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EQUICEVIAN POINTS AND EQUIANGULAR LINES OF A TRIANGLE IN AN ISOTROPIC PLANE

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Abstract. The concepts of equicevian points and equiangular lines of a triangle in an isotropic plane are studied in this paper. A number of significant properties of the introduced concepts are considered.

The isotropic (Galilean) plane is defined as a projective–metric plane with an absolute which consists of a line, *absolute line* ω, and a point on that line, *absolute point* Ω (see [7] and [9]). The lines through the point Ω are *isotropic lines*, and the points on the line ω are *isotropic points* (the points at infinity). Two lines through the same isotropic point are *parallel*, and two points on the same isotropic line are *parallel points*.
In an isotropic plane, the *distance* T_1T_2 between two non–parallel points $T_i = (x_i, y_i)$ $(i = 1, 2)$ is defined by $T_1T_2 = x_2 - x_1$. For two non-isotropic lines $y = k_1x + l_1$ and $y = k_2x + l_2$ the isotropic angle is defined by $k_2 - k_1$. The distances and angles are obviously oriented values.

In an isotropic plane, a triangle is admissible if none of its sides is an isotropic line. In [2], it is shown that each admissible triangle in an isotropic plane can be set, by a suitable choice of coordinates, in the so–called *standard position*, i.e., that its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$. We assume $a + b + c = 0$ – this guarantees positioning of the center of gravity G on the y–axis of the coordinate frame. With the abbreviations

$$p := abc, \qquad q := bc + ca + ab,$$

this gives

$$a^2 + b^2 + c^2 = -2q$$
$$q = bc - a^2.$$

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The point T, such that points $D = AT \cap BC$, $E = BT \cap CA$, $F = CT \cap AB$ satisfy the equations

$$AD = BE = CF = t, \qquad (t \neq 0), \tag{1}$$

will be called an *equicevian point* of the triangle ABC.

In Euclidean geometry, equicevian points are considered by Brocard [1], Neuberg [4] and Pounder [5].

Now, we will consider equicevian points of an admissible triangle in an isotropic plane.

Theorem 1. *Each admissible triangle of the isotropic plane has two equicevian points. The equicevian points of a standard triangle ABC are of the form*

$$T = \left(\frac{2}{3}\,t, \frac{3p}{t} - \frac{2}{3}q\right), \tag{2}$$

where

$$t = \pm\sqrt{-3\,q}. \tag{3}$$

Proof. Because of (1), the abscissa of the point D is $a + t$ and we get

$$\frac{BD}{CD} = \frac{t + a - b}{t + a - c},$$

and similarly,

$$\frac{CE}{AE} = \frac{t + b - c}{t + b - a}, \quad \frac{AF}{BF} = \frac{t + c - a}{t + c - b}.$$

According to Ceva's theorem that also holds in isotropic geometry because of its affine character, the lines AD, BE, CF pass through the point T if and only if $BD \cdot CE \cdot AF + CD \cdot AE \cdot BF = 0$, i.e.,

$$(t + a - b) \cdot (t + b - c) \cdot (t + c - a) + (t + a - c) \cdot (t + b - a) \cdot (t + c - b) = 0.$$

This condition can also be written in the form

$$2t^3 + 2t[(c - a)(a - b) + (a - b)(b - c) + (b - c)(c - a)] = 0,$$

i.e., $2t^3 + 6qt = 0$, since the expression enclosed in square brackets is equal to

$$bc + ca + ab - (a^2 + b^2 + c^2) = q - (-2q) = 3q.$$

As $t \neq 0$, we have $t^2 + 3q = 0$, which leads to (3). The line BC has the equation $y = -ax - bc$, and for the ordinate of the point D with the abscissa $a + t$ we get the value

$$-a(a + t) - bc = -at - (bc - q) - bc = -at + q - 2bc.$$

Therefore,

$$D = (a + t, q - 2bc - at). \tag{4}$$

Because of

$$(2q-3bc-at)a+2a^2t+3p-2aq=a^2t,$$

$$(2q-3bc-at)(a+t)+2a^2t+3p-2aq=t(a^2+2q-3bc-at)=t(q-2bc-at),$$

the points $A=(a,a^2)$ and D from (4) lie on the line with the equation

$$ty=(2q-3bc-at)x+2a^2t+3p-2aq, \tag{5}$$

and therefore it is the line AD. The point T from (2) also lies on this line. Indeed, owing to $t^2=-3q$, we obtain

$$\begin{aligned}&(2q-3bc-at)\cdot\frac{2}{3}t+2a^2t+3p-2aq\\ &=\frac{4}{3}qt+2(a^2-bc)t-2a\left(\frac{t^2}{3}+q\right)+3p=\frac{4}{3}qt-2qt+3p=t\left(\frac{3p}{t}-\frac{2}{3}q\right).\end{aligned}$$

□

Two equicevian points from Theorem 1 can be written in the form

$$T_1=\left(\frac{2}{3}t,\ \frac{3p}{t}-\frac{2}{3}q\right),\quad T_2=\left(-\frac{2}{3}t,\ -\frac{3p}{t}-\frac{2}{3}q\right), \tag{6}$$

where t achieves one of the values from (3); it does not matter which one. The point $G=\left(0,-\frac{2}{3}q\right)$ is the midpoint of the points T_1 and T_2 from (6), and we obtain the following.

