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GREEN PORTFOLIO OPTIMIZATION: PENALTIES FOR BROWN INVESTMENTS

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Dedicated to dear Prof. Dr. Mehmed Nurkanović on the occasion of his 65th birthday

ABSTRACT. In sustainable portfolio management, categorizing assets as "brown" or "green" based solely on ESG ratings can be misleading. A positive ESG score does not inherently indicate environmental responsibility unless it is evaluated relative to a meaningful benchmark. We propose a rescaled ESG rating system that measures each asset's environmental standing relative to a threshold set by policymakers, reflecting the urgency of the current climate crisis. In this system, assets are assigned positive scores if they exceed the threshold (green) and negative scores if they fall below it (brown), enhancing the interpretability of sustainability metrics in portfolio construction. However, a challenge arises when aggregating these scores into an overall portfolio rating. Under sustainable portfolio optimization developed in [11], short positions in brown assets, otherwise effectively betting against polluting companies, can paradoxically improve the portfolio's sustainability score. This creates a misleading incentive structure. To address this, we introduce a constraint that prohibits short positions in brown assets, ensuring that such investments do not positively impact the portfolio sustainability rating. While this restriction better aligns with environmental objectives, it also introduces complexity into the optimization process. To resolve this, we present an intuitive algorithm inspired by the active set method, which we refer to as *Green Portfolio Optimization*, capable of handling these constraints efficiently even in high-dimensional settings.

1. INTRODUCTION

As environmental challenges and global climate change intensify, sustainable investing plays an increasingly significant role in both financial decision-making and portfolio optimization. Many papers in the literature consider portfolios composed of "green" and "brown" assets, yet often without providing a precise or operational definition of this distinction. For example, [3,4] study a portfolio optimization problem involving "green" and "brown" assets under a utility maximization

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framework, where brown assets are penalized via higher effective risk aversion. In [5], the authors extend this setup by modeling ESG performance as a stochastic process. Similarly, [2] consider a utility-based portfolio problem with carbon constraints, in this case distinguishing between green and brown assets based on carbon scores. However, these studies rely on heuristic classifications based on ESG or carbon scores without formal thresholds or rules to define green and brown assets explicitly.

[7] argues in the following way: The ESG framework, originally designed to serve investors seeking social and environmental impact alongside financial returns, is increasingly being mainstreamed as a tool for assessing material credit risk. This shift is driven by the growing recognition of climate change as a systemic financial risk, prompting central banks and regulators to explore mechanisms to integrate ESG risks into financial oversight. One proposed mechanism is the use of a *benchmark-based classification*, in which assets are evaluated against a regulatory ESG threshold. [7] then argues that central banks could impose a *green-supporting factor* on assets scoring above the benchmark, providing them with regulatory relief, while those falling below the threshold would be subject to a *brown-penalizing factor*, increasing their regulatory or financial cost. Such a framework aims to correct market failures stemming from the underpricing of climate risks and would mark a shift from voluntary ESG adoption to a mandatory, rule-based regulatory regime.

Before we explain how we implement such brown-penalizing factors, we want to explain the sustainability setting of the paper. Our model remains grounded in the sustainable investment principles defined by the EU taxonomy, which outlines the environmental and social goals of the EU Green Deal [6, 13]. Recognizing that financial assets often fall somewhere between fully sustainable and entirely unsustainable, we adopt a more nuanced approach by incorporating sustainability ratings. As in [11], we assume the existence of a credible rating system to evaluate the sustainability of investments. One example could be the Bloomberg ESG score, which assesses firms using roughly 800 indicators and provides a rating between 0 and 100, where 100 stands for the most sustainable rating [9]. While much of the existing literature addresses sustainable integration using a one-period mean-variance framework in the spirit of Markowitz, see, for example, [14–16], our approach extends the sustainable continuous-time utility maximization framework developed in [11]. In [11], the portfolio assets are assigned sustainability ratings and the investor imposes a minimum acceptable sustainability threshold D on the portfolio. The sustainability ratings are here normalized to lie within the unit interval $[0,1]$. The aim is to construct an optimal portfolio that satisfies this constraint while maximizing expected utility over time. However, this approach alone does not differentiate between “brown“ and “green“ assets, as one might argue that simply stating an asset has a positive ESG rating does not necessarily exclude it from being a “brown“ asset.

To design and implement brown-penalizing factors in a portfolio optimization context, it is essential to first establish a clear, quantitative criterion for distinguishing between green and brown assets. Namely, the first step is to introduce a benchmark value $B \in [0,100]$, where now assets with scores below B are classified as brown, and those above or equal to B as green. This can be seen as a threshold imposed by the government/regulatory body on the basis of a current climate crisis situation. To reflect this distinction more clearly, we rescale the ratings so that green assets are assigned positive values and brown assets assigned negative ones. For example, setting $B = 40$, we map the sustainability scores in the interval $[-0.4, 0.6]$, where 0 represents the separation boundary, and the values are linearly scaled such that $[-0.4, 0)$ corresponds to brown assets and $[0, 0.6]$ to green ones.

However, this alone is insufficient to ensure that the sustainability of the portfolio aligns with our goals. Consider a green portfolio's rating $\tilde{R}^{\pi} = \sum_{i=1}^{n} \pi_i \tilde{R}_i$, where $\tilde{R}_i$ is the rating of asset i adjusted to the benchmark B and π_i represents the fraction of wealth invested in it. If both $\tilde{R}_i$ and π_i are negative, the product $\tilde{R}_i \pi_i > 0$ actually contributes positively to the green portfolio rating $\tilde{R}^{\pi}$. This implies that a negative rating with a corresponding negative investment unintentionally rewards investment in brown assets, thus undermining the intent of promoting environmental responsibility.

To address this, we introduce an additional constraint: for any asset i with $\tilde{R}_i < 0$, the portfolio weight must satisfy $\pi_i \geq 0$. This ensures that we do not hold short positions in brown assets, avoiding unintended positive contributions (regarding the portfolio rating) from such holdings. This means, with investments in brown assets we only allow negative contributions to the portfolio rating. This can be interpreted as a direct implementation of a brown-penalizing factor. Taken together, our approach aligns with both the benchmark-based classification and brown-penalizing factor argued in [7]. We name this approach *green portfolio optimization*.

