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GLOBAL DYNAMICS OF CERTAIN NON-SYMMETRIC SECOND ORDER DIFFERENCE EQUATION WITH QUADRATIC TERM

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Dedicated to the memory of Professor Harry Miller and Professor Fikret Vajzović

ABSTRACT. We investigate global dynamics of the equation

$$x_{n+1}=\frac{x_{n-1}+F}{ax_n^2+f},\ \ n=0,1,2,...,$$

where the parameters a,F and f are positive numbers and the initial conditions x_{-1},x_0 are arbitrary nonnegative numbers such that $x_{-1}+x_0>0$. The existence and local stability of the unique positive equilibrium are analyzed algebraically. We characterize the global dynamics of this equation with the basins of attraction of its equilibrium point and periodic solutions.

1. INTRODUCTION AND PRELIMINARIES

We investigate global behavior of the equation

$$x_{n+1}=\frac{x_{n-1}+F}{ax_n^2+f},\ n=0,1,2\ldots, \tag{1.1}$$

where parameters a,f and F are positive numbers and the initial conditions x_{-1},x_0 are arbitrary nonnegative numbers. For Equation (1.1) we will precisely define the basins of attraction of all attractors, which consist of the equilibrium point, period-two solution and points at infinity. The special case of Equation (1.1), where $F=0$,

$$x_{n+1}=\frac{x_{n-1}}{ax_n^2+f} \tag{1.2}$$

were studied in detail in [8]. The presence of parameter F in the Equation (1.1) excludes a scenario of coexistence of infinite number nonhyperbolic minimal-period two solutions which is possible in Equation (1.2) for some values of parameters. Both equations, (1.1) and (1.2), are special cases of equation

$$x_{n+1}=\frac{Ax_n^2+Ex_{n-1}+F}{ax_n^2+ex_{n-1}+f},\ \ n=0,1,2,... \tag{1.3}$$

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which was considered in [7]. The global asymptotic stability results were obtained in [7] for several special cases of Equation (1.3), where the right-hand side does not change its monotonicity. Some special second order quadratic fractional difference equations have been considered in the series of papers, see [1, 2, 5, 6, 11, 12, 16, 17]. Also, several global asymptotic results for some special cases of a general second order quadratic fractional difference equation were obtained in [9, 10]. Our investigation of the global character of Equation (1.1) will be based on the theory of competitive systems and difference inequalities.
We will use the following theorem for a general second order difference equation

$$x_{n+1} = f\left(x_n, x_{n-1}\right), \qquad n = 0, 1, 2, \ldots, \tag{1.4}$$

see [4].

Theorem 1.1. *Let $[a,b]$ be an interval of real numbers and assume that $f : [a,b] \times [a,b] \to [a,b]$ is a continuous function satisfying the following properties:*

(a) *$f(x,y)$ is non-increasing in first and non-decreasing in second variable.*
(b) *Equation (1.4) has no minimal period-two solutions in $[a,b]$.*

Then every solution of Equation (1.4) converges to $\overline{x}$.

Theorem 1.2. *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{x} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\left(Q_1(\overline{x}) \cup Q_3(\overline{x})\right)$ is nonempty (i.e., $\overline{x}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{x}$.

b. The Jacobian $J_T(\overline{x})$ of T at $\overline{x}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{x}$ that is invariant and a subset of the basin of attraction of $\overline{x}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{x}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 1.2 reduces just to $|\lambda| < 1$. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis.

Theorem 1.3. *(A) Assume the hypotheses of Theorem 1.2, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 1.2. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\begin{aligned} \mathcal{W}_- &:= \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\} \quad \text{and} \\ \mathcal{W}_+ &:= \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \end{aligned} \tag{1.5}$$

such that the following statements are true.

