

## CESÀRO MEANS OF SUBSEQUENCES OF DOUBLE SEQUENCES

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*In memory of Professor Harry I. Miller*

**ABSTRACT.** In this paper we characterize the convergence and  $(C, 1, 1)$  summability of a double sequence. In particular we study conditions under which the convergence or  $(C, 1, 1)$  summability of a double sequence carry over to that of its subsequences, and conversely, whether these properties for suitable subsequences imply them for the sequence itself. We show, for instance, that a bounded double sequence is  $(C, 1, 1)$  summable if and only if almost all of its subsequences are  $(C, 1, 1)$  summable.

### 1. INTRODUCTION

Establishing a one-to-one correspondence between the interval  $(0, 1]$  and the collection of all subsequences of a given sequence  $(s_n)$ , Buck and Pollard [2] proved that  $(s_n)$  is  $(C, 1)$  summable if almost all of subsequences are, but not conversely. Replacing  $(C, 1)$  matrix by  $p$ -Cesàro matrix similar problems have also been considered in [9].

In the present paper we consider analogous problems for double sequences.

A double sequence  $s = (s_{ij})$  is said to be Pringsheim convergent (i.e., it is convergent in Pringsheim's sense) to  $L$  if for every  $\varepsilon > 0$  there exists an  $N \in \mathbb{N}$  such that  $|s_{ij} - L| < \varepsilon$  whenever  $i, j \geq N$  ([10]). In this case  $L$  is called the Pringsheim limit of  $s$  and the space of such sequences is denoted by  $c^{(2)}$ . A double sequence  $s$  is bounded if there exists a positive number  $M$  such that  $|s_{ij}| < M$  for all  $i$  and  $j$ , i.e.,

$$\|s\|_{(\infty, 2)} = \sup_{i, j} |s_{ij}| < \infty.$$

We will denote the set of all bounded double sequences by  $l_{\infty}^{(2)}$ . Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

Throughout the paper convergence means the Pringsheim convergence.

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Four dimensional Cesàro matrix  $(C, 1, 1) = (c_{jk}^{nm})$  is defined by

$$c_{kj}^{nm} = \begin{cases} \frac{1}{nm} & , \quad 1 \leq j \leq n \text{ and } 1 \leq k \leq m \\ 0 & , \quad \text{otherwise} \end{cases} .$$

It is known that the  $(C, 1, 1)$  matrix is an *RH* regular, i.e., it sums every bounded convergent sequence to the same limit.

There exist several versions of the concept of subsequences for double sequences ([5], [11], [16]). We adopt the definition of [5, 6] on subsequences of double sequences throughout the paper.

Let  $X$  denote the set of all double sequences of 0's and 1's, that is

$$X = \{x = (x_{jk}) : x_{jk} \in \{0, 1\} \text{ for each } j, k \in \mathbb{N}\} .$$

Let  $\mathfrak{R}$  be the smallest  $\sigma$ -algebra of subsets of the set  $X$  which contains all sets of the form

$$\{x = (x_{jk}) \in X : x_{j_1 k_1} = a_1, \dots, x_{j_n k_n} = a_n\}$$

where each  $a_i \in \{0, 1\}$  and the pairs  $\{(j_i k_i)\}_{i=1}^n$  are pairwise distinct.

There exists a unique probability measure  $P$  on the set  $\mathfrak{R}$ , such that

$$P(\{x = (x_{jk}) \in X : x_{j_1 k_1} = a_1, \dots, x_{j_n k_n} = a_n\}) = \frac{1}{2^n}$$

for all choices of  $n$  and all pairwise disjoint pairs  $\{(j_i k_i)\}_{i=1}^n$ , and all choices of  $a_1, \dots, a_n$  ([5]).

Let  $s = (s_{jk})$  be a double sequence and  $x = (x_{jk}) \in X$ . Following [5] we define a subsequence of the sequence  $s$  by

$$s_{jk}(x) = \begin{cases} s_{jk} & , \quad \text{if } x_{jk} = 1 \\ * & , \quad \text{if } x_{jk} = 0 \end{cases} .$$

Mapping  $x \rightarrow s(x)$  is a bijection from the set  $X$  to the set of all the subsequences of the sequence  $s = (s_{jk})$ .