In cases where all brown asset positions of the sustainable optimum are already positive, no adjustment is needed. However, if this is not met, this constraint complicates the optimization problem, as then the solution cannot be expressed in a closed-form formula. We develop an algorithm capable of solving the problem even in such settings.

It should be mentioned that [2] developed an approach that uses a utility-based framework and with “brown“ and “green“ asset. However, our approach differs in a few aspects from theirs. Firstly, they consider only one “brown“ and one “green“ asset, where our approach allows any number of these assets. Additionally, the differentiation is not based on a benchmark system, as in our case. Further, their additional constraint is limiting the investment in the green assets (it needs to be greater than the unconstrained optimal investment), while we are limiting the investment in the brown asset (it has to be non-negative). And lastly, their approach does not assign a rating to the non-risky asset at all. It is also important to note that they focus on limiting the carbon risk from above, while we want to limit the sustainable portfolio rating from below.

The structure of the paper is as follows. Section 2 presents the mathematical framework and portfolio setup. Section 3 builds on this foundation to develop our green portfolio optimization approach and illustrates it with numerical examples. Finally, Section 4 summarizes the main findings and provides concluding remarks. For readers interested in the optimization methods that motivated our approach, Appendix A provides an overview of quadratic programming and the active set method.

2. Mathematical framework and portfolio setup

We operate within the probabilistic and financial framework laid out in [11], adapted for the purposes of this paper. Consider a complete probability space $(\Omega, \mathcal{F}, P)$, equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ that is right-continuous. It is assumed that this space supports a d-dimensional standard Brownian motion $W(t) = (W_1(t), \dots, W_d(t))$, adapted to the filtration.

Let $B(t) = e^{rt}$ for $t \in [0,T]$ denote the value of a risk-free asset, such as a bank account, accruing interest at a constant rate r. The prices of d risky assets (e.g., stocks), denoted by $S_i(t)$ for $i = 1, \dots, d$, evolve according to the stochastic differential equations

$$dS_i(t) = S_i(t)\left(b_i\,dt + \sum_{j=1}^{d} \sigma_{ij}\,dW_j(t)\right), \quad i = 1, \dots, d, \tag{2.1}$$

where $b = (b_1, \dots, b_d)'$ represents the vector of expected returns. The volatility matrix $\sigma = (\sigma_{ij})_{i,j\in\{1,\dots,d\}}$ is assumed to be of full rank which implies that $\sigma\sigma'$ is positive definite.

A portfolio strategy is a vector-valued process $\pi(t) = (\pi_1(t), \dots, \pi_d(t))'$, representing the proportion of wealth invested in each of the risky assets. The fraction allocated to the riskless asset is given by $\pi_0(t) = 1 - \sum_{i=1}^{d} \pi_i(t)$. We assume that $\pi(t)$ is progressively measurable with respect to $\mathbb{F}$ and square integrable componentwise.

We focus on self-financing investment strategies, where the associated wealth process $X^\pi(t)$ satisfies the stochastic differential equation

$$dX^\pi(t) = X^\pi(t)\left[\left(r + \pi(t)'(b - r\underline{1})\right)dt + \pi(t)'\sigma\,dW(t)\right], \quad X^\pi(0) = x, \tag{2.2}$$

with initial capital $x > 0$, and $\underline{1} = (1, \dots, 1)' \in \mathbb{R}^d$.

To incorporate sustainability into the investment model, we assume that each asset, including the risk-free one (indexed by $i = 0$), carries a (scaled) sustainability score $R_i(t) \in [0,1]$. These ratings may be constant over time or time-dependent. The overall sustainability rating of the portfolio is given by a weighted average,

$$R^{\pi}(t) = \sum_{i=0}^{d} \pi_i(t) R_i(t).$$

By substituting $\pi_0(t) = 1 - \underline{1}'\pi(t)$, we can express this more compactly as

$$R^{\pi}(t) = R_0(t) + \pi(t)'(R(t) - R_0(t)\underline{1}),$$

where $R(t) = (R_1(t), \ldots, R_d(t))'$.

A sustainability requirement is introduced via a demand process $D(t)$, which prescribes a lower bound on the average sustainability rating that must be met by the portfolio at each point in time. Hence, admissible portfolios must satisfy the so-called *sustainability constraint*

$$R^{\pi}(t) \geq D(t), \quad \forall t \in [0,T]. \tag{2.3}$$

Definition 2.1 (Benchmark, Green Adjusted Ratings and Demand)**.** *Let the benchmark be defined as $B(t) \in [0,1], \forall t \in [0,T]$. Then, the adjusted ratings and demand for the green portfolio are for $t \in [0,T]$ defined as:*

$$\begin{aligned} \tilde{R}_i(t) &= R_i(t) - B(t), i = 0, 1, ..., d \\ \tilde{D}(t) &= D(t) - B(t) \geq 0. \end{aligned} \tag{2.4}$$

Notice that these imply now that $\tilde{R}(t) = R(t) - B(t)\underline{1}$, and $D(t) \geq B(t), \forall t \in [0,T]$.

Remark 2.1 (On Stochastic Modeling of Sustainability Components). Our framework can accommodate stochastic demand, stochastic ratings, and a stochastic benchmark as well. In such cases, it becomes necessary to explicitly model the dynamics of these quantities and adjust the portfolio strategies accordingly. However, if demand and ratings are already adapted to the Brownian filtration, no additional modifications are required. Therefore, in Definition 2.1, we implicitly assume that the benchmark process $B(t)$ is adapted to the Brownian filtration.

Remark 2.2 (Choice of Sustainability Rating). Note that the ESG score mentioned in this paper serves merely as an example of a sustainability-related measure. The proposed framework can be applied using any sustainability score that can be rescaled to the $[0,1]$ interval. The specific choice of the score is left to the relevant regulatory or governing body, which would also determine the corresponding benchmark value. The benchmark should be selected consistently with the chosen scoring methodology to ensure meaningful and comparable evaluations.