(i) *$\mathcal{W}_-$ is invariant, and* $\operatorname{dist}(T^n(x), Q_2(\overline{\mathrm{x}})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_-$.
(ii) *$\mathcal{W}_+$ is invariant, and* $\operatorname{dist}(T^n(x), Q_4(\overline{\mathrm{x}})) \to 0$ *as* $n \to \infty$ *for every* $x \in \mathcal{W}_+$.
(B) *If, in addition to the hypotheses of part (A), $\overline{\mathrm{x}}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{\mathrm{x}}$, then T has no periodic points in the boundary of $Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}})$ except for $\overline{\mathrm{x}}$, and the following statements are true.*
(ii) *For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \operatorname{int} Q_2(\overline{\mathrm{x}})$ for $n \geq n_0$.*
(iv) *For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \operatorname{int} Q_4(\overline{\mathrm{x}})$ for $n \geq n_0$.*

If T is a map on a set $\mathcal{R}$ and if $\overline{\mathrm{x}}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{\mathrm{x}}\}$ and unstable set $\mathcal{W}^u(\overline{\mathrm{x}})$ of $\overline{\mathrm{x}}$ is the set

$$\Big\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x, \text{ and } \lim_{n\to-\infty} x_n = \overline{\mathrm{x}} \Big\}.$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{\mathrm{x}})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{\mathrm{x}})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{\mathrm{x}})$ and $\mathcal{W}^u(\overline{\mathrm{x}})$ are actually the global stable and unstable manifolds of $\overline{x}$.

Remark 1.1. We say that $f(u,v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Equation (1.4) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (1.4) is a strictly competitive map on $I \times I$, see [14].

Next result is one of the basic results on difference equation inequalities which we will use in this paper.

Theorem 1.4. *[3] Let $n \in N_{n_0}^+$ and $g(n,u,v)$ be a nondecreasing function in u and v for any fixed n. Suppose that for $n \geq n_0$, the inequalities*

$$y_{n+1} \leq g(n, y_n, y_{n-1}) \tag{1.6}$$

$$u_{n+1} \geq g(n, u_n, u_{n-1}) \tag{1.7}$$

hold. Then

$$y_{n_0-1} \leq u_{n_0-1}, \quad y_{n_0} \leq u_{n_0}$$

implies that

$$y_n \leq u_n \quad n \geq n_0.$$

The rest of this paper is organized as follows. The second section presents the local stability of the unique positive equilibrium solution. The third section gives conditions for existence of the minimal period-two solution and its local stability.

The fourth section presents global dynamics in certain regions of the parametric space.

2. Local stability analysis

In this section, we present the local stability of the unique positive equilibrium of Equation (1.1). The equilibrium points of Equation (1.1) are the positive solutions of the equation

$$\overline{x} = \frac{\overline{x} + F}{a\overline{x}^2 + f},$$

i.e.

$$a\overline{x}^3 + (f-1)\overline{x} - F = 0. \tag{2.1}$$

We will denote the left side of the previous relation by

$$G(x) = ax^3 + (f-1)x - F.$$

Then it holds

$$\begin{aligned} G'(x) &= 3ax^2 + f - 1 \text{ and} \\ G'(x) &= 0 \Leftrightarrow x_{\pm} = \pm\sqrt{\frac{1-f}{3a}}. \end{aligned}$$

Since $G(-\infty) = -\infty, G(0) = -F$, and $G(+\infty) = +\infty$, by using the above relations, it implies that there exists unique positive equilibrium point $\overline{x}$.

Next result uses standard local stability analysis, see [12] and [13].
Let

$$p = \frac{\partial f}{\partial u}(\overline{x},\overline{x}) \quad \text{and} \quad q = \frac{\partial f}{\partial v}(\overline{x},\overline{x})$$

denote the partial derivatives of $f(u,v)$ evaluated at the equilibrium $\overline{x}$ of Equation (1.4). Then the equation

$$y_{n+1} = py_n + qy_{n-1}, \quad n = 0,1,\dots \tag{2.2}$$

is called the linearized equation associated with Equation (1.4) about the equilibrium point $\overline{x}$.