An element  $x$  of  $X$  is said to be normal ([5]) if for each  $\varepsilon > 0$  there is a natural number  $N_\varepsilon$  such that for  $n, m \geq N_\varepsilon$  we have  $\left| \frac{1}{nm} \sum_{\substack{j \leq n \\ k \leq m}} x_{jk} - \frac{1}{2} \right| < \varepsilon$ . Let  $\eta$  denote the set of all elements  $x$  in  $X$  that are normal. This means that normal elements are  $(C, 1, 1)$ -summable to  $\frac{1}{2}$ . It is also known ([5]) that  $P(\eta) = 1$ . We also need the functions  $r_{j,k}(x) = 2x_{j,k} - 1$ , for  $x = (x_{jk}) \in X$ . Recall that the functions  $r_{j,k}$  are the Rademacher functions (see [5]).

## 2. SUBSEQUENCE CHARACTERIZATION OF CONVERGENCE AND CESÀRO SUMMABILITY

In this section we characterize the convergence and  $(C, 1, 1)$  summability of a double sequence. In particular we study conditions under which the convergence

or  $(C, 1, 1)$  summability of a double sequence carry over to that of its subsequences, and conversely, whether these properties for suitable subsequences imply them for the sequence itself. The results are analogues to those of Buck and Pollard [2] for single sequences. We note in passing that the summability properties of the set of second category subsequences may be found in [12].

**Theorem 2.1.** *If almost all of the subsequences of a double sequence  $s = (s_{jk})$  converges to  $L$ , then the sequence  $s = (s_{jk})$  itself converges to  $L$ .*

*Proof.* Assume that almost all of the subsequences of a double sequence  $s = (s_{jk})$  converges to  $L$ , i.e.,  $P(C) = 1$  where  $C = \{x \in X : s(x) \text{ converges to } L\}$ .

We use the technique given in [5]. Now given a sequence  $x = (x_{jk}) \in X$  we define a sequence  $\bar{x} = (\bar{x}_{jk})$  by

$$\bar{x}_{jk} = \begin{cases} 0 & , \quad x_{jk} = 1 \\ 1 & , \quad x_{jk} = 0 \end{cases} .$$

Let  $Y = C \cap \eta$  and  $\bar{Y} = \{(\bar{x}_{jk}) : x_{jk} \in Y\}$ . Therefore we have  $\bar{Y} = \bar{C} \cap \eta$  where  $\bar{C}$  is defined in the obvious way. Since the mapping  $(x_{jk}) \rightarrow (\bar{x}_{jk})$  preserves the measure  $P$ , we get  $P(\bar{Y}) = 1$  and hence  $P(Y \cap \bar{Y}) = 1$ . So  $Y \cap \bar{Y}$  is a non-empty set. If  $x = (x_{jk}) \in Y \cap \bar{Y}$ , then we have  $x \in C$ ,  $x \in \eta$  and  $\bar{x} \in C$ ,  $\bar{x} \in \eta$ . Since  $x, \bar{x} \in C$ , we have  $s(x) \rightarrow L$  and  $s(\bar{x}) \rightarrow L$  with  $x, \bar{x} \in \eta$ . This implies that the sequence  $s = (s_{jk})$  converges to  $L$ .  $\square$

We now turn our attention to the  $(C, 1, 1)$ -summability of subsequences.