Definition 2.2 (Max-offer Condition)**.** *In [11], we introduced the following auxiliary feasibility condition:*

$$D(t) \leq R^*(t) := \max\{R_0(t), R_1(t), \ldots, R_d(t)\}, \quad \forall t \in [0,T]. \tag{2.5}$$

This guarantees that admissible portfolios satisfying the sustainability constraint can exist without enforcing short positions. While in the results of [11] this can be exchanged for a weaker condition (as proposed in [12]), here we need particularly this assumption to ensure that our green portfolio optimization problem always has a solution.

Let $X^{\pi}(t)$ again denote the wealth process under the strategy π. We now define the *sustainable portfolio optimization problem* as follows:

$$\begin{aligned} &\max_{\pi(\cdot)\in A(x)} \mathbb{E}_{0,x}\left[U\left(X^{\pi}(T)\right)\right] \\ &\text{subject to} \quad R^{\pi}(t) \geq D(t), \quad \forall t \in [0,T], \end{aligned} \tag{2.6}$$

where U is a utility function, and $A(x)$ denotes the class of admissible portfolio strategies satisfying:

$$\mathbb{E}_{0,x}\left[U\left(X^{\pi}(T)\right)^{-}\right] < \infty, \tag{2.7}$$

with $f^{-}(x) = \max(-f(x),0)$ representing the negative part of the function f.

The unconstrained optimal portfolio under log-utility $U(x) = \ln(x)$ is given as (see e.g., [10], but also shown later in Section 3):

$$\pi^{*} = \left(\sigma\sigma'\right)^{-1}(b - r\underline{1}). \tag{2.8}$$

In [11] we derived the following explicit form of the sustainable optimal portfolio process π_S^{opt} for the logarithmic utility function:

Proposition 2.1 (Korn and Nurkanovic, 2023, [11])**.** *Let the max-offer condition* (2.5) *be satisfied. If the bond possesses a sustainability rating* $R_0 \geq 0$*, then the optimal portfolio process for problem* (2.6) *with* $U(x) = \ln(x)$ *is given by*

$$\pi_S^{opt}(t) = \begin{cases} \left(\sigma\sigma'\right)^{-1}(b - r\underline{1}), \\ \quad \text{if} R_0(t) + (R(t) - R_0(t)\underline{1})'\left(\sigma\sigma'\right)^{-1}(b - r\underline{1}) \geq D(t) \\ \left(\sigma\sigma'\right)^{-1}\Bigg[(b - r\underline{1}) \\ + \frac{D(t) - R_0(t) - (b - r\underline{1})'\left(\sigma\sigma'\right)^{-1}(R(t) - R_0(t)\underline{1})}{(R(t) - R_0(t)\underline{1})'\left(\sigma\sigma'\right)^{-1}(R(t) - R_0(t)\underline{1})}(R(t) - R_0(t)\underline{1})\Bigg], \\ \quad \text{otherwise.} \end{cases} \tag{2.9}$$

Definition 2.3 (Set of Green/Brown Assets)**.** *The set of brown assets, denoted as* I_{brown}*, represents all indices i with* $\tilde{R}_i < 0$*, and* I_{green} *denotes the set of indices j for which we have non-negative ratings* $\tilde{R}_j \geq 0$ *which are considered green.*

Definition 2.4 (Set of Green Portfolios)**.** *The set of green portfolios, denoted* $A_G(x)$*, is the collection of all sustainable strategies* $\pi(\cdot) \in A_S(x)$ *that additionally satisfy* $\pi_i \geq 0$ *for all* $i \in I_{brown}$*.*

Finally, we define the *green portfolio optimization problem* as follows

$$\begin{aligned} &\max_{\pi(\cdot)\in A(x)} \mathbb{E}_{0,x}\left[U\left(X^{\pi}(T)\right)\right] \\ \text{subject to} \quad &\tilde{R}^{\pi}(t) \geq \tilde{D}(t), \quad \forall t \in [0,T], \\ \text{and} \quad &\pi_i(t) \geq 0, \quad \forall i \in I_{brown}(t), \quad \forall t \in [0,T]. \end{aligned} \tag{2.10}$$

Any further technical conditions will be stated as needed in the following sections.

3. GREEN PORTFOLIO OPTIMIZATION: PUNISHING INVESTMENT IN BROWN ASSETS

As we use the log-utility function $U(x) = \ln(x)$, we consider the portfolio problem under the following constraints

$$\begin{aligned} &\max_{\pi(\cdot)\in A(x)} \mathbb{E}_{0,x}\left[\ln\left(X^{\pi}(T)\right)\right] \\ \text{subject to} \quad &\tilde{R}^{\pi}(t) \geq \tilde{D}(t), \quad \forall t \in [0,T], \\ \text{and} \quad &\pi_i(t) \geq 0, \quad \forall i \in I_{brown}, \quad \forall t \in [0,T]. \end{aligned} \tag{3.1}$$

where we have implicitly assumed that the max offer condition (2.5) is satisfied. Remember, the wealth equation of $X^{\pi}(t)$ is given in (2.2). Now, applying the Itô's formula to $\ln(X^{\pi}(T))$, and then taking expectation of both sides, leads to

$$\begin{aligned} \mathbb{E}_{0,x}\left(\ln\left(X^{\pi}(T)\right)\right) = \ln(x) + \mathbb{E}_{0,x}\int_0^T \left(r + \pi(t)'(b - r\underline{1}) - \frac{1}{2}\pi(t)'\sigma\sigma'\pi(t)\right)dt \\ + \sigma\mathbb{E}_{0,x}\left(\int_0^T \pi(t)dW(t)\right). \end{aligned}$$

We assume $\pi(t)$ is progressively measurable with respect to the filtration $\mathbb{F}$ and square integrable in the sense that we have $\mathbb{E}\left(\int_0^T \pi^2(s)ds\right) < \infty$. Then, it follows $\mathbb{E}\left(\int_0^T \pi(t)dW(t)\right) = 0$ and we get

$$\begin{aligned} \mathbb{E}_{0,x}\left(\ln\left(X^{\pi}(T)\right)\right) &= \ln(x) \\ &+ \mathbb{E}_{0,x}\int_0^T \left(r + \pi(t)'(b - r\underline{1}) - \frac{1}{2}\pi(t)'\sigma\sigma'\pi(t)\right)dt. \end{aligned} \tag{3.2}$$

Now, maximizing the left side is equivalent to pointwise maximization under the integral on the right side for fixed $t \in [0,T]$ and every $\omega \in \Omega$. Not considering the constraints, this yields the unconstrained optimal portfolio process π^* given by

$$\pi^*(t) = \pi^* := (\sigma\sigma')^{-1}(b - r\underline{1}) \tag{3.3}$$

with $\underline{1} = (1,...,1)' \in \mathbb{R}^d$. If the unconstrained optimal portfolio already satisfies the constraints, then it is also the optimal sustainable portfolio. If this is not the case then we apply elementary Lagrangian multiplier considerations to the integrand ω-wise for every $t \in [0,T]$ and get the result from [11].