Proposition 2.1. *(a) If both roots of the quadratic equation*

$$\lambda^2 - p\lambda - q = 0 \tag{2.3}$$

lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\overline{x}$ of Equation (1.4) *is locally asymptotically stable.*
(b) If at least one of the roots of Equation (2.3) *has absolute value greater than one, then the equilibrium $\overline{x}$ of Equation* (1.4) *is unstable.*
(c) A necessary and sufficient condition for both roots of Equation (2.3) *to lie in the open unit disk $|\lambda| < 1$, is*

$$|p| < 1 - q < 2. \tag{2.4}$$

In this case the locally asymptotically stable equilibrium $\overline{x}$ *is also called a sink.*
(d) A necessary and sufficient condition for both roots of Equation (2.3) *to have absolute value greater than one is*

$$|q| > 1 \quad \textit{and} \quad |p| < |1-q|.$$

In this case $\overline{x}$ *is a repeller.*
(e) A necessary and sufficient condition for one root of Equation (2.3) *to have absolute value greater than one and for the other to have absolute value less than one is*

$$p^2 + 4q > 0 \quad \textit{and} \quad |p| > |1-q|.$$

In this case the unstable equilibrium $\overline{x}$ *is called a saddle point.*
(f) A necessary and sufficient condition for a root of Equation (2.3) *to have absolute value equal to one is*

$$|p| = |1-q| \quad \textit{or} \quad (q = -1 \ i \ |p| \le 2).$$

In this case the equilibrium $\overline{x}$ *is called a nonhyperbolic point.*

Now, we prove the following lemma.

Lemma 2.1.

(1) *If* $f > 1$, *then the unique equilibrium point* $\overline{x}$ *of Equation (1.1) is:*
- i) *]locally asymptotically stable if* $2(f-1)\sqrt{a(f-1)} - aF > 0$,
- ii) *a nonhyperbolic point if* $2(f-1)\sqrt{a(f-1)} - aF = 0$,
- iii) *a saddle point if* $2(f-1)\sqrt{a(f-1)} - aF < 0$.

(2) *If* $f \le 1$, *then the unique equilibrium point* $\overline{x}$ *of Equation (1.1) is a saddle point.*

Proof. Denote as

$$H(u,v) = \frac{v+F}{au^2+f}.$$

Then we have

$$p = H_u'(\overline{x}) = \frac{-2a\overline{x}(\overline{x}+F)}{(a\overline{x}^2+f)^2}, \quad q = -H_v'(\overline{x}) = \frac{-1}{a\overline{x}^2+f} < 0,$$

and

$$p-1-q = \frac{-2a\overline{x}(\overline{x}+F)}{(a\overline{x}^2+f)^2} - 1 + \frac{1}{a\overline{x}^2+f} = \frac{-3a\overline{x}^2+1-f}{a\overline{x}^2+f} = -\frac{G'(\overline{x})}{a\overline{x}^2+f},$$

$$p+1+q = \frac{-a\overline{x}^2+f-1}{a\overline{x}^2+f}.$$

Since the function $G(x)$ is increasing when it passes through the equilibrium point $\overline{x}$, that is $G'(x) > 0$, so it implies $p-1-q < 0$. Hence, we need to determine the sign of the term $p+1+q$. Since the denominator is obviously positive, the sign of the expression depends on the sign of the numerator.

$$-ax^2+f-1 \quad = \quad 0 \Rightarrow x_{\pm} = \pm\sqrt{\frac{f-1}{a}}.$$

1. Let $f > 1$. Since

$$2a(f-1)\sqrt{a(f-1)} - a^2F > 0 \Leftrightarrow G(x_+) > 0 \Leftrightarrow \overline{x} < x_+ \Rightarrow p+1+q > 0,$$

the unique equilibrium point $\overline{x}$ is locally asymptotically stable. Analogously,

$$2a(f-1)\sqrt{a(f-1)} - a^2F = 0 \Leftrightarrow G(x_+) = 0 \Leftrightarrow \overline{x} = x_+ \Rightarrow p+1+q = 0,$$

which implies that the equilibrium point is nonhyperbolic. Finally, if

$$2a(f-1)\sqrt{a(f-1)} - a^2F < 0 \Leftrightarrow G(x_+) < 0 \Leftrightarrow \overline{x} > x_+ \Rightarrow p+1+q < 0,$$

which implies that the equilibrium point is a saddle point.
2. If $f \leq 1$, then $p+q+1 = \frac{-a\overline{x}^2+f-1}{a\overline{x}^2+f} < 0$ and the statement is true. □

3. Period-Two solutions

Now we present results about existence and local stability of minimal period-two solutions of Equation (1.1).