Theorem 2. *The equicevian points of an admissible triangle of the isotropic plane are symmetrical with respect to its centroid.*

Theorem 3. *In addition to* (1)*, an equicevian point T of the admissible triangle ABC of the isotropic plane, with the cevians AD, BE, CF, satisfies the equalities*

$$AT+BT+CT=2t,\ \ TD+TE+TF=t,\ \ AT+BT+CT=2(TD+TE+TF). \tag{7}$$

Proof. From (2) and (4) we immediately obtain the equalities $AT=\frac{2}{3}t-a$, $TD=\frac{1}{3}t+a$ which, with analogous equalities for the other two cevians, owing to $a+b+c=0$, which proves equalities (7). □

This theorem is analogous to the Euclidean result of Sastry [8].

According to [6], the circumscribed Steiner ellipse of the triangle ABC has the equation

$$q^2x^2-9pxy-3qy^2-6pqx-4q^2y+9p^2=0.$$

If we put $x = \frac{2}{3}t$ in this equation, where $t^2 = -3q$, then after division by $-3q$, since

$$-\frac{3p^2}{q} = \frac{p^2t^2}{q^2},$$

we get the equation

$$\frac{4}{9}q^2 + \frac{2p}{q}ty + y^2 + \frac{4}{3}pt + \frac{4}{3}qy + \frac{p^2t^2}{q^2} = 0,$$

i.e., the equation

$$\left(y + \frac{pt}{q} + \frac{2}{3}q\right)^2 = 0$$

with the double solution

$$y = -\frac{p\,t}{q} - \frac{2}{3}\,q = \frac{3p}{t} - \frac{2}{3}\,q.$$

This means that the isotropic lines $x = \frac{2}{3}\,t$, where t is given by (3), touch the circumscribed Steiner ellipse at the points T given by (2), i.e., according to [6], these points are the foci of this ellipse. Therefore, we have proved the following.

Theorem 4. *The equicevian points of an admissible triangle of the isotropic plane are the foci of its circumscribed Steiner ellipse and they lie on its Steiner axis.*

The line $\mathcal{T}$ such that the points $D' = \mathcal{T} \cap BC$, $E' = \mathcal{T} \cap CA$, $F' = \mathcal{T} \cap AB$ satisfy the equalities

$$\angle(BC, AD') = \angle(CA, BE') = \angle(AB, CF') = \varphi \;\; (\text{with}\, \varphi \neq 0) \tag{8}$$

will be called an *equiangular line* of the triangle ABC.

Theorem 5. *Each admissible triangle of the isotropic plane has two equiangular lines. The standard triangle ABC has the equiangular lines $\mathcal{T}_1$ and $\mathcal{T}_2$ given by the equations*

$$\mathcal{T}_i \quad \ldots \quad y = -\frac{2}{3}tx - \frac{3p}{t} - \frac{q}{3} \quad (i = 1, 2), \tag{9}$$

where the two values of t are given by formula (3)*.*

Proof. Because of (8), the line AD' has slope $\varphi - a$, and its equation is

$$y = (\varphi - a)x - a\varphi + 2a^2 \tag{10}$$

since the point $A = (a, a^2)$ lies on line (10). From equation (10) and the equation $y = -ax - bc$ of the line BC for the abscissa x of the point D' we get the equation $\varphi x = a\varphi - 2a^2 - bc$, i.e., $\varphi x = a\varphi + 2q - 3bc$, with the solution $x = a + \frac{1}{\varphi}(2q - 3bc)$. With this value for x from the equation of the line BC for the y ordinate of the point D' we obtain

$$y = -a^2 - \frac{a}{\varphi}(2q - 3bc) - bc = q - 2bc + \frac{1}{\varphi}(3p - aq).$$

Therefore, the point D' is the first of the three analogous points

$$D' = \left(a + \frac{1}{\varphi}(2q - 3bc),\ q - 2bc + \frac{1}{\varphi}(3p - 2aq)\right),$$

$$E' = \left(b + \frac{1}{\varphi}(2q - 3ca),\ q - 2ca + \frac{1}{\varphi}(3p - 2bq)\right),$$

$$F' = \left(c + \frac{1}{\varphi}(2q - 3ab),\ q - 2ab + \frac{1}{\varphi}(3p - 2cq)\right).$$