More precisely, even after including all of the constraints our optimization problem turns out to be a deterministic problem we have to solve for every t (and every ω), and it looks as the following

$$\begin{gathered} \max_{\pi(t)} \left(r + \pi(t)'(b - r\underline{1}) - \tfrac{1}{2}\pi(t)'\sigma\sigma'\pi(t)\right), \\ \tilde{R}^{\pi}(t) \geq \tilde{D}(t), \\ \pi_i(t) \geq 0 \quad \text{for i } \in I_{brown}(t). \end{gathered} \tag{3.4}$$

Remember that we have a closed-form solution if we are not imposing the last set of constraints. Notice how the function which we are maximizing is a function with a negative sign from the QP problem (A.1). Thus, our approach will be motivated by the active set method (see Section A.3). Recall that if we ignore the non-negativity constraints, we actually have a closed form solution π_S^{opt}.

Remark 3.1 (Rough idea for solving the green portfolio optimization problem). We first apply the result from Proposition 2.1, and if the additional green constraints are satisfied, i.e., $\pi_i \geq 0 \;$ for i $\in I_{brown}$, then we have solved the green portfolio optimization problem. However, if there exists $i \in I_{brown}$ with $\pi_i < 0$, then we set $\pi_i = 0$ and start again with the sustainable portfolio optimization without considering that asset. In what follows we will explain in detail how to deal with this when we have one, two or more brown assets included.

From now on, for simplicity, we will assume constant ratings and demand in $[0,T]$, i.e., we assume

$$R_i(t) = R_i \in [0,1], \tag{3.5}$$

$$D(t) = D \in [0,1], \tag{3.6}$$

$$B(t) = B \in [0,1] \;\; \forall t \in [0,T]. \tag{3.7}$$

Such assumption was named as Assumption A1 in [12], and in that paper, in the remark afterwards, one can read more about its significance.

3.1. **One brown asset**

One green asset. Let us say that we have only one green asset and one brown asset, i.e. $\tilde{R}_1 \geq 0$ and $\tilde{R}_2 < 0$. If π_S^{opt} already satisfies $\pi_2 \geq 0$, then that is our solution, i.e. $\pi_G^* = \pi_S^{opt}$. However, if it does not, then we set $\pi_2 = 0$ and apply the formula for π_S^{opt} without the brown asset, i.e., by returning to the previous constrained problem with only the bank account and one green asset. The solution in that case is given as $\pi_G^* = \left(\frac{\tilde{D}-\tilde{R}_0}{\tilde{R}_1-\tilde{R}_0}, 0\right)$. Let us for a moment analyze this. Consider different cases:

- $\tilde{D} \geq \tilde{R}_0, \tilde{R}_1 > \tilde{R}_0$ implies $\pi_1 \geq 0$ – due to the max offer condition, here we would need at least the rating of S_1 to be greater than the demand.

- $\tilde{D} \leq \tilde{R}_0, \tilde{R}_1 < \tilde{R}_0$ implies $\pi_1 \geq 0$ – the max offer condition is always satisfied in this case, which thus still allows a positive position in the green asset.
- $\tilde{D} > \tilde{R}_0, \tilde{R}_1 < \tilde{R}_0$ would imply $\pi_1 < 0$, but as the max offer condition cannot be satisfied here, this case is excluded.
- $\tilde{D} < \tilde{R}_0, \tilde{R}_1 > \tilde{R}_0$ would imply $\pi_1 < 0$, but notice how that would not make sense as $\tilde{R}_1$ is big enough and so is $\tilde{R}_0$. This actually never happens, because in this case already the unconstrained portfolio π_1^* satisfies the demand and is our solution.

As the last two cases are not possible, that means, that in the case of one green and one brown risky asset we always have non-negative investment in the green asset. This, however, does not have to be true for the higher dimensional case.

Remark 3.2 (Short selling of green assets vs. short selling of brown assets). In our green portfolio optimization framework, short selling green assets is preferable to short selling brown assets. Short selling brown assets is excluded because, due to $\tilde{R}_i < 0$ and $\pi_i < 0$, it would increase the portfolio sustainability rating. If permitted, this could allow a portfolio consisting solely of shorted brown assets to meet the sustainability requirement, which is clearly undesirable.

In contrast, green assets have positive sustainability ratings. Thus, short selling green assets reduces the overall portfolio rating. As a result, even short selling of green assets will be minimized in the optimization, since it negatively affects the portfolio sustainability profile. On the other hand, in a market consisting solely of green assets, short selling of a green asset due to its bad risk-return characteristics becomes a normal case.

In what follows we will give two examples – one where the sustainable portfolio is already the green one, and one where they differ. Those examples will also be represented in respective figures 1 and 2. Both figures illustrate the relationship between green, sustainable, and unconstrained portfolio choices. The x- and y-axes represent π_1 (the fraction of wealth invested in the green asset) and π_2 (the fraction of wealth invested in the brown asset), respectively. The feasible set of green portfolios is represented as the green shaded area, while the light blue shaded area represents the feasible set for sustainable portfolios. The feasible region of green assets is a subset of the sustainable portfolios region. The three different optima are represented as well. The green portfolio optimum is represented as a green diamond, the sustainable optimum as a blue square, while the unconstrained optimum is represented by a red dot. Finally, the ellipses represent the level-lines of the growth function that we are maximizing.

