(\widetilde{x}, \widetilde{y})), \\ D(fu, G(\widetilde{x}, \widetilde{y})) + D(g\widetilde{x}, F(u, v)), \\ d(fv, g\widetilde{y}), D(fv, F(v, u)), D(g\widetilde{y}, G(\widetilde{y}, \widetilde{x})), \\ D(fv, G(\widetilde{y}, \widetilde{x})) + D(g\widetilde{y}, F(v, u)) \end{pmatrix} \leq 0.$$

From (ψ_1), we have

$$\psi \begin{pmatrix} D(u, G(\widetilde{x}, \widetilde{y})), \\ 0, 0, D(u, G(\widetilde{x}, \widetilde{y})), D(u, G(\widetilde{x}, \widetilde{y})) + 0, \\ 0, 0, D(v, G(\widetilde{y}, \widetilde{x})), D(v, G(\widetilde{y}, \widetilde{x})) + 0 \end{pmatrix} \leq 0.$$

Hence, by (ψ_2), we have $D(u, G(\widetilde{x}, \widetilde{y})) = D(v, G(\widetilde{y}, \widetilde{x})) = 0$. Thus

$$u = g\widetilde{x} \in G(\widetilde{x}, \widetilde{y}) \text{ and } v = g\widetilde{y} \in G(\widetilde{y}, \widetilde{x}),$$

that is, $(\widetilde{x}, \widetilde{y})$ is a coupled coincidence point of G and g. Hence $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$. From $w-$compatibility of $\{G, g\}$, we have $gG(\widetilde{x}, \widetilde{y}) \subseteq G(g\widetilde{x}, g\widetilde{y})$, hence $g^2\widetilde{x} \in G(g\widetilde{x}, g\widetilde{y})$ and $g^2\widetilde{y} \in G(g\widetilde{y}, g\widetilde{x})$, that is, $gu \in G(u, v)$ and $gv \in G(v, u)$. Now, by condition (1.2), we get

$$\psi \begin{pmatrix} H(F(u, v), G(u, v)), \\ d(fu, gu), D(fu, F(u, v)), D(gu, G(u, v)), \\ D(fu, G(u, v)) + D(gu, F(u, v)), \\ d(fv, gv), D(fv, F(v, u)), D(gv, G(v, u)), \\ D(fv, G(v, u)) + D(gv, F(v, u)) \end{pmatrix} \leq 0.$$

From (ψ_2) and triangle inequality, we have

$$\psi \begin{pmatrix} d(u, gu), \\ d(u, gu), 0, 0, 2d(u, gu), \\ d(v, gv), 0, 0, 2d(v, gv) \end{pmatrix} \leq 0.$$

Hence, by (ψ_3), we have $d(u, gu) = d(v, gv) = 0$. Thus

$$u = gu \in G(u, v) \text{ and } v = gv \in G(v, u).$$

Therefore (u, v) is a common coupled fixed point of F, G, f, g. The proof is similar when $g(X)$ is assumed to be a closed subset of X.   □

Put $f = g$ in Theorem 7, we get the following result:

Corollary 8. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying (2.1), (2.2), (2.3) and*
  (8.1) *$\{F, g\}$ and $\{G, g\}$ are $w-$compatible.*
  Then F, G, g have a common coupled fixed point.

Put $F = G$ and $f = g$ in Theorem 7, we get the following result:

Corollary 9. *Let (X, d) be a complete metric space. Assume $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying* (2.3), (3.1), (3.2) *and*

(9.1) $\{F, g\}$ *is $w-$compatible.*

Then F and g have a common coupled fixed point.

Examples 1-3 and Theorem 7 imply the following:

Corollary 10. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $f, g : X \to X$ be mappings satisfying* (1.1), (4.1), *and* (7.1) *and* (7.2), *then F, G, f, g have a common coupled fixed point.*

Examples 1-3 and Corollary 9 imply the following:

Corollary 11. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying* (2.1), (2.3), (5.1) *and* (8.1), *then F, G, g have a common coupled fixed point.*

Examples 1-3 and Corollary 9 imply the following:

Corollary 12. *Let (X, d) be a complete metric space. Assume $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying* (2.3), (3.1), (6.1) *and* (9.1), *then F and g have a common coupled fixed point.*

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