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QUASI-ASYMPTOTICALLY ALMOST PERIODIC VECTOR-VALUED GENERALIZED FUNCTIONS

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Dedicated to the memory of Academician Fikret Vajzović

ABSTRACT. In this paper are introduced the notions of a quasi-asympto-tically almost periodic distributions and quasi-asymptotically almost periodic ultradistributions with values in a Banach space, as well as some other generalizations of these concepts. Furthermore, some applications of the introduced concepts in the analysis of systems of ordinary differential equations are provided.

1. INTRODUCTION AND PRELIMINARIES

The main goal of this paper is the selection and structural analysis of various classes of almost and asymptotically almost periodic or automorphic distributions and ultradistributions.
The concept of almost periodicity was introduced in [3] and later this theory is generalized by many other mathematicians. We put $I=\mathbb{R}$ or $I=[0,\infty)$, and $f: I\rightarrow X$ be continuous function. Given $\varepsilon>0$, we call $\tau>0$ an ε-period for $f(\cdot)$ iff

$$\|f(t+\tau)-f(t)\|\leq\varepsilon,\quad t\in I.$$

The set constituted of all ε-periods for $f(\cdot)$ is denoted by $\vartheta(f,\varepsilon)$. It is said that $f(\cdot)$ is almost periodic, (AP) for short, iff for each $\varepsilon>0$ the set $\vartheta(f,\varepsilon)$ is relatively dense in I, which means that there exists $l>0$ such that any subinterval of I of length l meets $\vartheta(f,\varepsilon)$. The vector space consisting of all almost periodic functions is denoted by $AP(I:X)$.

The notion of a scalar-valued asymptotically almost periodic ((AAP) in short) distribution has been introduced by I. Cioranescu in [12], while the notion of a vector-valued (AAP) distributions has been considered by D. N. Cheban [9] following a different approach (see also I. K. Dontvi [14] and A. Halanay, D. Wexler [15]). Some contributions have been also given by B. Stanković [29]- [30].

In our recent joint research study [26], we have analyzed the notions of an almost automorphic ((AAut) in short) distributions and an almost automorphic

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(AAut) ultradistributions in Banach space; the notion of an (AP) ultradistribution in Banach space has been recently analyzed by M. Kostić [24] within the framework of Komatsu's theory of ultradistributions, with the corresponding sequences not satisfying the condition (M.3); see also the papers by I. Cioranescu [11] and M. C. Gómez-Collado [17] for first results in this direction. As mentioned in the abstract, the main aim of this paper is to introduce the notions of a quasi-asymptotically almost periodic ((Q-AP) in short) (ultra)distribution in Banach space, as well as to provide some applications in the qualitative analysis of vector-valued (ultra)distributional solutions to systems of ordinary differential equations (the notion of a (Q-AP) ultradistribution seems to be not considered elsewhere even in scalar-valued case). In such a way, we expand and contemplate the results obtained in [5]- [6], [9], [10]- [12], [14]- [15] and [29]- [30]; see also [33]- [34] for some other results about the existence and uniqueness of various types of generalized (AP) solutions of nonlinear Volterra integro-differential equations.

The organization of paper is briefly described as follows. After giving some preliminary results and definitions from the theory of vector-valued ultradistributions (Subsection 1.1), in Section 2 we analyze the notions of (Q-AP) and Stepanov p-quasi-asymptotically almost periodic ((S^pQ-AAP) in short) vector-valued (ultra)distributions. Here, we recognize the importance of condition $T * \varphi \in (Q-AP)(\mathbb{R} : X)$, $\varphi \in \mathcal{D}$ for a bounded vector-valued distribution $T \in \mathcal{D}'_{L^1}(X)$, in contrast with the considerations of I. Cioranescu [12] and C. Bouzar, F. Z. Tchouar [5], where the above inclusions are required to be valid only for the test functions belonging to the space $\mathcal{D}_0$. The main result of paper is Theorem 3.9, where we state an important structural characterization for the class of (Q-AP) vector-valued ultradistributions. The last section of paper is reserved for certain applications to systems of ordinary differential equations in distribution and ultradistribution spaces.

The standard notation is used throughout the paper. By $(X, \|\cdot\|)$ we denote a non-trivial complex Banach space. The abbreviations $C_b(I : X)$ and $C(K : X)$, where K is a non-empty compact subset of $\mathbb{R}$, stand for the spaces consisting of all bounded continuous functions $I \mapsto X$ and all continuous functions $K \mapsto X$, respectively. Both spaces are Banach endowed with sup-norm. By $C_0([0,\infty) : X)$ we denote the closed subspace of $C_b([0,\infty) : X)$ consisting of functions vanishing at plus infinity.

We say that a continuous function $f : \mathbb{R} \to X$ is (AAP) iff there is a function $q \in C_0([0,\infty) : X)$ and an (AP) function $g : \mathbb{R} \to X$ such that $f(t) = g(t) + q(t)$, $t \geq 0$. By $AAP(\mathbb{R} : X)$, we denote the vector space consisting of all (AAP) functions. See [9], [13]- [14], [18], [23], [33] and references cited therein for more details on the subject.

Let $1 \leq p < \infty$. Then we say that a function $f \in L^p_{loc}(\mathbb{R} : X)$ is Stepanov p-bounded, S^p-bounded shortly, iff

$$\|f\|_{S^p} := \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(s)\|^p \, ds \right)^{1/p} < \infty.$$

The space $L^p_S(\mathbb{R} : X)$ consisted of all S^p-bounded functions becomes a Banach space equipped with the above norm. A function $f \in L^p_S(\mathbb{R} : X)$ is said to be Stepanov p-almost periodic, (S^p-AP) shortly, iff the function $\hat{f} : \mathbb{R} \to L^p([0,1] : X)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in \mathbb{R}, \, s \in [0,1]$$

is almost periodic. Following H. R. Henríquez [16], we say that a function $f \in L^p_S(\mathbb{R} : X)$ is Stepanov p-asymptotically almost periodic, (S^p-AAP) shortly, iff there are two locally p-integrable functions $g : \mathbb{R} \to X$ and $q : [0,\infty) \to X$ satisfying the following conditions:

(i) $g(\cdot)$ is (S^p-AP),
(ii) $\hat{q}(\cdot)$ belongs to the class $C_0([0,\infty) : L^p([0,1] : X))$,
(iii) $f(t) = g(t) + q(t)$ for all $t \geq 0$.

By $AAPS^p(\mathbb{R} : X)$ we denote the space consisting of all (S^p-AAP) functions.

The notion of an almost automorphic function (AAut) on topological group was introduced and further analyzed in the landmark papers by W. A. Veech [31]- [32] between 1965 and 1967. For more details about (AP) and (AAut) functions with values in Banach spaces, we refer the reader to the monographs [13] by T. Diagana and [18] by G. M. N'Guérékata.