Example 3.1 (Sustainable portfolio optimum which is also green)**.** *To illustrate our findings with some numbers we choose the following figures:*

$$D = 0.45, r = 0.01, b_1 = 0.04, b_2 = 0.05, \sigma_1 = 0.2, \sigma_2 = 0.3,$$
$$R_0 = 0.55, R_1 = 0.35, R_2 = 0.1, B = 0.3,$$

meaning that S_1 is green and S_2 is brown (since $R_1 > B$ and $R_2 < B$). This then yields

$$\pi^* = (0.75, 0.44),\ R^{\pi^*} = 0.2 < 0.45 = D,$$

implying the unconstrained optimal portfolio does not satisfy the sustainability constraint. Now, we obtain the optimal admissible sustainable portfolio as

$$\pi_S^{opt} = (0.365, 0.059),\ R^{\pi_S^{opt}} = 0.45 = D.$$

Hence, both the portfolio return decreases from 0.05 *to* 0.02 *and the portfolio volatility decreases from* 0.2 *to* 0.075 *for the optimal sustainable portfolio. Further, as a consequence of the sustainability constraint, the optimal growth rate decreases from* 0.03 *to* 0.02*.*

Now, our ratings and demand become

$$\tilde{D} = 0.15, \tilde{R}_0 = 0.25, \tilde{R}_1 = 0.05, \tilde{R}_2 = -0.2.$$

The sustainable optimum π_S^{opt} stays the same, and as we already have a positive position for the brown asset, that is also our green optimum, i.e. $\pi_G^* = \pi_S^{opt}$. We represent this graphically in (1).

Example 3.2 (Green portfolio optimum different from the sustainable optimum)**.** *Compared to the previous example, let us only lower the rating of the riskless asset from* 0.55 *to* 0.5 *and keep all the other parameters as in the previous example. The unconstrained portfolio is given in the same way and it still does not satisfy the sustainability constraint. Now, we obtain the optimal admissible sustainable portfolio as*

$$\pi_S^{opt} = (0.365, -0.011),\ R_0 = 0.45 = D.$$

The optimal growth rate decreases from 0.03 *to* 0.02*. Unfortunately, this portfolio is not green, because we have a negative position in the brown asset.*

In the green investment our ratings and demand become

$$\tilde{D} = 0.15, \tilde{R}_0 = 0.25, \tilde{R}_1 = 0.05, \tilde{R}_2 = -0.2,$$

since as in the previous example we shift the ratings and demand for $B = 0.3$. The π_S^{opt} stays the same, but due to the negative position in the brown asset, as already mentioned, it is not green. Thus, we set $\pi_{G,2}^* = 0$, and get $\pi_{G,1}^* = \frac{\tilde{D}-\tilde{R}_0}{\tilde{R}_1-\tilde{R}_0}$, i.e. we get

$$\pi_G^* = (0.333, 0),\ \tilde{R}_0 = 0.15 = \tilde{D}.$$

The graphical representation can be found in Figure 2.

Comparison of Figure 1 and Figure 2: The difference between the figures in Figure 1 and 2 is in the rating of the non-risky asset. That is, in the first one we have $R_0 = 0.55$, while in the second one a slightly lower $R_0 = 0.5$. Due to the smaller rating in the second case, one can notice that the regions of green and sustainable portfolios are much smaller in the second case compared to the first.

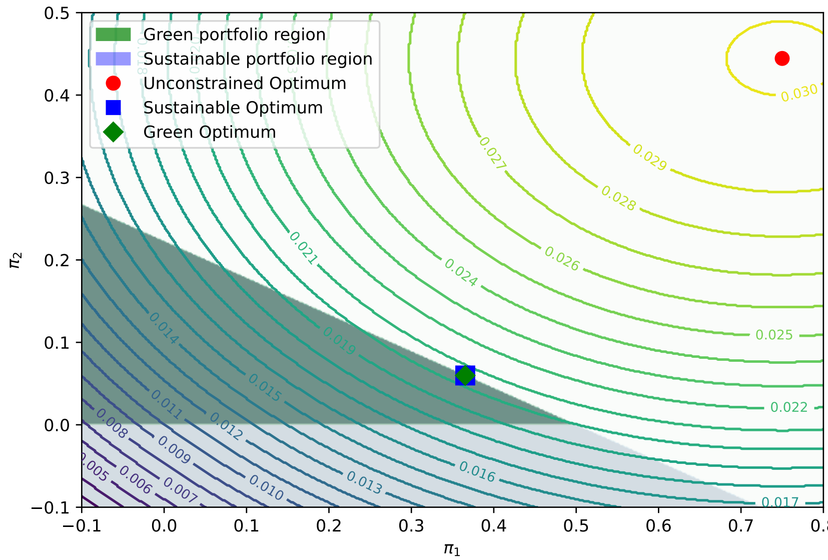

FIGURE 1. The plot represents values of π_1 (the fraction of wealth invested in the green asset) and π_2 (the fraction of wealth invested in the brown asset) as x and y axes, respectively. The feasible set of green portfolios is represented as the green shaded area, while the light blue shaded area represents the feasible set for sustainable portfolios, for figures given as in Example 1. The green optimum, represented as green diamond is in this case matching the sustainable optimum, represented as blue square. The unconstrained optimum is represented by the red dot. The ellipses represent the value level lines of the growth function that we are maximizing.

Multiple green assets Let us now assume that we have multiple green assets, e.g., from n total assets, let $n-1$ be green, i.e., $\tilde{R}_i \geq 0, i = 1, \ldots, n-1$, and one brown asset, i.e. $\tilde{R}_n < 0$. Again, we only have one additional constraint compared to the sustainable optimization. The procedure would again be to check if our π_S^{opt} satisfies the brown non-negativity constraint. If it does, then that is our solution, i.e., $\pi_G^* = \pi_S^{opt}$. However, if that is not the case, then we again set $\pi_n = 0$ and apply $\pi_S^{opt,(n)}$, which is the formula from sustainable optimization applied on the rest of the green assets, i.e., now on a lower dimension of $n-1$, without considering the