Theorem 3.1. *Assume that* $f > 1$. *If* $aF^2 - 4(f-1)^3 > 0$, *then Equation* (1.1) *has a minimal period-two solution*

$$\{\phi,\psi,\phi,\psi,\ldots\} \text{ and } \{\psi,\phi,\psi,\phi\ldots\} \tag{3.1}$$

where

$$\phi = \frac{aF - \sqrt{a^2F^2 - 4a(f-1)^3}}{2a(f-1)}, \psi = \frac{aF + \sqrt{a^2F^2 - 4a(f-1)^3}}{2a(f-1)}, \tag{3.2}$$

which is locally asymtotically stable.

Proof. Suppose that there exists a minimal period-two solution $\{\phi,\psi,\phi,\psi,\ldots\}$ of Equation (1.1), where ϕ and ψ are distinct nonnegative real numbers such that $\phi^2+\psi^2 \neq 0$. Then we have the following system:

$$\left.\begin{array}{l} \phi = \frac{\phi+F}{a\psi^2+f} \\ \psi = \frac{\psi+F}{a\phi^2+f} \end{array}\right\} \Leftrightarrow \left.\begin{array}{r} a\phi\psi^2 + f\phi = \phi + F \\ a\phi^2\psi + f\psi = \psi + F \end{array}\right\} \tag{3.3}$$

which is equivalent to

$$(\phi-\psi)(f-1-a\phi\psi) = 0.$$

Since $\phi \neq \psi$, we have that

$$\phi\psi = \frac{f-1}{a} \Rightarrow \phi \quad = \quad \frac{f-1}{a\psi}, f > 1. \tag{3.4}$$

Substituting (3.4) in (3.3) we obtain

$$a\left(\frac{f-1}{a\psi}\right)^2\psi + f\psi = \psi + F,$$

i.e.

$$a(f-1)\psi^2 - aF\psi + (f-1)^2 = 0,$$

from which

$$\psi_\pm = \frac{aF \pm \sqrt{D}}{2a(f-1)} \text{ and } \phi_\pm = \frac{f-1}{a}\frac{2a(f-1)}{aF \pm \sqrt{D}} = \frac{2(f-1)^2}{aF \pm \sqrt{D}},$$

where $D = a^2F^2 - 4a(f-1)^3$. One can prove that $\psi_\pm = \phi_\mp$. So if $f > 1$ and $D > 0$ there exists minimal period-two solution $\{\psi_+, \psi_-, \psi_+, \psi_-, ...\}$, where ψ_- and ψ_+ are given by (3.2). By substitution $x_{n-1} = u_n$, $x_n = v_n$, Equation (1.1) becomes the system of equations

$$\begin{aligned} u_{n+1} &= v_n, \\ v_{n+1} &= \tfrac{u_n+F}{av_n^2+f}. \end{aligned} \tag{3.5}$$

The map T is of the form

$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ h(u,v) \end{pmatrix},$$

where $h(u,v) = \dfrac{u+F}{av^2+f}$. The second iteration of the map T is

$$T^2\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} h(u,v) \\ h(v, h(u,v)) \end{pmatrix} = \begin{pmatrix} G(u,v) \\ H(u,v) \end{pmatrix} = \begin{pmatrix} \frac{u+F}{av^2+f} \\ \frac{(v+F)(av^2+f)^2}{a(u+F)^2+f(av^2+f)^2} \end{pmatrix}.$$