Following [25], we can recall the definition of (Q-AP) functions. Let we suppose that $I = [0,\infty)$ or $I = \mathbb{R}$. It is said that a bounded continuous function $f : I \to X$ is (Q-AP) iff for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon,\tau) > 0$ such that

$$\|f(t+\tau) - f(t)\| \leq \varepsilon, \quad \text{provided } t \in I \text{ and } |t| \geq M(\varepsilon,\tau). \tag{1.1}$$

We denote by $Q-AP(I : X)$ the set consisting of all (Q-AP) functions from I into X. It is not relevant whether we write (1.1) or $\|f(t+\tau) - f(t)\| \leq \varepsilon$, provided $t \in I$, $|t| \geq M(\varepsilon,\tau)$ and $|t+\tau| \geq M(\varepsilon,\tau)$. So, we can easily seen that the class $AAP(I : X)$ is contained in the class $Q-AAP(I : X)$, since the number M depends only on ε and not on τ for (AAP) functions. Further, we will use the shorthand:
(S): "there exists finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number".

Let $f \in L^p_S(I : X)$. It is said that $f(\cdot)$ is (S^p-Q-AP), iff for each $\varepsilon > 0$ there exists a finite number $L(\varepsilon) > 0$ such that any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains at least one number $\tau \in I'$ satisfying that there exists a finite number $M(\varepsilon,\tau) > 0$ such that

$$\int_{t}^{t+1} \| f(s+t) - f(s) \|^{p} \, ds \leq \varepsilon^{p}, \quad \text{provided } t \in I, \ |t| \geq M(\varepsilon, \tau). \tag{1.2}$$

Denote by $S^pQ-AP(I:X)$ the set consisting of all (S^p-Q-AP) functions from I into X. From the definition, follows that $Q-AP(I:X) \subseteq S^pQ-AP(I:X)$. This inclusion is strict, since the function $f(t)=\chi_{[-1,\infty)}(t)$, $t\in\mathbb{R}$ is in $S^pQ-AP(\mathbb{R}:X)$ but not in class $Q-AP(\mathbb{R}:X)$, because $f(\cdot)$ is not continuous. Furthermore, any (S^p-AAP) is (S^p-Q-AP), so $S^pAAP(I:X) \subseteq S^pQ-AP(I:X)$. If $1\leq p<p'<\infty$, then $S^{p'}Q-AP(I:X) \subseteq S^pQ-AP(I:X)$ and for every function $f \in L^p_S(I:X)$, we have that $f(\cdot)$ is (S^p-Q-AP) iff the function $\check{f}: I \to L^p([0,1])$ defined by $\check{f}(t)(s) = f(t+s)$ is (Q-AP).

Let us recall that a continuous function $f: I \to X$ is said to be (AAut) i for every real sequence (b_n) there exist a subsequence (a_n) of (b_n) and a map $g: I \to X$ such that $\lim_{n\to\infty} f(t+a_n) = g(t)$ and $\lim_{n\to\infty} g(t-a_n) = f(t)$, pointwise for $t \in I$. The space of all automorphic functions $f: I \to X$ will be denoted by $AA(I:X)$.

A bounded continuous function $f: I \to X$ is said to be (AAAut) if there exist two functions $h \in AAut(I:X)$ and $q \in C_0(I:X)$ such that $f = h+q$ on I. The notion of Stepanov p-almost automorphy(S^p-AAut) has been introduced by G. M. N'Guérékata and A. Pankov in [19]: A function $f \in L^p_{loc}(I:X)$ is called (S^p-AAut) iff for every real sequence (a_n), there exists a subsequence (a_{n_k}) and a function $g \in L^p_{loc}(I:X)$ such that

$$\lim_{k\to\infty} \int_t^{t+1} \left\| f\left(a_{n_k}+s\right) - g(s) \right\|^p ds = 0$$

and

$$\lim_{k\to\infty} \int_t^{t+1} \left\| g\left(s-a_{n_k}\right) - f(s) \right\|^p ds = 0$$

for each $t \in I$. The vector space of all (S^p-AAut) functions will be denoted by $S^pAAut(I:X)$. A function $f \in L^p_{loc}(I:X)$ is called (S^p-AAAut) iff there exists an (S^p-AAut)function $g(\cdot)$ and a function $q \in L^p_S(I:X)$ such that $f(t) = g(t)+q(t)$, $t \geq 0$ and $\hat{q} \in C_0(I:L^p([0,1]:X))$. The vector space consisting of all (S^p-AAAut) functions will be denoted by $S^pAAAut(I:X)$.

Concerning distribution spaces, we will use the following elementary notion (cf. L. Schwartz [28] for more details). By $\mathcal{D}(X) = \mathcal{D}(\mathbb{R}:X)$ we denote the Schwartz space of test functions with values in X, by $\mathcal{S}(X) = \mathcal{S}(\mathbb{R}:X)$ we denote the space of rapidly decreasing functions with values in X, and by $\mathcal{E}(X) = \mathcal{E}(\mathbb{R}:X)$ we denote the space of all infinitely differentiable functions with values in X; $\mathcal{D} \equiv \mathcal{D}(\mathbb{C})$, $\mathcal{S} \equiv \mathcal{S}(\mathbb{C})$ and $\mathcal{E} \equiv \mathcal{E}(\mathbb{C})$. Here $\mathcal{S}(\mathbb{C})$ denotes the space of complex valued rapidly decreasing functions on $\mathbb{R}$. The same for $\mathcal{E}(\mathbb{C})$ and $\mathcal{D}(\mathbb{C})$. The spaces of all linear continuous mappings from $\mathcal{D}$, $\mathcal{S}$ and $\mathcal{E}$ into X will be denoted by $\mathcal{D}'(X)$, $\mathcal{S}'(X)$ and $\mathcal{E}'(X)$, respectively. Set $\mathcal{D}_0 := \{\varphi \in \mathcal{D} : \operatorname{supp}(\varphi) \subseteq [0,\infty)\}$.

1.1. **Vector-valued ultradistributions**

In this section the approach of Komatsu to the vector-valued ultradistributions will be followed, with the sequence (M_p) of positive real numbers satisfying $M_0 = 1$ and the following conditions: (M.1): $M_p^2 \leq M_{p+1}M_{p-1}, \; p \in \mathbb{N}$, (M.2): $M_p \leq AH^p \sup_{0\leq i\leq p} M_i M_{p-i}, \; p \in \mathbb{N}$, for some $A, \; H > 1$, (M.3'): $\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty$. Any use of the condition
(M.3): $\sup_{p\in\mathbb{N}} \sum_{q=p+1}^{\infty} \frac{M_{q-1}M_{p+1}}{pM_pM_q} < \infty$, which is slightly stronger than (M.3$'$), will be explicitly emphasized.