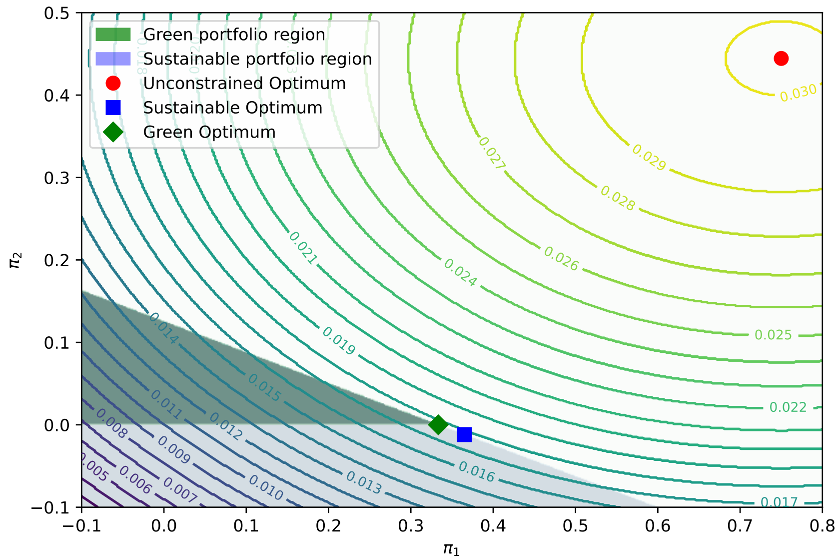

FIGURE 2. The plot represents values of π_1 (the fraction of wealth invested in the green asset) and π_2 (the fraction of wealth invested in the brown asset) as x and y axes, respectively. The feasible set of green portfolios is represented as the green shaded area, while the light blue shaded area represents the feasible set for sustainable portfolios, for figures given as in Example 2. The green optimum, represented as green diamond is in this case matching the sustainable optimum, represented as the blue square. The unconstrained optimum is represented by the red dot. The ellipses represent the value level lines of the growth function that we are maximizing.

brown asset n. Here, (n) in the superscript of the portfolio strategy means that we are considering the portfolio problem without asset n.

3.2. **Two brown assets**

In Figure 3 we give an algorithmic representation when we deal with $n-2$ green assets, i.e., $\tilde{R}_i \geq 0$, for $i = 1, \ldots, n-2$ and two brown assets, i.e. $\tilde{R}_{n-1}, \tilde{R}_n < 0$. Let us explain what exactly we have there. Firstly, we start by calculating π_S^{opt}, and from there one of the three scenarios can happen, each colored in a different color in the algorithmic scheme.

Case 1: Sustainable optimum does not violate brown non-negativity constraints
In this case π_S^{opt} already satisfies the brown non-negativity constraints, and our problem is solved, i.e. $\pi_G^* = \pi_S^{opt}$.
Case 2: Sustainable optimum violates exactly one brown non-negativity constraint
Here we assume π_S^{opt} violates exactly one of the brown non-negativity constraints, either for asset $n-1$ or asset n. We denote that asset by i, and then set $\pi_{S,i}^{opt} = 0$, and now do the sustainable portfolio optimization on one dimension below without that asset, i.e., we calculate $\pi_S^{opt,(i)}$. Now, the following two cases can happen:

- In case $\pi_S^{opt,(i)}$ does not violate the non-negativity constraint for the remaining brown asset, let us denote it by j, then $\pi_S^{opt,(i)}$ together with $\pi_{S,i}^{opt} = 0$ is our green optimum.
- However, if $\pi_S^{opt,(i)}$ does violate the non-negativity constraint for the brown asset j, then we need to set $\pi_{G,j}^* = 0$ along with $\pi_{G,i}^* = 0$, and calculate the sustainable optimum without considering either of the brown assets, i.e., $\pi_S^{opt,(i,j)}$, which is then the green optimum.

In the graphic illustration, for writing simplicity, let us w.l.o.g. assume $i = n$.
Case 3: Sustainable optimum violates both brown non-negativity constraints
Here, π_S^{opt} immediately violates both of the non-negativity constraints. In this case we will have to repeat Case 2 two times, once starting from $i = n-1$ and the other time with $i = n$. Thus, from the two subcases we get (at most) two possible solutions denoted by $\pi_G^{*,1}$ and $\pi_G^{*,2}$. We compare their objective function, and choose the solution which gives the highest objective.

3.3. **Multiple brown assets**

The case with multiple brown assets is carried out in the same fashion as when we had two brown assets, where we just end up having more cases. Starting again from sustainable optimal portfolio solution π_S^{opt}, it adjusts the portfolio iteratively to ensure that the specified non-negativity constraints for brown assets are satisfied. The main idea is summarized within the following steps.

- **Initialization and Check:** The algorithm first checks if the sustainable solution already satisfies the non-negativity constraints. If it does, this solution is directly outputted as optimal.
- **Iterative Constraint Satisfaction:** If the constraints are violated, the algorithm evaluates combinations of ”forcing” some constrained assets to zero, solving the reduced optimization problem for the remaining free assets at each iteration. It prioritizes combinations with fewer forced-zero constraints, progressing systematically.

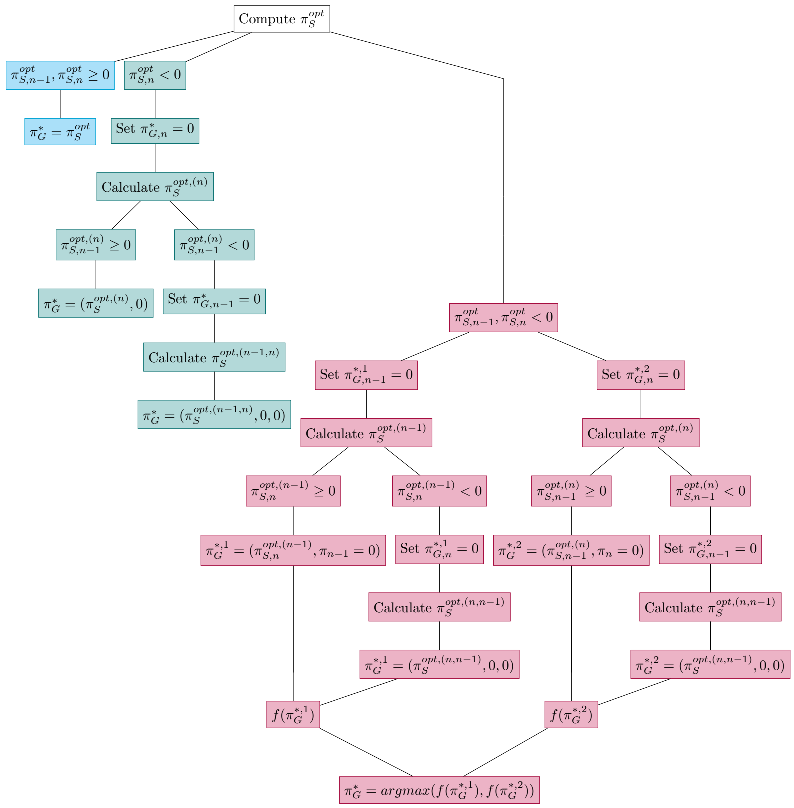

FIGURE 3. Algorithm for green optimization when considering two brown assets and multiple green ones.