The Jacobian matrix of the map T^2 at the points (ϕ, ψ) is of the form

$$J_{T^2}(\phi,\psi) = \begin{pmatrix} \frac{\partial G}{\partial u}(\phi,\psi) & \frac{\partial G}{\partial v}(\phi,\psi) \\ \frac{\partial H}{\partial u}(\phi,\psi) & \frac{\partial H}{\partial v}(\phi,\psi) \end{pmatrix}$$

where

$$\frac{\partial G}{\partial u}(\phi,\psi) = \frac{1}{a\Psi^2+f},$$

$$\frac{\partial G}{\partial v}(\phi,\psi) = -\frac{(\phi+F)2a\psi}{(a\psi^2+f)^2} = -\frac{\phi(a\psi^2+f)2a\psi}{(a\psi^2+f)^2} = -\frac{2a\phi\psi}{a\psi^2+f},$$

$$\begin{aligned} \frac{\partial H}{\partial u}(\phi,\psi) = &\quad -\frac{(\psi+F)2a\frac{\phi+F}{a\psi^2+f}\frac{1}{a\psi^2+f}}{\left(a\left(\frac{\Phi+F}{a\psi^2+f}\right)^2+f\right)^2} = -\frac{2a\Psi\left(a\phi^2+f\right)\phi\left(a\psi^2+f\right)\frac{1}{\left(a\psi^2+f\right)^2}}{\left(\frac{a(\phi+F)^2+f\left(a\psi^2+f\right)^2}{\left(a\psi^2+f\right)^2}\right)^2} \\ = &\quad -\frac{2a\phi\psi\left(a\phi^2+f\right)\left(a\psi^2+f\right)^3}{\left(a\psi^2+f\right)^4\left(a\phi^2+f\right)^4} = -\frac{2a\phi\psi}{(a\psi^2+f)(a\phi^2+f)}, \end{aligned}$$

and

$$\frac{\partial H}{\partial v}(\phi,\psi) = \frac{\left(ah^2(\phi,\psi)+f\right) - (\psi+F)2ah(\phi,\psi)\frac{\partial h}{\partial v}(\phi,\psi)}{\left(ah^2(\phi,\psi)+f\right)^2} =$$

$$= \frac{1}{\left(ah^2(\phi,\psi)+f\right)^2} + \frac{(\Psi+F)2a\frac{\phi+F}{a\psi^2+f}\frac{2a\psi(\phi+F)}{(a\psi^2+f)^2}}{\frac{(a\psi^2+f)^4(a\phi^2+f)^2}{(a\psi^2+f)^4}}$$

$$= \frac{1}{a\phi^2+f} + \frac{4a^2\phi^2\psi^2}{(a\Psi^2+f)(a\Phi^2+f)}$$

Now we have

$$\begin{aligned} Tr_{J_{T^2}} = p \quad &= \tfrac{\partial G}{\partial u}(\Phi,\Psi) + \tfrac{\partial H}{\partial v}(\Phi,\Psi) = \tfrac{1}{a\Psi^2+f} + \tfrac{1}{a\Phi^2+f} + \tfrac{4a^2\Phi^2\Psi^2}{(a\Psi^2+f)(a\Phi^2+f)} \\ &= \tfrac{\left(4a^2\Phi^2\Psi^2+a\Phi^2+a\Psi^2+2f\right)}{(a\Phi^2+f)(a\Psi^2+f)}, \end{aligned}$$

and

$$\begin{aligned} Det_{J_{T^2}} = q \quad &= \tfrac{1}{a\Psi^2+f}\left(\tfrac{1}{a\Phi^2+f} + \tfrac{4a^2\Phi^2\Psi^2}{(a\Psi^2+f)(a\Phi^2+f)}\right) - \tfrac{2a\Phi\Psi}{a\Psi^2+f}\left(\tfrac{2a\Phi\Psi}{(a\Psi^2+f)(a\Phi^2+f)}\right) \\ &= \tfrac{1}{(a\Phi^2+f)(a\Psi^2+f)}. \end{aligned}$$

Notice that $p > 0$ and $q < 1$, so we just need to show that $p < 1+q$.