Let us recall that the Gevrey sequence $(p!^s)$, $s > 1$, satisfies the above conditions. Set $m_p := \frac{M_p}{M_{p-1}}$, $p \in \mathbb{N}$.

The space of Beurling, resp., Roumieu ultradifferentiable functions, is defined by $\mathcal{D}^{(M_p)} := \text{indlim}_{K\Subset\Subset\mathbb{R}} \mathcal{D}_K^{(M_p)}$, resp., $\mathcal{D}^{\{M_p\}} := \text{indlim}_{K\Subset\Subset\mathbb{R}} \mathcal{D}_K^{\{M_p\}}$, where
$\mathcal{D}_K^{(M_p)} := \text{projlim}_{h\to\infty} \mathcal{D}_K^{M_p,h}$, resp., $\mathcal{D}_K^{\{M_p\}} := \text{indlim}_{h\to 0} \mathcal{D}_K^{M_p,h}$,

$$\mathcal{D}_K^{M_p,h} := \left\{ \phi \in C^\infty(\mathbb{R}) : \text{supp}\phi \subseteq K, \; \|\phi\|_{M_p,h,K} < \infty \right\}$$

and

$$\|\phi\|_{M_p,h,K} := \sup \left\{ \frac{h^p |\phi^{(p)}(t)|}{M_p} : t \in K, \; p \in \mathbb{N}_0 \right\}.$$

Henceforward, the asterisk $*$ is used to denote both, the Beurling case (M_p) or the Roumieu case $\{M_p\}$. Set $\mathcal{D}_0^* := \{\varphi \in \mathcal{D}^* : \text{supp}(\varphi) \subseteq [0,\infty)\}$. The space consisted of all continuous linear functions from $\mathcal{D}^*$ into X, denoted by $\mathcal{D}'^*(X) := L(\mathcal{D}^* : X)$, is said to be the space of all X-valued ultradistributions of $*$-class. We also need the notion of space $\mathcal{E}^*(X)$, defined as $\mathcal{E}^*(X) := \text{indlim}_{K\Subset\Subset\mathbb{R}} \mathcal{E}_K^*(X)$, where in Beurling case $\mathcal{E}_K^{(M_p)}(X) := \text{projlim}_{h\to\infty} \mathcal{E}_K^{M_p,h}(X)$, resp., in Roumieu case $\mathcal{E}_K^{\{M_p\}}(X) := \text{indlim}_{h\to 0} \mathcal{E}_K^{M_p,h}(X)$, and

$$\mathcal{E}_K^{M_p,h}(X) := \left\{ \phi \in C^\infty(\mathbb{R} : X) : \sup_{p\geq 0} \frac{h^p \|\phi^{(p)}\|_{C(K:X)}}{M_p} < \infty \right\}.$$

The space consisting of all linear continuous mappings $\mathcal{E}^*(\mathbb{C}) \to X$ is denoted by $\mathcal{E}'^*(X)$; $\mathcal{E}'^* := \mathcal{E}'^*(\mathbb{C})$. Notation $\mathcal{E}(\mathbb{C})$ means that we consider ultradifferentiable functions on $\mathbb{R}$ with values in $\mathbb{C}$. An entire function of the form $P(\lambda) = \sum_{p=0}^{\infty} a_p \lambda^p$, $\lambda \in \mathbb{C}$, (a_p are complex numbers as well) is of class (M_p), resp., of class $\{M_p\}$, if there exist $l > 0$ and $C > 0$, resp., for every $l > 0$ there exists a constant $C > 0$, such that $|a_p| \leq Cl^p/M_p$, $p \in \mathbb{N}$ ([20]). The corresponding ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ is said to be of class (M_p), resp., of class $\{M_p\}$; it acts as a continuous linear operator between the spaces $\mathcal{D}^*$ and $\mathcal{D}^*$ ($\mathcal{D}'^*$ and $\mathcal{D}'^*$). The convolution of Banach space valued ultradistributions and scalar-valued ultradifferentiable functions of the same class will be taken in the sense of considerations given on page 685 of [22]. Let remind ourselves that, for every $f \in \mathcal{D}'^*(X)$ and

$\varphi \in \mathcal{D}^*$, we have $f * \varphi \in \mathcal{E}^*(X)$ as well as that the linear mapping $\varphi \mapsto \cdot * \varphi : \mathcal{D}'^*(X) \to \mathcal{E}^*(X)$ is continuous. The convolution of an X-valued ultradistribution $f(\cdot)$ and a scalar-valued ultradistibution $g \in \mathcal{E}'^*$ with compact support, defined by the identity $\langle f * g, \phi \rangle = \langle f, \check{g} * \phi \rangle$, where $\check{g}(x) = g(-x)$ and ϕ is a test function in $\mathcal{D}^*$, is an X-valued ultradistribution and the mapping $g * \cdot : \mathcal{D}'^*(X) \to \mathcal{D}'^*(X)$ is continuous. Set $\langle T_h, \varphi \rangle := \langle T, \varphi(\cdot - h) \rangle$, $T \in \mathcal{D}'^*(X)$, $\varphi \in \mathcal{D}^*$ ($h > 0$). We will use a similar definition for vector-valued distributions.

Assume that the sequence (M_p) satisfies (M.1), (M.2) and (M.3). Then

$$P_l(x) = (1 + x^2) \prod_{p \in \mathbb{N}} \left(1 + \frac{x^2}{l^2 m_p^2} \right),$$

resp.

$$P_{r_p}(x) = (1 + x^2) \prod_{p \in \mathbb{N}} \left(1 + \frac{x^2}{r_p^2 m_p^2} \right),$$

defines an ultradifferential operator of class (M_p), resp., of class $\{M_p\}$; here, (r_p) is a sequence of positive real numbers tending to infinity. The family consisting of all such sequences will be denoted by R henceforth. For more details on the subject, the reader may consult [20]- [22].