- **Objective Evaluation:** For each feasible candidate solution, it computes the objective function value. The solution with the highest objective value is retained as the best feasible solution for that iteration.
- **Output the Solution:** Once a feasible solution is found, the algorithm outputs the optimal constrained portfolio.

This approach is explained in detail in Algorithm 1.

Remark 3.3 (Why is this approach justified?). The algorithms we presented are justified, firstly because we go through all of the combinations where we set violated non-negative constraints to be active, and secondly since we are using π_S^{opt} which satisfies the sustainability constraint. The matrix H, in our case $\sigma\sigma'$ is indeed positive definite, implying that the KKT conditions are both necessary and sufficient and the solution is unique. See A.1, A.2 and Lemma A.1.

Algorithm 1 Portfolio Optimization with k Non-negativity Constraints on Brown Assets

Input: Unconstrained optimal solution π_S^{opt}; indices of brown assets with non-negativity constraints $I_{\text{brown}} = \{i_1, i_2, \ldots, i_k\}$.
Initialize: Set $\pi_G^* = \pi_S^{opt}$ and mark all constraints in I_{brown} as active.
Step 1: Check if all non-negativity constraints are satisfied:
if $\pi_{S,i} \geq 0$ for all $i \in I_{\text{brown}}$ **then**
    All constraints are satisfied. Output $\pi_G^* = \pi_S^{opt}$ as the solution and **stop**.
end if
Step 2: Iterative Constraint Satisfaction
for $m = 1$ to k **do**
    Evaluate all combinations of setting m assets in I_{brown} to zero.
    for each combination $C \subset I_{\text{brown}}$ with $|C| = m$ **do**
        Set $\pi_i = 0$ for all $i \in C$.
        Solve the reduced optimization problem for remaining free assets, yielding candidate solution $\pi^{(C)}$.
        Check if $\pi^{(C)}$ satisfies $\pi_i \geq 0$ for all $i \in I_{\text{brown}} \setminus C$.
        if all remaining non-negativity constraints are satisfied **then**
            Compute the objective function value for $\pi^{(C)}$: Objective$^{(C)} = f(\pi^{(C)})$.
            Store $\pi^{(C)}$ as a feasible candidate solution.
        end if
    end for
    if at least one feasible solution is found in this iteration **then**
        Select the feasible solution π_G^* with the highest objective value among all candidates from this iteration.
        Output π_G^* as the optimal constrained solution and **stop**.
    end if
end for

Remark 3.4 (Why not use the standard active set method?). In the active set method one has to deal with solving different systems of equations, while our approach is quite straight forward since we have a closed formula for π_S^{opt} which is the only quantity we calculate. This is by far easier to implement and it is quite intuitive as well.

Remark 3.5 (Reducing running time of the algorithm). One can notice already in the case of two brown assets that some steps overlap. For instance, looking at Figure 3, we see that $\pi_S^{opt,(n-1,n)}$ was calculated three times, or to be exact, once in Case 2, and two times in Case 3. Case 2 is basically a part of the Case 3. This means that the algorithm could be written in such a way that these overlapping steps are saved and reused, instead of calculated every time, which would then reduce the running time. The more brown sets there are, the more drastically would the running time be reduced.

4. Conclusion

In this paper, we have proposed a framework for sustainable portfolio optimization that addresses key deficiencies in the standard use of sustainability ratings. Recognizing that sustainability scores in isolation may offer a misleading representation of environmental impact, we introduced a benchmark-based rescaling that classifies assets as “green” or “brown” relative to a policy-driven threshold. This adjustment allows for a more transparent and normatively grounded interpretation of sustainability.

A central issue identified in benchmark-constrained portfolio construction is the unintended positive contribution of short positions in brown assets to the overall sustainability rating. To mitigate this distortion, we imposed a non-negativity constraint on such positions. While this additional constraint aligns the optimization problem with sustainability objectives, it introduces new complexity by restricting the feasible investment set.

To address this, we proposed a computationally efficient algorithm inspired by the active set method, leveraging the closed-form solution of the unconstrained problem. This approach systematically enforces the green constraints by iteratively adjusting the set of active (i.e., zero-weighted) brown assets until a feasible and optimal solution is found. The algorithm remains tractable in high-dimensional settings and avoids the need for solving full quadratic programs at each step.

Our findings lead to several important conclusions:

- By setting the minimum level of sustainability for an asset to be considered as green we prevent ambiguous interpretations of sustainability.
- Shorting brown assets can artificially inflate sustainability ratings, which necessitates explicit constraints to preserve the integrity of the portfolio green objective.
- The proposed non-negativity constraint of portfolio positions in the brown assets provides a simple and effective correction that aligns the optimization output with the environmental intent, even though it reduces the flexibility of the portfolio construction.
- Our algorithm provides a practical and scalable solution to the constrained optimization problem, offering clear advantages over standard active set methods in this context.

Overall, this work contributes both a theoretical advancement and a practical toolset for constructing environmentally responsible portfolios. Future research may explore stochastic or dynamic benchmark thresholds, integration of broader ESG dimensions, or empirical testing on real-world asset data to validate the proposed methodology.