**Theorem 2.2.** *If almost all subsequences of  $s = (s_{jk})$  are  $(C, 1, 1)$ -summable to a value  $L$  then the sequence  $s = (s_{jk})$  is  $(C, 1, 1)$ -summable to  $L$ .*

*Proof.* If almost all subsequences of  $(s_{jk})$  are  $(C, 1, 1)$ -summable to a value  $L$  then the set  $G = \{x \in X : s(x) \text{ is } (C, 1, 1)\text{-summable to } L\}$  has probability measure 1. Using the same type of argument in Theorem 2.1, if  $x \in G \cap \eta$  then we get  $\bar{x} \in G \cap \eta$ . Hence we obtain

$$s(x) \rightarrow L(C, 1, 1)$$

and

$$s(\bar{x}) \rightarrow L(C, 1, 1),$$

with  $x, \bar{x} \in \eta$ . That is

$$\lim_{n,m \rightarrow \infty} \frac{\sum_{j,k=1,1}^{n,m} s_{jk} x_{jk}}{\sum_{j,k=1,1}^{n,m} x_{jk}} = L$$

and similarly we get

$$\lim_{n,m \rightarrow \infty} \frac{\sum_{j,k=1,1}^{n,m} s_{jk} \bar{x}_{jk}}{\sum_{j,k=1,1}^{n,m} \bar{x}_{jk}} = L.$$

Also since  $x, \bar{x} \in \eta$ , we have

$$\lim_{n,m \rightarrow \infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} x_{jk} = \frac{1}{2} \text{ and } \lim_{n,m \rightarrow \infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} \bar{x}_{jk} = \frac{1}{2}.$$

On the other hand the  $(C, 1, 1)$ -summability of the sequence  $(s_{jk})$  is equivalent to the existence of the limit of the following expression

$$\frac{\sum_{j,k=1,1}^{n,m} s_{jk}}{nm} = \frac{\sum_{j,k=1,1}^{n,m} x_{jk}}{nm} \frac{\sum_{j,k=1,1}^{n,m} s_{jk} x_{jk}}{\sum_{j,k=1,1}^{n,m} x_{jk}} + \frac{\sum_{j,k=1,1}^{n,m} \bar{x}_{jk}}{nm} \frac{\sum_{j,k=1,1}^{n,m} s_{jk} \bar{x}_{jk}}{\sum_{j,k=1,1}^{n,m} \bar{x}_{jk}},$$

so we get  $\lim_{n,m \rightarrow \infty} \frac{\sum_{j,k=1,1}^{n,m} s_{jk}}{nm} = \frac{L}{2} + \frac{L}{2} = L$ , which means that the sequence  $(s_{jk})$  is  $(C, 1, 1)$ -summable to  $L$ . □

In order to get the converse of Theorem 2.2, we need the following lemmas. The first one is an analog of the Khintchine inequality [3].

**Lemma 2.3.** *Let  $t_{nm}(x) = \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x)$ ,  $B_{nm} = \sum_{j,k=1,1}^{n,m} s_{jk}^2$ . Then the following inequality*

$$E \left( (t_{nm})^{2r} \right) \leq \frac{(2r)!}{2^r r!} (B_{nm})^r$$

is fulfilled, where  $r$  is a positive integer.

*Proof.*  $E \left( (t_{nm})^{2r} \right) = \sum_{v_1 + \dots + v_i = 2r} A_{v_1, \dots, v_i} s_{j_1 k_1}^{v_1} \dots s_{j_i k_i}^{v_i} E \left[ r_{j_1 k_1}^{v_1}(x) \dots r_{j_i k_i}^{v_i}(x) \right]$

and  $1 \leq j_1, \dots, j_i \leq n$ ,  $1 \leq k_1, \dots, k_i \leq m$  where  $\sum_{\mu=1}^i v_\mu = 2r$ ,  $A_{v_1, \dots, v_i} = \frac{(v_1 + \dots + v_i)!}{v_1! \dots v_i!}$ .

We have

$$E \left[ r_{j_1 k_1}^{v_1}(x) \dots r_{j_i k_i}^{v_i}(x) \right] = \begin{cases} 1 & , \quad v_1, \dots, v_i \text{ even} \\ 0 & , \quad \text{otherwise} \end{cases}$$

and hence

$$E \left( (t_{nm})^{2r} \right) = \sum_{p_1 + \dots + p_i = r} A_{2p_1, \dots, 2p_i} s_{j_1 k_1}^{2p_1} \dots s_{j_i k_i}^{2p_i}$$