The spaces of tempered ultradistributions of Beurling, resp., Roumieu type, are defined by S. Pilipović [27] as duals of the corresponding test spaces

$$\mathcal{S}^{(M_p)} := \text{projlim}_{h \to \infty} \mathcal{S}^{M_p,h}, \text{ resp., } \mathcal{S}^{\{M_p\}} := \text{indlim}_{h \to 0} \mathcal{S}^{M_p,h},$$

where

$$\mathcal{S}^{M_p,h} := \left\{ \phi \in C^\infty(\mathbb{R}) : \|\phi\|_{M_p,h} < \infty \right\} \ (h > 0),$$

$$\|\phi\|_{M_p,h} := \sup \left\{ \frac{h^{\alpha+\beta}}{M_\alpha M_\beta} (1 + t^2)^{\beta/2} |\phi^{(\alpha)}(t)| : t \in \mathbb{R},\ \alpha,\ \beta \in \mathbb{N}_0 \right\}.$$

2. Quasi-asymptotical almost periodicity of vector-valued distributions

We refer the reader to [5], [10] and [26] for the basic results about vector-valued (AP) distributions. Let $1 \leq p \leq \infty$. Then by $\mathcal{D}_{L^p}(X)$ we denote the vector space consisting of all infinitely differentiable functions $f : \mathbb{R} \to X$ satisfying that for each number $j \in \mathbb{N}_0$ we have $f^{(j)} \in L^p(\mathbb{R} : X)$. The Fréchet topology on $\mathcal{D}_{L^p}(X)$ is induced by the following system of seminorms

$$\|f\|_k := \sum_{j=0}^{k} \left\| f^{(j)} \right\|_{L^p(\mathbb{R})}, \quad f \in \mathcal{D}_{L^p(X)} \ \ (k \in \mathbb{N}).$$

If $X = \mathbb{C}$, then the above space is simply denoted by $\mathcal{D}_{L^p}$. A linear continuous mapping $f : \mathcal{D}_{L^1} \to X$ is said to be a bounded X-valued distribution; the space consisting of such vector-valued distributions will be denoted by $\mathcal{D}'_{L^1}(X)$. Endowed with the strong topology, $\mathcal{D}'_{L^1}(X)$ becomes a complete locally convex space. For every $f \in \mathcal{D}'_{L^1}(X)$, we have that $f_{|\mathcal{S}} : \mathcal{S} \to X$ is a tempered X-valued distribution ([24]).

The space of bounded vector-valued distributions tending to zero at plus infinity, $B'_{+,0}(X)$ for short, is defined by

$$B'_{+,0}(X) := \Big\{ T \in \mathcal{D}'_{L^1}(X) : \lim_{h\to+\infty} \big\langle T_h, \varphi \big\rangle = 0 \text{ for all } \varphi \in \mathcal{D} \Big\}.$$

It can be simply verified that the structural characterization for the space $B'_{+,0}(\mathbb{C})$, (scalar valued case usually we will denote by $B'_{+,0}$) proved in [12, Proposition 1], is still valid in vector-valued case as well that the space $B'_{+,0}(\mathbb{C})$ is closed under differentiation.

Let $T \in \mathcal{D}'_{L^1}(X)$. Then the following assertions are equivalent ([26]):

(i) $T * \varphi \in AP(\mathbb{R} : X)$, $\varphi \in \mathcal{D}$, resp., $T * \varphi \in AA(\mathbb{R} : X)$, $\varphi \in \mathcal{D}$.

(ii) There exist an integer $k \in \mathbb{N}$ and (AP), resp. (AAut) functions $f_j(\cdot) : [0,\infty) \to X$ $(1 \le j \le k)$ such that $T = \sum_{j=0}^{k} f_j^{(j)}$.

We say that a distribution $T \in \mathcal{D}'_{L^1}(X)$ is (AP), resp. (AAut), if T satisfies any of the above two equivalent conditions. By $B'_{AP}(X)$, $B'_{AA}(X)$ we denote the space consisting of all (AP), resp. (AAut), distributions.

Definition 2.1. *A distribution $T \in \mathcal{D}'_{L^1}(X)$ is said to be (AAP), resp. (AAAut), if there exist an (AP), resp. (AAut), distribution $T_{ap} \in B'_{AP}(X)$, resp. $T_{aa} \in B'_{AA}(X)$, and a bounded distribution tending to zero at plus infinity $Q \in B'_{+,0}(X)$ such that $\langle T, \varphi\rangle = \langle T_{ap}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0$, resp. $\langle T, \varphi\rangle = \langle T_{aa}, \varphi\rangle + \langle Q, \varphi\rangle$, $\varphi \in \mathcal{D}_0$.*

By $B'_{AAP}(X)$, resp. $B'_{AAAut}(X)$, we denote the vector space consisting of all (AAP), resp. (AAAut) distributions.

It is well known that the representation $T = T_{ap} + Q$ is unique in almost periodic case.

Definition 2.2. *A distribution $T \in \mathcal{D}'_{L^1}(X)$ is said to be (Q-AP) distribution if $T * \varphi \in Q-AP(\mathbb{R} : X)$, for all $\varphi \in \mathcal{D}$. The space of all (Q-AP) distributions will be denoted by $\mathcal{D}'_{Q-AP}(X)$.*

Theorem 2.1. *The following statements hold:*

i) $\mathcal{D}'_{Q-AP}(X) \cap \mathcal{D}'_{AAAut}(X) = \mathcal{D}'_{AAP}(X)$;
$[\mathcal{D}'_{AAAut}(X) \setminus \mathcal{D}'_{AAP}(X)] \cap \mathcal{D}'_{Q-AP}(X) = \emptyset$;

ii) $\mathcal{D}'_{AA}(X) \cap \mathcal{D}'_{Q-AAP}(X) = \mathcal{D}'_{AP}(X)$.

Proof. The proof use the same arguments like in [25]. For completeness, we will give in sequel.

Let we consider only the case when $I = \mathbb{R}$. The other case $I = [0,\infty)$ is analogous. Since $\mathcal{D}'_{AAP}(X) \subseteq \mathcal{D}'_{AAAut}(X) \cap \mathcal{D}'_{Q-AP}(X)$, we will prove the opposite inclusion. Let $T \in \mathcal{D}'_{AAAut} \cap \mathcal{D}'_{Q-AP}(X)$ and $\varphi \in \mathcal{D}$ is arbitrary. Then $T * \varphi \in AAAut(\mathbb{R} : X) \cap Q-AP(\mathbb{R} : X)$. Since $T * \varphi \in AAAut(\mathbb{R} : X)$, there exist two functions $f \in AAut(\mathbb{R} : X)$ and $g \in C_0(\mathbb{R} : X)$, such that $T * \varphi = f + g$ on $\mathbb{R}$ and for every $\varepsilon > 0$,

(S) holds and there exists a number $M(\varepsilon,\tau) > 0$ such that

$$\|(f(t+\tau) - f(t)) + (g(t+\tau) - g(t))\| \leq \varepsilon, \text{ for } t \in \mathbb{R} \text{ and } |t| \geq M(\varepsilon,\tau). \tag{2.1}$$

Let $\varepsilon > 0$ be fixed and the real number τ satisfies (2.1) for $|\tau| \geq M(\varepsilon,\tau)$. By $g \in \mathcal{C}_0(\mathbb{R} : X)$, there exists a finite number $M_1(\varepsilon,\tau) \geq M(\varepsilon,\tau)$ such that $\|g(t+\tau) - g(t)\| \leq \frac{\varepsilon}{2}$ for $t \in \mathbb{R}$, $|t| \geq M_1(\varepsilon,\tau)$. We define the function $F : \mathbb{R} \to X$, $F(t) = f(t+\tau) - f(t)$, $t \in \mathbb{R}$. The space $AAut(\mathbb{R} : X)$ is translation invariant, $F \in AAut(\mathbb{R} : X)$. Using supremum formula, we have

$$\sup_{t\in\mathbb{R}} \|F(t)\| = \sup_{t\geq M_1(\varepsilon,\tau)} \|F(t)\| = \sup_{t\geq M_1(\varepsilon,\tau)} \|f(t+\tau) - f(t)\| \leq \frac{\varepsilon}{2}.$$