$$\begin{aligned} p < 1+q \quad &\Longleftrightarrow \quad \tfrac{1}{a\Psi^2+f} + \tfrac{1}{a\Phi^2+f} + \tfrac{4a^2\Phi^2\Psi^2}{(a\Psi^2+f)(a\Phi^2+f)} < 1 + \tfrac{1}{(a\Phi^2+f)(a\Psi^2+f)} \\ &\Longleftrightarrow \quad a\Phi^2 + f + a\Psi^2 + f + 4a^2\Phi^2\Psi^2 < \left(a\Phi^2+f\right)\left(a\Psi^2+f\right) + 1 \\ &\Longleftrightarrow \quad a\Phi^2(1-f) + a\Psi^2(1-f) - (1-f)^2 + 3a^2\Phi^2\Psi^2 < 0 \\ &\Longleftrightarrow \quad a(1-f)(\Phi^2+\Psi^2) - (1-f)^2 + 3a^2\left(\tfrac{f-1}{a}\right)^2 < 0 \\ &\Longleftrightarrow \quad (1-f)\tfrac{4a^2F^2-8a(f-1)^3}{4a(f-1)^2} + 2(f-1)^2 < 0 \\ &\Longleftrightarrow \quad \tfrac{-a^2F^2+2a(f-1)^3}{a(f-1)} + 2(f-1)^2 < 0 \\ &\Longleftrightarrow \quad \tfrac{a^2F-4a(f-1)^3}{a(1-f)} < 0, \end{aligned}$$

which is true since $f > 1$ and $D = a^2F - 4a(f-1)^3 > 0$. □

4. GLOBAL DYNAMICS

In this section, we present global dynamic results for Equation (1.1). Every solution of Equation (1.1) satisfies

$$x_{n+1} \le \frac{x_{n-1}}{f} + \frac{F}{f}, \quad n = 0,1,\ldots$$

which in view of Theorem 1.4, means that $x_n \le z_n, \quad n = 0,1,\ldots$, where $\{z_n\}$ satisfy

$$z_{n+1} = \frac{z_{n-1}}{f} + \frac{F}{f}. \tag{4.1}$$

So we obtain that $x_n \le \dfrac{F}{f-1}$ if $f > 1$ since

$$z_n = \frac{F}{f-1} + \frac{C_1}{\sqrt{f}^n} + \frac{(-1)^n C_2}{\sqrt{f}^n}, \quad n = 0,1,\ldots$$

So, every solution of Equation (1.1) is bounded if $f > 1$ and in that case $[L,U] = [0, \frac{F}{f-1}]$ is an invariant interval for solutions of the Equation (1.1).

Theorem 4.1. *If $f > 1$ and $D = a^2F - 4a(f-1)^3 > 0$ then there exist equilibrium point $\bar{x}$ which is a saddle point and the minimal period-two solution defined by (3.1) and (3.2) which is locally asymptotically stable. There exists a set $C \subset R = [0,\infty) \times [0,\infty)$ such that $(\bar{x}_+, \bar{x}_+) \in C$, and $W^s((\bar{x}_+, \bar{x}_+)) = C$ is an invariant subset of the basin of attraction of $(\bar{x}_+, \bar{x}_+)$. The set C is a graph of a strictly increasing continuous function of the first variable on an interval and separates R into two connected and invariant components $W_-(\bar{x}_+, \bar{x}_+)$ and $W_+(\bar{x}_+, \bar{x}_+)$, which satisfy that*

(i) if $(x_{-1}, x_0) \in W_+(\bar{x}_+, \bar{x}_+)$, then

$$\lim_{n\to\infty} x_{2n} = \frac{aF - \sqrt{a^2F^2 - 4a(f-1)^3}}{2a(f-1)}$$

and

$$\lim_{n\to\infty} x_{2n+1} = \frac{aF + \sqrt{a^2F^2 - 4a(f-1)^3}}{2a(f-1)};$$

(ii) if $(x_{-1}, x_0) \in W_-(\bar{x}_+, \bar{x}_+)$, then

$$\lim_{n\to\infty} x_{2n} = \frac{aF + \sqrt{a^2F^2 - 4a(f-1)^3}}{2a(f-1)}$$

and

$$\lim_{n\to\infty} x_{2n+1} = \frac{aF - \sqrt{a^2F^2 - 4a(f-1)^3}}{2a(f-1)}.$$

For visual representation see Figure 1.