A. QUADRATIC PROGRAMMING PROBLEMS

A **quadratic programming (QP) problem** is an optimization problem with a (convex) quadratic objective function, and affine constraint functions, as defined by the following

$$\begin{aligned} \min_{\mathbf{x}\in\mathbb{R}^n} \quad & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}'\mathbf{H}\mathbf{x} + \mathbf{c}'\mathbf{x} \\ \text{subject to} \quad & [\mathbf{A}]_i\mathbf{x} \leq [\mathbf{b}]_i, \end{aligned} \tag{A.1}$$

where $x \in \mathbb{R}^n$ represents the vector over which we minimize the objective function f. The objective function is defined through a quadratic term $\mathbf{H} \in \mathbb{R}^{n\times n}$ and a linear term $\mathbf{c} \in \mathbb{R}^n$. The notation $[\cdot]_i$ represents the ith row of a matrix, and by $\mathbf{A} \in \mathbb{R}^{mxn}$ and $\mathbf{b} \in \mathbb{R}^m$ the feasible set is given as a polyhedron. Here we will briefly cover this topic with a type of algorithm called active set method which will be enough for our needs in green portfolio optimization. For further reference, see, e.g. [1], [8], [17].

A.1. Feasibility

A quadratic program (QP) in the form of (A.1) is deemed **infeasible** if no solution exists, which can happen in two distinct ways:

(1) **Primal Infeasibility**: This occurs when there is no point that satisfies the constraints $Ax \leq b$, meaning the feasible set is empty, i.e.,

$$\{x \in \mathbb{R}^n : Ax \leq b\} = \emptyset.$$

(2) **Dual Infeasibility**: This happens when H (the Hessian matrix of the quadratic term) is singular, and there exist points within the feasible region that make the objective function arbitrarily small. In such cases, the problem has no finite minimum and is considered *dual infeasible* or *unbounded*.

A.2. Optimality

A solution to the QP in (A.1), denoted as x^*, satisfies a set of conditions known as the **Karush-Kuhn-Tucker (KKT) conditions**, given as:

$$Hx^* + A'\lambda + c = 0, \quad \lambda \geq 0, \quad Ax^* \leq b, \quad \lambda'(Ax^* - b) = 0,$$

where $\lambda \in \mathbb{R}^m$ represents the dual variables (or multipliers).

In general, the KKT conditions are **necessary** for optimality, but for convex problems (e.g., when H is positive semi-definite), they are also **sufficient**. Additionally:

(1) If H is positive definite, the solution x^* is unique.
(2) If H is only positive semi-definite, there may be multiple solutions.

Among the conditions, complementary slackness $\lambda'(Ax^* - b) = 0$ introduces nonlinearity, which makes solving QPs more challenging. In **active-set methods**, which we introduce in the next subsection, this condition is maintained throughout all iterations using a *working set*, while the other conditions are gradually satisfied.

A.3. **The Active Set Method**

The main challenge in solving the QP in problem (A.1) arises due to the presence of inequality constraints $Ax \leq b$. If the QP instead involved only equality constraints (referred to as an equality-constrained QP or EQP), it could be solved simply by resolving a system of linear equations. Specifically, the minimizer x^* of the EQP

$$\min_x \frac{1}{2}x'Hx + c'x \quad \text{subject to } Ex = d,$$

is the solution to the following linear system, also known as the **KKT system**:

$$\begin{bmatrix} H & E' \\ E & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -f \\ d \end{bmatrix}.$$

In the case of EQPs, only **stationarity** and **primal feasibility** are necessary conditions for optimality. However, when inequality constraints are introduced, additional conditions such as **dual feasibility** and **complementary slackness** also become essential.

The simplicity of solving EQPs forms the foundation for **active-set methods**. An important observation underlying these methods is that, at the optimal solution x^*, only those inequality constraints that hold as equalities are relevant. This motivates the following definitions:

Definition A.1 (Active Constraint)**.** *An inequality constraint $a'x \leq c$ is said to be **active** at a point $\tilde{x} \in \mathbb{R}^n$ if it is satisfied as an equality, i.e., $a'\tilde{x} = c$.*

Definition A.2 (Active set)**.** *The **active set** at a point $x \in \mathbb{R}^n$, denoted as $\mathcal{A}(x)$, is the set of indices of all inequality constraints that are active at x, formally defined as:*

$$\mathcal{A}(x) = \{i \in \{1, \ldots, m\} : [A]_i x = [b]_i\}.$$

The following lemma establishes the significance of the active set at x^*. Intuitively, it shows that removing inactive constraints at x^* does not affect the solution. The lemma itself and the following remark can be found in more detail in [1].

Lemma A.1 (Sufficiency of the Active Set, [1])**.** *Let x^* be the solution to* (A.1)*, and let $\mathcal{A}^* = \mathcal{A}(x^*)$ be the active set at x^*. Then, x^* is also the solution to the EQP:*

$$\min_x \frac{1}{2}x'Hx + c'x \quad \textit{subject to } [A]_i x = [b]_i, \forall i \in \mathcal{A}^*.$$

Remark A.1 (Key Insight from Lemma A.1). The result of Lemma A.1 emphasizes that the active set contains all the information needed to determine the solution x^*. That is, if A^* was known, the solution to Problem (A.1) would reduce to solving a single system of linear equations. This observation underpins the primary goal of active-set methods: identifying A^*. To achieve this, these methods iteratively refine an estimate of A^*, referred to as the *working set*, denoted by W. The working set is updated by adding or removing constraints, with the choice of constraints dictated by solving an equality-constrained quadratic program (EQP) defined by the current working set W. Thus, the overall quadratic program (QP) is solved through a sequence of EQPs, where each EQP corresponds to the current state of W.

Active Set Method Algorithm Description The active set method is described via the following steps.

(1) **Initialization:** Start with a feasible point $\mathbf{x}_0$ and an initial active set $\mathcal{A}_0$.
(2) **Solve Subproblem:** Solve the equality-constrained QP using the active constraints.
(3) **Optimality Check:** Verify Lagrange multipliers and constraint feasibility.
(4) **Update Active Set:** Modify $\mathcal{A}$ by adding/removing constraints.
(5) **Repeat:** Iterate until KKT conditions are satisfied.

This algorithm of the active set method motivates our approach developed in Algorithm 1 for solving *green portfolio optimization*.

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