So, $\|f(t+\tau) - f(t)\| \leq \frac{\varepsilon}{2}$ for all $t \in \mathbb{R}$, so f is (AP) function. Hence, $\mathcal{D}'_{AAP}(X) \subseteq \mathcal{D}'_{Q-AP}(X)$, so the equation in (i) holds. The equation in (ii), follows from the proof of (i). $\square$

Definition 2.3. *Let* $\omega \in I$.

a) *A bounded continuous function* $f : I \to X$ *is said to be asymptotically* ω*-almost periodic ((*AAp_ω*) in short) if there exist a function* $g \in \mathcal{C}_\omega(I : X)$ *and a function* $q \in \mathcal{C}_0(I : X)$ *such that* $f(t) = g(t) + q(t)$ *for all* $t \in I$;
b) *A bounded continuous function* $f : I \to X$ *is said to be S-asymptotically* ω*-periodic ((*$S-AP_\omega$*) in short) if* $\lim_{|t|\to\infty} \|f(t+\omega) - f(t)\| = 0$. *Denote by* $S-AP_\omega(I : X)$ *the space consisting of all such functions;*
c) *A Stepanov p-bounded function* $f(\cdot)$ *is said to be Stepanov p-asymptotically* ω*-periodic ((*$S^p - AP_\omega$*) in short) if*

$$\lim_{|t|\to\infty} \int_t^{t+1} \|f(s+\omega) - f(s)\|^p \, ds = 0.$$

The space of all ($S^p - AP_\omega$*) functions is denoted by* $S^p - AP_\omega(I : X)$. *Note that* $S - AP_\omega(I : X) \subseteq S^p - AP_\omega(I : X)$.

Definition 2.4. *Let* $\omega \in I$ *and* $T \in \mathcal{D}'_{L^1}(X)$.

a) *A distribution* T *is said to be (*AAp_ω*) if* $T * \varphi \in AAP_\omega(I : X)$ *for every* $\varphi \in \mathcal{D}$. *The space of all asymptotically* ω*-almost periodic distributions is denoted by* $\mathcal{D}'_{AAP_\omega}(X)$.
b) *A distribution* T *is said to be (*$S^p - AP_\omega$*) if* $T * \varphi \in S^p - AP_\omega(I : X)$ *for every* $\varphi \in \mathcal{D}$. *The space of all (*$S^p - AP_\omega$*) distributions is denoted by* $\mathcal{D}'_{S^p-AP_\omega}(X)$.
c) *A distribution* T *is said to be (*$S - AP_\omega$*) if* $T * \varphi \in S - AP_\omega(I : X)$, *for every* $\varphi \in \mathcal{D}$. *The space of all (*$S - AP_\omega$*) distributions is denoted by* $\mathcal{D}'_{S-AP_\omega}(X)$.
d) *A distribution* T *is said to be (*$S^p - AAAut$*) if* $T * \varphi \in S^p - AAAut(I : X)$, *for every* $\varphi \in \mathcal{D}$. *The space of all (*$S^p - AAAut$*) distributions is denoted by* $\mathcal{D}'_{S^pAAA}(X)$.
e) *A distribution* T *is said to be (*$S^p - AAut$*) if* $T * \varphi \in S^p - AAut(I : X)$, *for every* $\varphi \in \mathcal{D}$. *The space of all (*$S^p - AAut$*) distributions is denoted by* $\mathcal{D}'_{S^p-AAut}(X)$.

Definition 2.5. *Let $T \in \mathcal{D}'_{L^1}(X)$.*

a) A distribution T is said to be (S^pQ-AP) if $T * \varphi \in S^pQ-AP(I:X)$, for every $\varphi \in \mathcal{D}$. The space of all (S^pQ-AP) distributions is denoted by $\mathcal{D}'_{S^pQ-AP}(X)$.

b) A distribution T is said to be ($SQ-P_\omega$) if $T * \varphi \in SQ-AP_\omega(I:X)$, for every $\varphi \in \mathcal{D}$. The space of all ($SQ-P_\omega$) distributions is denoted by $\mathcal{D}'_{SQ-AP_\omega}(X)$.

c) A distribution T is said to be ($Q-AP_\omega$) if $T * \varphi \in Q-AP_\omega(I:X)$, for every $\varphi \in \mathcal{D}$. The space of all ($Q-AP_\omega$) distributions is denoted by $\mathcal{D}'_{Q-AP_\omega}(X)$.

d) A distribution T is said to be (S^pQ-AP_ω) if $T * \varphi \in S^pQ-AP_\omega(I:X)$, for every $\varphi \in \mathcal{D}$. The space of all (S^pQ-AP_ω) distributions is denoted by $\mathcal{D}'_{S^pQ-AP_\omega}(X)$.

e) A distribution T is said to be (S^pQ-AP) if $T * \varphi \in S^pQ-AP(I:X)$, for every $\varphi \in \mathcal{D}$. The space of all (S^pQ-AP) distributions is denoted by $\mathcal{D}'_{S^pQ-AP}(X)$.

The next theorem follows from [25, Proposition 2.7].

Theorem 2.2. *Let $\omega \in I$. It holds $\mathcal{D}'_{S-AP_\omega}(X) \subseteq \mathcal{D}'_{Q-AP}(X)$.*

Proof. Let $\varepsilon > 0$ be given and $T \in \mathcal{D}'_{S-AP_\omega}(X)$. Take $L(\varepsilon) = 2\omega$. Then for any interval $I' \subseteq I$ of length $L(\varepsilon)$ contains a number $\tau = n\omega > 0$ such that

$$\|T * \varphi(t+\omega) - T * \varphi(t)\| < \frac{\varepsilon}{2\omega}, \quad \text{for} \quad |t| \geq M(\varepsilon, n).$$

Hence,

$$\|T * \varphi(t+n\omega) - T * \varphi(t)\| \leq \sum_{k=0}^{n-1} \|f(t+\tau-k\omega) - f(t+\omega-(k+1)\omega)\| \leq \frac{n\varepsilon}{n\omega} = \varepsilon,$$

for $|t| \geq M(\varepsilon, n) + n\omega$, so the conclusion of the theorem follows. □