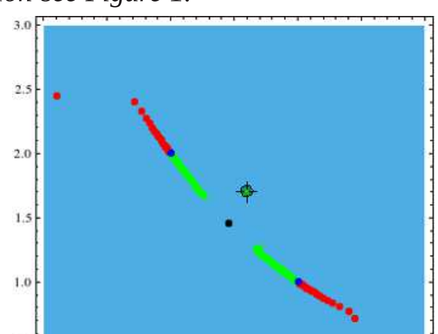

FIGURE 1. *Global dynamics of Equation* (1.1) *for $f = 5$, $F = 12$, $a = 2$ and inital conditions $(x_0, x_{-1}) = (0.3, 0.1)$ - red and $(x_0, x_{-1}) = (1.6, 1.7)$ - green.*

Proof. It is clear that the point $(\bar{x},\bar{x})$ and the period-two solutions (ϕ,ψ) and (ψ,ϕ) are the equilibrium points of the map T^2. Since the map T^2 is competitive, by Theorems 1.2 and 1.3, there exists a curve $\mathcal{C}$ through $(\bar{x},\bar{x})$ that is invariant and a subset of the basin of attraction of $(\bar{x},\bar{x})$ and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. If $(u_0,v_0)\in\mathcal{W}_+((\bar{x},\bar{x}))$, then by Theorem 1.3, $T^{2n}(u_0,v_0)\in\mathcal{W}_+((\bar{x},\bar{x}))$, and $T^{2n+1}(u_0,v_0)\in\mathcal{W}_-((\bar{x},\bar{x})$ for all $n\in\{0,1,2,...\}$. So we obtain that

$$\lim_{n\to\infty}T^{2n}(u_0,v_0)=(\psi,\phi)\ \text{and}\ \lim_{n\to\infty}T^{2n+1}(u_0,v_0)=(\phi,\psi)\,.$$

If $(u_0,v_0)\in\mathcal{W}_-((\bar{x},\bar{x}))$, then $T^{2n}(u_0,v_0)\in\mathcal{W}_-((\bar{x},\bar{x}))$ and $T^{2n+1}(u_0,v_0)\in\mathcal{W}_+((\bar{x},\bar{x})$ for all $n\in\{0,1,2,...\}$ which yields

$$\lim_{n\to\infty}T^{2n}(u_0,v_0)=(\phi,\psi)\ \text{and}\ \lim_{n\to\infty}T^{2n+1}(u_0,v_0)=(\psi,\phi)\,.$$

Consequently, if $(x_{-1},x_0)\in\mathcal{W}_+((\bar{x},\bar{x}))$, then

$$\lim_{n\to\infty}T^{2n}(x_{-1},x_0)=(\psi,\phi)\ \text{and}\ \lim_{n\to\infty}T^{2n+1}(x_{-1},x_0)=(\phi,\psi)\,,$$

which means that $\lim\limits_{n\to\infty}x_{2n}=\phi$ and $\lim\limits_{n\to\infty}x_{2n+1}=\psi$. If $(x_{-1},x_0)\in\mathcal{W}_-((\bar{x},\bar{x}))$, then

$$\lim_{n\to\infty}T^{2n}(x_{-1},x_0)=(\psi,\phi)\ \text{and}\ \lim_{n\to\infty}T^{2n+1}(x_{-1},x_0)=(\phi,\psi)\,,$$

which means that $\lim\limits_{n\to\infty}x_{2n}=\psi$ and $\lim\limits_{n\to\infty}x_{2n+1}=\phi$, where

$$\phi=\frac{aF-\sqrt{a^2F^2-4a(f-1)^3}}{2a(f-1)}\ \text{and}\ \psi=\frac{aF+\sqrt{a^2F^2-4a(f-1)^3}}{2a(f-1)}.$$

□

Theorem 4.2. *If $f>1$ and $D\le 0$, then the unique equilibrium solution of Equation* (1.1) *is globally asymptotically stable.*