where  $\sum_{u=1}^i p_u = r$  such that  $p_1, \dots, p_i$  are positive integers. On the other hand it is well known that

$$(2p_1)! \dots (2p_i)! \geq 2^{p_1} p_1! \dots 2^{p_i} p_i!. \text{ So we have}$$

$$\begin{aligned}
E\left((t_{nm})^{2r}\right) &= \sum_{p_1+\dots+p_r=r} \frac{(2r)!}{2^r p_1! \dots p_r!} \frac{r!}{r!} s_{j_1 k_1}^{2p_1} \dots s_{j_r k_r}^{2p_r} \\
&\leq \frac{(2r)!}{2^r r!} \sum_{p_1+\dots+p_r=r} \frac{r!}{p_1! \dots p_r!} s_{j_1 k_1}^{2p_1} \dots s_{j_r k_r}^{2p_r} \\
&= \frac{(2r)!}{2^r r!} (B_{nm})^r.
\end{aligned}$$

This completes the proof.  $\square$

The next result is an analog of the Marcinkiewicz-Zygmund inequality [15].

**Lemma 2.4.** Let  $t_{nm}(x) = \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x)$ ,  $B_{nm} = \sum_{j,k=1,1}^{n,m} s_{jk}^2$  and  $t_{nm}^*(x) = \max_{\substack{1 \leq j \leq n \\ 1 \leq k \leq m}} |t_{jk}|$ . Then for  $a > 0$  the following inequality

$$E\left(e^{at_{nm}^*(x)}\right) \leq 32e^{a^2 \frac{B_{nm}}{2}}$$

holds.

*Proof.* Observe that

$$\begin{aligned}
e^{at_{nm}^*(x)} &\leq 2 \frac{e^{at_{nm}^*(x)} + e^{-at_{nm}^*(x)}}{2} \\
&= 2 \left\{ \frac{\sum_{r=0}^{\infty} \frac{(at_{nm}^*(x))^r}{r!} + \sum_{r=0}^{\infty} \frac{(-1)^r (at_{nm}^*(x))^r}{r!}}{2} \right\} \\
&= 2 \left\{ 1 + \sum_{r=1}^{\infty} \frac{(at_{nm}^*(x))^{2r}}{(2r)!} \right\}.
\end{aligned}$$

Using the last inequality we get

$$\begin{aligned}
E\left(e^{at_{nm}^*(x)}\right) &\leq 2 \left\{ 1 + \sum_{r=1}^{\infty} \frac{E\left[(at_{nm}^*(x))^{2r}\right]}{(2r)!} \right\} \\
&\leq 2 \left\{ 1 + \sum_{r=1}^{\infty} \frac{1}{(2r)!} a^{2r} E\left[\left(\max_{\substack{1 \leq j \leq n \\ 1 \leq k \leq m}} |t_{jk}|\right)^{2r}\right] \right\}.
\end{aligned}$$

Since  $E(r_{jk}(x)) = 0$ ,  $X_{jk} := s_{jk} r_{jk}$  for  $j \geq 1$ ,  $k \geq 1$  is an array of martingale differences ( see [13]). Hence using the Doob inequality [4] for multiple sequences and considering Lemma 2.3, we get

$$\begin{aligned}
 E\left(e^{a t_{nm}^*(x)}\right) &\leq 2\left\{1+\sum_{r=1}^{\infty} \frac{\left(\frac{2r}{2r-1}\right)^{4r} a^{2r} E\left[\left(|t_{nm}|\right)^{2r}\right]}{(2r)!}\right\} \\
 &\leq 2\left\{1+\sum_{r=1}^{\infty} \frac{\left(\frac{2r}{2r-1}\right)^{4r} a^{2r} \frac{(2r)!}{2^r r!}\left(B_{nm}\right)^r}{(2r)!}\right\} \\
 &= 2\left\{1+\sum_{r=1}^{\infty}\left(\frac{2r}{2r-1}\right)^{4r} \frac{\left(\frac{a^2 B_{nm}}{2}\right)^r}{(r)!}\right\} \\
 &\leq 32 \sum_{r=0}^{\infty} \frac{\left(\frac{a^2 B_{nm}}{2}\right)^r}{(r)!}=32 e^{a^2 \frac{B_{nm}}{2}} . \quad \square
 \end{aligned}$$

The next result which is an analog of Theorem 1 of [14] for double sequences is the converse of the present Theorem 2.2.