Theorem 2.3. *Let $\omega \in I$. Then*

$$\mathcal{D}'_{S-AP_\omega}(X) \cap \mathcal{D}'_{AAAut}(X) \subseteq \mathcal{D}'_{AP_\omega}(X).$$

Proof. Let $T \in \mathcal{D}'_{S-AP_\omega}(X) \cap \mathcal{D}'_{AAAut}(X) \subseteq \mathcal{D}'_{AP_\omega}(X)$ and $\varphi \in \mathcal{D}$. Then by the definitions of $\mathcal{D}'_{S-AP_\omega}(X)$ and $\mathcal{D}'_{AAAut}(X)$, $T * \varphi \in S-AP_\omega(I:X) \cap AAAut(I:X)$. Since, $T * \varphi \in AAAut(I:X)$, there exist $g \in AAut(I:X)$ and $h \in C_0([0,\infty):X)$ such that $T * \varphi = g + h$. It is suffices to prove that $g \in P_\omega(I:X)$. Since, $C_0([0,\infty):X) \subseteq S-AP_\omega(X)$ it follows that $g = T * \varphi - h \in S-AP_\omega(X)$. Hence,

$$\lim_{t\to\infty} \|g(t+\omega) - g(t)\| = 0. \tag{2.2}$$

Now, since g is (AAut), we can find a subsequence (t_k) of (t_n) such that for all $t \in \mathbb{R}$,

$$\lim_{m\to\infty} \lim_{k\to\infty} g(t+\omega-t_{n_k}-t_{n_m}) - g(t+t_{n_k}-t_{n_m}) = g(t+\omega) - g(t).$$

From (2.2) and $\lim_{k\to\infty}(t_{n_k} - t_{n_m}) = +\infty$, we have

$$\lim_{k\to\infty} g(t+\omega+t_{n_k}-t_{n_m}) - g(t+t_{n_k}-t_{n_m}) = 0,$$

for all $t \in I$ so $g(t+\omega) - g(t) = 0$, for all $t \in I$. This implies $g(t+\omega) - g(t) = 0$ for all $t \in I$, so the proof is finished. □

Now, using the technique in the previous result we can give the following:

Theorem 2.4. *The following statements hold:*

i) $\mathcal{D}'_{S^p-AAAut}(X) \cap \mathcal{D}'_{S^pQ-AP}(X) = \mathcal{D}'_{S^p-AAP}(X)$
$[\mathcal{D}'_{S^p-AAAut}(X) \backslash \mathcal{D}'_{S^p-AAP}(X)] \cap \mathcal{D}'_{S^pQ-AP}(X) = \emptyset$*;*

ii) $\mathcal{D}'_{S^p-AAut}(X) \cap \mathcal{D}'_{S^pQ-AP}(X) = \mathcal{D}'_{S^p-AP}(X)$.

The proof is quit similar like of Theorem 2.3, so we omit it.

Theorem 2.5. *It holds that* $\mathcal{D}'_{S^p-AP_\omega}(X) \subseteq \mathcal{D}'_{S^pQ-AP}(X)$.

Theorem 2.6. *Let* $T \in \mathcal{D}'_{L^1}(X)$. *The then following holds:*

i) Let $T \in \mathcal{D}'_{Q-AP}(X)$*, resp.,* $T \in \mathcal{D}'_{S^pQ-AP}(X)$*. Then* cT *is a (Q-AP) distribution, resp., (*S^PQ-AP*) distribution, for any* $c \in \mathbb{C}$*;*

ii) If (T_n) *is a sequence in* $\mathcal{D}'_{Q-AP}(X)$*, resp.,* $\mathcal{D}'_{S^pQ-AP}(X)$ *and* $T_n \to T$ *uniformly in* $\mathcal{D}'_{Q-AP}(X)$*, resp.,* $\mathcal{D}'_{S^pQ-AP}(X)$*, then* $T \in \mathcal{D}'_{Q-AP}(X)$*, resp.,* $T \in \mathcal{D}'_{S^pQ-AP}(X)$*;*

iii) Any translation $T_h = \langle T, \varphi(\cdot - h)\rangle$ *of* $T \in \mathcal{D}'_{Q-AP}(X)$ *(*$T \in \mathcal{D}'_{S^pQ-AP}(X)$*) is again in* $\mathcal{D}'_{Q-AP}(X)$ *(*$\mathcal{D}'_{S^pQ-AP}(X)$*).*

Proof. The case when $T \in \mathcal{D}'_{S^pQ-AP}(X)$ is similar with the case when $T \in \mathcal{D}'_{Q-AP}(X)$, so we skip it.
i) Let $T \in \mathcal{D}'_{Q-AP}(X)$ and $\varphi \in \mathcal{D}$. Then $T * \varphi \in Q-AP(I : X)$. Since, $(cT) * \varphi = c(T * \varphi)$, using [25, Theorem 2.13 (i)], the statement of i) follows.
ii) Let $\varepsilon > 0$ be given. Then there exists a distribution T_{n_0} such that $|T * \varphi(x) - T_{n_0} * \varphi(x)| < \frac{\varepsilon}{3}$, for all $x \in I$, $|x| \geq M(\varepsilon, \tau)$, for some $M(\varepsilon, \tau) > 0$ and for all $\varphi \in \mathcal{D}$. Let $E_q(\varepsilon, T)$, be the set of all translation numbers of T belonging to $\varepsilon \geq 0$ if $|T * \varphi(x+\tau) - T * \varphi(x)| \leq \varepsilon$, for all $\varphi \in \mathcal{D}$. Put $\tau \in E(\frac{1}{3}\varepsilon, T_{n_0} * \varphi(x))$. Then,

$$\begin{aligned}|T * \varphi(x+\tau) - T * \varphi(x) &\leq |T * \varphi(x+\tau) - T_{n_0} * \varphi(x+\tau)| \\ &+ |T_{n_0} * \varphi(x+\tau) - T_{n_0} * \varphi(x)| + |T_{n_0} * \varphi(x) - T * \varphi(x)| < \varepsilon.\end{aligned}$$

Since $E_q(\varepsilon, T * \varphi(x)) \supset E_q(\frac{1}{3}\varepsilon, T_{n_0} * \varphi(x))$, it follows $E_q(\varepsilon, T * \varphi(x))$ is relatively dense and $\varepsilon > 0$ is arbitrary, we conclude that T is (Q-AP) distribution.
iii) Let $T \in \mathcal{D}'_{Q-AP}(X)$ and $\varphi \in \mathcal{D}$ Then $T * \varphi \in Q-AP(I : X)$. Now, since $T_h * \varphi = \langle T, \varphi(\cdot - h)\rangle = (\tau_h T) * \varphi = \tau_h(T * \varphi)$, by [25, Theorem 2.13 (v)], $\tau_h(T * \varphi) \in Q-AAP(I : X)$. Hence $T_h \in \mathcal{D}'_{Q-AP}(X)$. $\square$

Further on, we would like to observe that the following structural result holds in vector-valued case:

Theorem 2.7. *Let* $T \in \mathcal{D}'_{L^1}(X)$. *Then the following assertions are equivalent:*

(i) $T \in D'_{Q-AP}(X)$.