The line $D'E'$ has slope

$$\frac{-2ca - \frac{1}{\varphi} \cdot 2bq + 2bc + \frac{1}{\varphi} \cdot 2aq}{b - \frac{1}{\varphi} \cdot 3ca - a + \frac{1}{\varphi} \cdot 3bc}$$

$$= \frac{2\left(\frac{q}{\varphi} - c\right)(a - b)}{-\left(\frac{3c}{\varphi} + 1\right)(a - b)} = \frac{2\left(c - \frac{q}{\varphi}\right)}{1 + \frac{3c}{\varphi}} = \frac{2(c\varphi - q)}{\varphi + 3c},$$

and analogously the line $D'F'$ has slope

$$\frac{2(b\varphi - q)}{\varphi + 3b}.$$

Hence, the points D', E', F' lie on one line provided that $(c\varphi - q)(\varphi + 3b) = (b\varphi - q)(\varphi + 3c)$, which can also be written in the form $(c - b)(\varphi^2 + 3q) = 0$. Therefore, this condition is $\varphi^2 + 3q = 0$, i.e., for example $\varphi = -t$, where t is given by (3). With these values, each of the two obtained lines have slope

$$\frac{2(-ct - q)}{-t + 3c} = \frac{2ct + \frac{2}{3} \cdot 3q}{t - 3c} = \frac{2ct - \frac{2}{3}t^2}{t - 3c} = -\frac{2}{3}t.$$

Now it remains to show that the point

$$D' = \left(a - \frac{1}{t}(2q - 3bc),\ q - 2bc + \frac{1}{t}(2aq - 3p)\right)$$

lies on line (9). Since $-\frac{t}{3} = \frac{q}{t}$, we obtain

$$-\frac{2}{3}t\left[a - \frac{1}{t}(2q - 3bc)\right] - \frac{3p}{t} - \frac{q}{3}$$

$$= \frac{2aq}{t} + \frac{2}{3}(2q - 3bc) - \frac{3p}{t} - \frac{q}{3} = q - 2bc + \frac{1}{t}(2aq - 3p).$$

□

The equations of the lines $\mathcal{T}_1$ and $\mathcal{T}_2$ can be written in the form

$$y = -\frac{2}{3}tx - \frac{3p}{t} - \frac{q}{3}, \quad y = \frac{2}{3}tx + \frac{3p}{t} - \frac{q}{3}, \tag{11}$$

where t achieves one of the values in (3). A bisector of the lines in (11) has the equation $y = -\frac{q}{3}$, and this is, by [2], the equation of the orthic axis of the triangle ABC, and we have:

Theorem 6. *The equiangular lines of an admissible triangle of the isotropic plane are symmetrical with respect to its orthic axis.*

Theorem 7. *In addition to* (8)*, for the equiangular line $\mathcal{T}$ of an admissible triangle ABC of the isotropic plane and its intersections D', E', F' with the lines BC, CA, AB, the following equalities hold:*

$$\angle(BC, \mathcal{T}) + \angle(CA, \mathcal{T}) + \angle(AB, \mathcal{T}) = 2\varphi,$$

$$\angle(\mathcal{T}, AD') + \angle(\mathcal{T}, BE') + \angle(\mathcal{T}, CF') = \varphi, \tag{12}$$

$$\angle(BC,\mathcal{T})+\angle(CA,\mathcal{T})+\angle(AB,\mathcal{T}) = 2[\angle(\mathcal{T}, AD')+\angle(\mathcal{T}, BE')+\angle(\mathcal{T}, CF')].$$

Proof. The lines BC, $\mathcal{T}$ and AD' have slopes $-a$, $\frac{2}{3}\varphi$, $\varphi - a$, so for example

$$\angle(BC, \mathcal{T}) = \frac{2}{3}\varphi + a, \qquad \angle(\mathcal{T}, AD') = \frac{\varphi}{3} - a.$$

With analogous equalities

$$\angle(CA, \mathcal{T}) = \frac{2}{3}\varphi + b, \quad \angle(AB, \mathcal{T}) = \frac{2}{3}\varphi + c,$$
$$\angle(\mathcal{T}, BE') = \frac{\varphi}{3} - b, \quad \angle(\mathcal{T}, CF') = \frac{\varphi}{3} - c,$$

by addition, because of $a + b + c = 0$, equalities (12) follow. □

According to (3), with $y = -\frac{q}{3}$, from (9) we get

$$x = -\frac{9p}{2t^2} = \frac{9p}{6q} = \frac{3p}{2q}.$$

Therefore, the point $K = \left(\frac{3p}{2q}, -\frac{q}{3}\right)$ lies on the lines $\mathcal{T}_1$ i $\mathcal{T}_2$, and the point K is, owing to [3], the symmedian center of the triangle ABC, i.e., we have the following.

Theorem 8. *Equiangular lines of an admissible triangle of the isotropic plane meet at its symmedian center.*

The concept of the equicevian point and the equicevian line of a triangle of the isotropic plane are mutually dual and therefore Theorems 5–8 are dual to Theorems 1–4. However, the proofs of all these statements are provided by means of point coordinates. Though, strictly speaking, the proofs of Theorems 6–8 could be omitted because they are not necessary. It is not done, since then it is necessary to explain the duality of the concept of the centroid and the orthic axis as well as the duality of the concept of the Steiner axis and the symmedian center of a triangle of the isotropic plane.

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