**Theorem 2.5.** *If the sequence  $(s_{jk})$  is  $(C, 1, 1)$ -summable to a value  $L$  and*

$$\sum_{j,k=1,1}^{n,m} s_{jk}^2 = o\left(\frac{n^2 m^2}{\log \log nm}\right)$$

*then almost all subsequences of  $(s_{jk})$  are  $(C, 1, 1)$ -summable to  $L$ .*

*Proof.* The  $(C, 1, 1)$ -summability of almost all subsequences of  $(s_{jk})$  is equivalent to the convergence of the following expression

$$\frac{\sum_{j,k=1,1}^{n,m} s_{jk} x_{jk}}{\sum_{j,k=1,1}^{n,m} x_{jk}} \quad \text{for almost all } x.$$

We can rewrite the above expression as follows for almost all  $x$

$$\frac{\sum_{j,k=1,1}^{n,m} s_{jk} \left(\frac{1+r_{jk}(x)}{2}\right)}{\sum_{j,k=1,1}^{n,m} \left(\frac{1+r_{jk}(x)}{2}\right)} = \frac{\frac{1}{2nm} \sum_{j,k=1,1}^{n,m} s_{jk} + \frac{1}{2nm} \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x)}{\frac{1}{nm} \sum_{j,k=1,1}^{n,m} \left(\frac{1+r_{jk}(x)}{2}\right)}. \quad (2.1)$$

Since  $P(\eta) = 1$ , observe that the denominator of (2.1) converges to  $\frac{1}{2}$  for almost all  $x$ . To complete the proof, it suffices to establish that

$$\frac{1}{nm} \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x) \rightarrow 0, \quad (\text{as } n, m \rightarrow \infty) \text{ for almost all } x.$$

Let  $\varepsilon > 0$  and define

$$E_{jk} := \{x : \text{there exists } (n, m) \text{ with } 2^{j-1} < n \leq 2^j, 2^{k-1} < m \leq 2^k \text{ such that } |t_{nm}(x)| \geq nm\varepsilon\}$$

and let

$$G_{jk} = \left\{ x : t_{2^j, 2^k}^*(x) > 2^{j-1} 2^{k-1} \varepsilon \right\}.$$

Notice that  $E_{jk} \subset G_{jk}$ . The proof will be completed if we prove that for every  $\varepsilon > 0$ ,

$\sum_{j,k=1,1}^{\infty, \infty} P(G_{jk}) < \infty$ . Now using Lemma 2.4 we have

$$P(G_{jk}) e^{a2^{j-1}2^{k-1}\varepsilon} \leq \int_X e^{at_{2^j, 2^k}^*(x)} dP(x) = E\left(e^{at_{2^j, 2^k}^*(x)}\right) \leq 32e^{a^2 \frac{B_{2^j 2^k}}{2}}.$$

Hence

$$P(G_{jk}) \leq 32e^{\frac{a^2 B_{2^j 2^k}}{2} - a2^{j-1}2^{k-1}\varepsilon}.$$

Taking  $a = \frac{2^{j-1}2^{k-1}\varepsilon}{B_{2^j 2^k}}$ , we have

$$\begin{aligned} P(G_{jk}) &\leq 32e^{-\frac{\varepsilon^2 2^{2(j-1)} 2^{2(k-1)}}{2B_{2^j 2^k}}} \\ &= 32e^{-\frac{\varepsilon^2 (2^j)^2 (2^k)^2}{32B_{2^j 2^k}}}. \end{aligned} \quad (2.2)$$

On the other hand it follows from the hypothesis that

$$\begin{aligned} \frac{B_{2^j 2^k}}{(2^j)^2 (2^k)^2} &= o\left(\frac{1}{\log \log 2^j 2^k}\right) \\ \frac{B_{2^j 2^k}}{(2^j)^2 (2^k)^2} &\leq \frac{\varepsilon^2}{96 \log \log 2^j 2^k}. \end{aligned}$$