(ii) There exist an integer $k \in \mathbb{N}$ *and (Q-AP) functions* $f_j(\cdot) : \mathbb{R} \to X$ *(*$0 \leq j \leq k$*) such that* $T = \sum_{j=0}^{k} f_j^{(j)}$ *on* $[0, \infty)$.

(iii) *There exist* $S \in \mathcal{D}'_{L^1}(X)$, $k \in \mathbb{N}$ *and bounded (Q-AP) functions* $f_j(\cdot) : \mathbb{R} \to X$ *(*$0 \leq j \leq k$*) such that* $S = \sum_{j=0}^{k} f_j^{(j)}$ *on* $\mathbb{R}$, *and* $\langle S, \varphi \rangle = \langle T, \varphi \rangle$ *for all* $\varphi \in \mathcal{D}_0$.

(iv) *There exists* $S \in \mathcal{D}'_{L^1}(X)$ *such that* $\langle S, \varphi \rangle = \langle T, \varphi \rangle$ *for all* $\varphi \in \mathcal{D}_0$ *and* $S * \varphi \in Q - AP(\mathbb{R} : X)$, $\varphi \in \mathcal{D}$.

Proof. The equivalence of (i)-(ii) can be proved as in (Q-AP) scalar-valued case (see [12, Theorem I, Proposition 1]).

The implication (ii) $\Rightarrow$ (iii) trivially follows from the fact that the expression $S = \sum_{j=0}^{k} f_j^{(j)}$ defines an element from $\mathcal{D}'_{L^1}(X)$. Since the space $\mathrm{A} \equiv Q - AP(\mathbb{R} : X) \cap C_b(\mathbb{R} : X)$ is uniformly closed (and therefore, C^∞-uniformly closed), closed under addition and $\mathrm{A} * \mathcal{D} \subseteq \mathrm{A}$ (see [1] for the notion), so we have that (iii) implies (iv). The implication (iv) $\Rightarrow$ (ii) is trivial, hence we have the equivalence of assertions (i)-(iv). $\square$

Concerning the assertions of Theorem 2.7, it is worth noting the following:

Remark 2.1. The validity of (iv) for some $S \in \mathcal{D}'_{L^1}(X)$ implies its validity for S replaced therein with $S_Q = S + Q$, where $Q \in \mathcal{D}'_{L^1}(X)$ and $\mathrm{supp}(Q) \subseteq (-\infty, 0]$. For this, it suffices to observe that $(Q * \varphi)(x) = \langle Q, \varphi(x - \cdot) \rangle = 0$ for all $x \geq \sup(\mathrm{supp}(\varphi))$, $\varphi \in \mathcal{D}$.

3. Quasi-Asymptotical almost periodicity of vector-valued ultradistributions

For any $h > 0$, we define

$$\mathcal{D}_{L^1}^{M_p,h} := \left\{ f \in \mathcal{D}_{L^1} \; ; \; \|f\|_{1,h} := \sup_{p \in \mathbb{N}_0} \frac{h^p \|f^{(p)}\|_1}{M_p} < \infty \right\}.$$

Then $(\mathcal{D}_{L^1}^{M_p,h}, \|\cdot\|_{1,h})$ is a Banach space and the space of all X-valued bounded Beurling ultradistributions of class (M_p), resp., X-valued bounded Roumieu ultradistributions of class $\{M_p\}$, is defined to be the space consisting of all linear continuous mappings from $\mathcal{D}_{L^1}^{(M_p)}$, resp., $\mathcal{D}_{L^1}^{\{M_p\}}$, into X, where

$$\mathcal{D}_{L^1}^{(M_p)} := \mathrm{projlim}_{h \to +\infty} \mathcal{D}_{L^1}^{M_p,h},$$

resp.,

$$\mathcal{D}_{L^1}^{\{M_p\}} := \mathrm{indlim}_{h \to 0+} \mathcal{D}_{L^1}^{M_p,h}.$$

These spaces, equipped with the strong topologies, will be shortly denoted by $\mathcal{D}'_{L^1}(X)$, resp., $\mathcal{D}_{L^1}^{\prime\{M_p\}}(X)$. We will use the shorthand $\mathcal{D}_{L^1}^{\prime *}(X)$ to denote either $\mathcal{D}_{L^1}^{\prime(M_p)}(X)$ or $\mathcal{D}_{L^1}^{\prime\{M_p\}}(X)$; $\mathcal{D}_{L^1}^{\prime *} \equiv \mathcal{D}_{L^1}^{\prime *}(\mathbb{C})$. As it can be easily proved, $\mathcal{D}_{L^1}^{\prime *}(X)$ is a complete locally convex space.

It is well known that $\mathcal{D}^{(M_p)}$, resp. $\mathcal{D}^{\{M_p\}}$, is a dense subspace of $\mathcal{D}_{L^1}^{(M_p)}$, resp., $\mathcal{D}_{L^1}^{\{M_p\}}$, as well as that $\mathcal{D}_{L^1}^{(M_p)} \subseteq \mathcal{D}_{L^1}^{\{M_p\}}$. It can be simply proved that $f_{|\mathcal{S}^{(M_p)}}$:

$\mathcal{S}^{(M_p)} \to X$, resp., $f_{|\mathcal{S}^{\{M_p\}}} : \mathcal{S}^{\{M_p\}} \to X$, is a tempered X-valued ultradistribution of class (M_p), resp., of class $\{M_p\}$. The space $\mathcal{D}'^{*}_{L^1}(X)$ is closed under the action of ultradifferential operators of $*$-class.

The space of bounded vector-valued ultradistributions tending to zero at plus infinity, $\mathcal{D}'^{*}_{+,0}(X)$ for short, is defined by

$$\mathcal{D}'^{*}_{+,0}(X) := \Big\{ T \in \mathcal{D}'^{*}_{L^1}(X) : \lim_{h\to+\infty} \big\langle T_h, \varphi \big\rangle = 0 \text{ for all } \varphi \in \mathcal{D}^* \Big\}.$$

Let $T \in \mathcal{D}'^{*}_{L^1}(X)$. Then we say that T is (AP), resp. (AAut), if T satisfies: $T * \varphi \in AP(\mathbb{R} : X)$, $\varphi \in \mathcal{D}^*$, resp., $T * \varphi \in AAut(\mathbb{R} : X)$, $\varphi \in \mathcal{D}^*$. By $\mathcal{D}'^{*}_{AP}(X)$, we denote the vector space consisting of all almost periodic ultradistributions of $*$-class.