Proof. The proof follows from Theorems 1.1 and 3.1 and Lemma 2.1. □

Theorem 4.3. *If $f\le 1$, then Equation* (1.1) *has a unique equilibrium point $\bar{x}$ which is a saddle point and has no minimal period-two solutions. There exists a set C which is an invariant subset of the basin of attraction of $(\bar{x},\bar{x})$. The set C is a graph of a strictly increasing continuous function of the first variable on an interval and separates R into two connected and invariant components $W_-(\bar{x}_+,\bar{x}_+)$ and $W_+(\bar{x}_+,\bar{x}_+)$, which satisfy that*

(i) *(i) if $(x_{-1},x_0)\in W_-(\bar{x}_+,\bar{x}_+)$, then*

$$\lim_{n\to\infty}x_{2n}=\infty\ \text{ and }\ \lim_{n\to\infty}x_{2n+1}=0;$$

(ii) *(ii) if $(x_{-1},x_0)\in W_+(\bar{x}_+,\bar{x}_+)$, then*

$$\lim_{n\to\infty}x_{2n}=0\ \text{ and }\ \lim_{n\to\infty}x_{2n+1}=\infty.$$

See Figure 2. for visual representation.

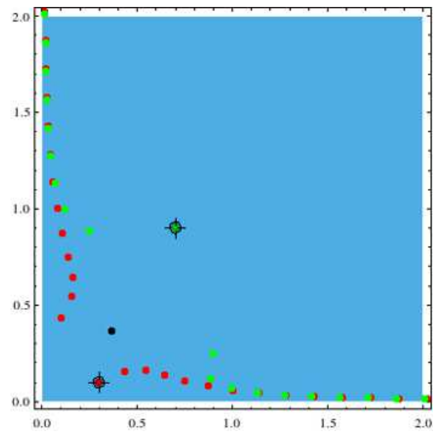

FIGURE 2. *Global dynamics of Equation* (1.1) *for* $f = 1$, $F = 0.15$, $a = 3$ *and inital conditions* $(x_0, x_{-1}) = (0.3, 0.1)$ - *red* *and* $(x_0, x_{-1}) = (0.7, 0.9)$ - *green.*

Proof. The point $(\overline{x}, \overline{x})$ is a saddle point for the strictly competitive map T^2 as well. The existence of the set C with the stated properties follows from Lemma 2.1, Theorems 1.2, 1.3 and 3.1. Equation (1.1) is equivalent to the system of difference equations (3.5), which can be decomposed into the system of the even-indexed and odd-indexed terms as follows:

$$\begin{cases} u_{2n} = v_{2n-1}, \\ u_{2n+1} = v_{2n}, \\ v_{2n} = \dfrac{u_{2n-1} + F}{av_{2n-1}^2 + f}, \\ v_{2n+1} = \dfrac{u_{2n} + F}{av_{2n}^2 + f}. \end{cases} \tag{4.2}$$

Now, using (4.2) we obtain

i) if $(u_0, v_0) \in \mathcal{W}_-$, then

$$(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \to (0, \infty)$$

and

$$(u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \to (\infty, 0);$$

ii) if $(u_0, v_0) \in \mathcal{W}_+$, then

$$(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \to (\infty, 0)$$

and

$$(u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \to (0, \infty);$$

Consequently,

i) if $(x_{-1},x_0)\in\mathcal{W}_-((\overline{x},\overline{x}))$, then
$T^{2n}((x_{-1},x_0))\to(0,\infty)$ and $T^{2n+1}((x_{-1},x_0))\to(\infty,0)$, that is

$$\lim_{n\to\infty}x_{2n}=\infty \text{ and } \lim_{n\to\infty}x_{2n+1}=0.$$

ii) if $(x_{-1},x_0)\in\mathcal{W}_+((\overline{x},\overline{x}))$, then
$T^{2n+1}((x_{-1},x_0))\to(0,\infty)$ and $T^{2n}((x_{-1},x_0))\to(\infty,0)$, that is

$$\lim_{n\to\infty}x_{2n}=0 \text{ and } \lim_{n\to\infty}x_{2n+1}=\infty.$$

□

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