Then (2.2) yields that

$$\begin{aligned} P(G_{jk}) &\leq 32e^{-\frac{\varepsilon^2}{32} \frac{96 \log \log 2^j 2^k}{\varepsilon^2}} \\ &= 32e^{-3 \log \log 2^j 2^k} \\ &= \frac{32}{[(j+k) \log 2]^3}. \end{aligned}$$

Since  $\sum \frac{1}{[(j+k) \log 2]^3} < \infty$  (see [1]),

$$\sum_{j,k=1,1}^{\infty, \infty} P(G_{jk}) \leq 32 \sum_{j,k=1,1}^{\infty, \infty} \frac{1}{[(j+k) \log 2]^3} < \infty.$$

Hence we obtain  $\lim_{j,k \rightarrow \infty} P(G_{jk}) = 0$  and also  $\lim_{j,k \rightarrow \infty} P(E_{jk}) = 0$ . This completes the proof.  $\square$

Now we are in a position to give a criterion.

**Corollary 2.1.** *A bounded double sequence  $(s_{jk})$  is  $(C, 1, 1)$ -summable if and only if the almost all subsequences are  $(C, 1, 1)$ -summable.*

**Theorem 2.6.** *If*

$$\lim_{n,m \rightarrow \infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x) = 0 \quad \text{for almost all } x \quad (2.3)$$

then

$$\lim_{n,m \rightarrow \infty} \frac{1}{n^2 m^2} \sum_{j,k=1,1}^{n,m} s_{jk}^2 = 0$$

holds.

*Proof.* Let  $E[p, q] = \{(j, k) : p \leq j \leq n \text{ or } q \leq k \leq m\}$  and

$$T_{p,q,n,m}(x) = \sum_{(j,k) \in E[p,q]} s_{jk} r_{jk}(x).$$

Hence

$$T_{p,q,n,m}^2(x) = \sum_{(j,k) \in E[p,q]} s_{jk}^2 + 2 \sum_{\substack{(j_1, k_1), (j_2, k_2) \in E[p,q] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} s_{j_1 k_1} s_{j_2 k_2} r_{j_1 k_1}(x) r_{j_2 k_2}(x).$$

Because of the Egoroff theorem there exists a set  $D \subset X$  with positive measure such that the limit in (2.3) exists uniformly on  $D$ . Therefore

$$\int_D T_{p,q,n,m}^2(x) dP(x) = P(D) \sum_{(j,k) \in E[p,q]} s_{jk}^2 + K, \quad (2.4)$$

where

$$K = 2 \sum_{\substack{(j_1, k_1), (j_2, k_2) \in E[p,q] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} s_{j_1 k_1} s_{j_2 k_2} \int_D r_{j_1 k_1}(x) r_{j_2 k_2}(x) dP(x).$$

By the Hölder inequality we have

$$|K| \leq 2 \left( \sum_{\substack{(j_1, k_1), (j_2, k_2) \in E[p,q] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} s_{j_1 k_1}^2 s_{j_2 k_2}^2 \right)^{\frac{1}{2}} \left( \sum_{\substack{(j_1, k_1), (j_2, k_2) \in E[p,q] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} v_{j_1 k_1, j_2 k_2}^2 \right)^{\frac{1}{2}} \quad (2.5)$$

where  $v_{j_1 k_1, j_2 k_2} = \int_D r_{j_1 k_1}(x) r_{j_2 k_2}(x) dP(x)$ . We know that the functions  $r_{j_1 k_1}(x)$  and  $r_{j_2 k_2}(x)$  are orthogonal on  $X$  (see [5]). So by the Bessel inequality for double sequences we get

$$\sum_{\substack{\leq j_1 < j_2 \leq \infty \\ 1 \leq k_1 < k_2 \leq \infty}} v_{j_1 k_1, j_2 k_2}^2 \leq \int_X (\chi_D(x))^2 dP(x) = P(D).$$

For sufficiently large  $p$  and  $q$ , we have

$$\left( \sum_{\substack{(j_1, k_1), (j_2, k_2) \in E[p, q] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} v_{j_1 k_1 j_2 k_2}^2 \right)^{\frac{1}{2}} \leq \frac{P(D)}{4}.$$

It follows from (2.5) that

$$|K| \leq \left( \sum_{\substack{(j_1, k_1), (j_2, k_2) \in E[p, q] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} s_{j_1 k_1}^2 s_{j_2 k_2}^2 \right)^{\frac{1}{2}} \frac{P(D)}{2} \leq \frac{P(D)}{2} \sum_{(j_1, k_1) \in E[p, q]} s_{j_1 k_1}^2.$$

Combining this with (2.4) we get

$$\begin{aligned} \int_D T_{p, q, n, m}^2(x) dP(x) &= P(D) \sum_{(j, k) \in E[p, q]} s_{jk}^2 + K \\ &\geq \frac{P(D)}{2} \sum_{(j, k) \in E[p, q]} s_{jk}^2. \end{aligned}$$

By (2.3) we have that

$$\lim_{n, m \rightarrow \infty} \frac{1}{n^2 m^2} \sum_{(j, k) \in E[p, q]} s_{jk}^2 = 0 \text{ and } \lim_{n, m \rightarrow \infty} \frac{1}{n^2 m^2} \sum_{j, k=1, 1}^{n, m} s_{jk}^2 = 0.$$

This completes the proof.  $\square$

In the next example we present a sequence so that it is  $(C, 1, 1)$  summable but almost none of its subsequences are  $(C, 1, 1)$  summable.

**Example 2.1.** Consider the double sequence  $s_{jk} = (-1)^j (-1)^k \sqrt{j} \sqrt{k}$ . Then

$$\sum_{j=1}^{\infty} \frac{(-1)^j \sqrt{j}}{j} = \sum_{j=1}^{\infty} \frac{(-1)^j}{\sqrt{j}} \text{ is convergent in the ordinary sense,}$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k}}{k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \text{ is convergent in the ordinary sense.}$$

On the other hand the double series  $\sum_{j, k=1, 1}^{\infty, \infty} \frac{(-1)^j (-1)^k}{\sqrt{j} \sqrt{k}}$  is convergent (see [1], page 90). Also since

$$\sum_{j=1}^{\infty} \frac{(-1)^j (-1)^k}{\sqrt{j} \sqrt{k}} \text{ is convergent for } j = 1, 2, \dots$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^j (-1)^k}{\sqrt{j} \sqrt{k}} \text{ is convergent for } k = 1, 2, \dots$$

then the double series  $\sum_{j,k=1,1}^{\infty,\infty} \frac{(-1)^j(-1)^k}{\sqrt{j}\sqrt{k}}$  is convergent in the restricted sense by Theorem 1 of [7]. Since the series  $\sum_{j,k=1,1}^{\infty,\infty} \frac{(-1)^j(-1)^k\sqrt{j}\sqrt{k}}{jk}$  is convergent in the restricted sense, we get that the sequence  $\left\{ \frac{1}{nm} \sum_{j,k=1,1}^{n,m} (-1)^j(-1)^k \sqrt{j}\sqrt{k} \right\}$  converges to 0 in the Pringsheim sense [8]. Hence the sequence  $\left( (-1)^j(-1)^k \sqrt{j}\sqrt{k} \right)$  is  $(C, 1, 1)$ -summable to 0. On the other hand, since

$$\frac{1}{n^2m^2} \sum_{j,k=1,1}^{n,m} jk = \frac{1}{n^2m^2} \frac{n(n+1)}{2} \frac{m(m+1)}{2} \rightarrow \frac{1}{4} \neq 0$$

by Theorem 2.6

$$\lim_{n,m} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} (-1)^j(-1)^k \sqrt{j}\sqrt{k} r_{jk}(x) \neq 0$$

so almost none of its subsequences are  $(C, 1, 1)$ -summable to zero.

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