Definition 3.1. *An ultradistribution $T \in \mathcal{D}'^{*}_{L^1}(X)$ is said to be (AAP) if there exist an (AP) ultradistribution $T_{ap} \in \mathcal{D}'^{*}_{AP}(X)$, and a bounded ultradistribution tending to zero at plus infinity $Q \in \mathcal{D}'^{*}_{+,0}(X)$ such that $\langle T, \varphi \rangle = \langle T_{ap}, \varphi \rangle + \langle Q, \varphi \rangle$, $\varphi \in \mathcal{D}^{*}_{0}$.*

By $\mathcal{D}'^{*}_{AAP}(X)$, we denote the vector space consisting of all (AAP) ultradistributions of $*$-class.

Likewise in distribution case, decomposition of an (AAP) ultradistribution of $*$-class into its (AP) part and bounded, tending to zero at plus infinity part, is unique. The space $\mathcal{D}'^{*}_{AAP}(X)$ is closed under the action of ultradifferential operators of $*$-class. This follows from the fact that this is true for the space $\mathcal{D}'^{*}_{AP}(X)$, (see [24] and [26]), as well as that, for every $Q \in \mathcal{D}'^{*}_{+,0}(X)$ and for every ultradifferential operator $P(D)$ of $*$-class, we have $\langle P(D)Q, \varphi(\cdot - h)\rangle = \langle Q, [P(D)\varphi](\cdot - h)\rangle$, $h \in \mathbb{R}$.

For the sequel, we need the following preparation. Let $\mathrm{A} \subseteq \mathcal{D}'^{*}(X)$. Following B. Basit and H. Güenzler [1], whose examinations have been carried out in distributional case, we have recently introduced the following notion in [24]:

$$\mathcal{D}'^{*}_{\mathrm{A}}(X) := \Big\{ T \in \mathcal{D}'^{*}(X) : T * \varphi \in \mathrm{A} \text{ for all } \varphi \in \mathcal{D}^* \Big\}.$$

Then $\mathcal{D}'^{*}_{\mathrm{A}}(X) = \mathcal{D}'^{*}_{\mathbb{A}\cap B}(X)$, for any set $B \subseteq L^1_{loc}(\mathbb{R} : X)$ that contains $C^\infty(\mathbb{R} : X)$, as well as the set $\mathcal{D}'^{*}_{\mathrm{A}}(X)$ is closed under the action of ultradifferential operators of $*$-class. Furthermore, the following holds [24]:

(i) Assume that there exist an ultradifferential operator $P(D) = \sum_{p=0}^{\infty} a_p D^p$ of class (M_p), resp., of class $\{M_p\}$, and $f,\, g \in \mathcal{D}'^{*}_{\mathrm{A}}(X)$ such that $T = P(D)f + g$. If A is closed under addition, then $T \in \mathcal{D}'^{*}_{\mathrm{A}}(X)$.

(ii) If $\mathrm{A} \cap C(\mathbb{R} : X)$ is closed under uniform convergence, $T \in \mathcal{D}'_{L^1}((M_p) : X)$ and $T * \varphi \in \mathrm{A}$, $\varphi \in \mathcal{D}^{(M_p)}$, then there is a number $h > 0$ such that for each compact set $K \subseteq \mathbb{R}$ we have $T * \varphi \in \mathrm{A}$, $\varphi \in \mathcal{D}^{M_p,h}_{K}$.

(iii) Assume that $T \in \mathcal{D}'^{(M_p)}(X)$ and there exists $h > 0$ such that for each compact set $K \subseteq \mathbb{R}$ we have $T * \varphi \in \mathrm{A}$, $\varphi \in \mathcal{D}^{M_p,h}_{K}$. If (M_p) additionally satisfies (M.3), then there exist $l > 0$ and two elements $f,\, g \in \mathrm{A}$ such that $T = P(D)f + g$.

Definition 3.2. *An ultradistribution $T \in \mathcal{D}'^{*}_{L^1}(X)$ is said to be (Q-AP) ultradistribution if $T * \varphi \in Q - AP(\mathbb{R} : X)$, of class $*$ for all $\varphi \in \mathcal{D}^{*}$. The space of all (Q-AP) ultradistributions will be denoted by $\mathcal{D}'^{*}_{Q-AP}(X)$.*

Definition 3.3. *Let $\omega \in I$ and $T \in \mathcal{D}'^{*}_{L^1}(X)$.*

a) *An ultradistribution T is said to be (AP_{ω}) if $T * \varphi \in AP_{\omega}(I : X)$ for every $\varphi \in \mathcal{D}^{*}$. The space of all ($AP_{\omega}$) ultradistributions is denoted by $\mathcal{D}'^{*}_{AP_{\omega}}(X)$.*
b) *An ultradistribution T is said to be ($S^p - AP_{\omega}$) if $T * \varphi \in S^p - AP_{\omega}(I : X)$ for every $\varphi \in \mathcal{D}^{*}$. The space of all ($S^p - AP_{\omega}$) ultradistributions is denoted by $\mathcal{D}'^{*}_{S^p-AP_{\omega}}(X)$.*
c) *An ultradistribution T is said to be ($S - AP_{\omega}$) if $T * \varphi \in S - AP_{\omega}(I : X)$, for every $\varphi \in \mathcal{D}^{*}$. The space of all ($S - AP_{\omega}$) ultradistributions is denoted by $\mathcal{D}'^{*}_{S-AP_{\omega}}(X)$.*
d) *An ultradistribution T is said to be ($S^p - AAAut$) if $T * \varphi \in S^p - AAAut(I : X)$, for every $\varphi \in \mathcal{D}^{*}$. The space of all ($S^p - AAAut$) ultradistributions is denoted by $\mathcal{D}'^{*}_{S^p-AAAut}(X)$.*
e) *An ultradistribution T is said to be ($S^p - AAut$) if $T * \varphi \in S^p - AAut(I : X)$, for every $\varphi \in \mathcal{D}^{*}$. The space of all ($S^p - AAut$) ultradistributions is denoted by $\mathcal{D}'^{*}_{S^p-AAut}(X)$.*

Definition 3.4. *Let $T \in \mathcal{D}'_{L^1}(X)$.*

a) *An ultradistribution T is said to be ($S^pQ - AP$) if $T * \varphi \in S^pQ - AP(I : X)$, for every $\varphi \in \mathcal{D}^{*}$. The space of all ($S^pQ - AP$) ultradistributions is denoted by $\mathcal{D}'^{*}_{S^pQ-AP}(X)$.*
b) *An ultradistribution T is said to be ($SQ - AP_{\omega}$) if $T * \varphi \in SQ - AP_{\omega}(I : X)$, for every $\varphi \in \mathcal{D}^{*}$. The space of all ($SQ - AP_{\omega}$) ultradistributions is denoted by $\mathcal{D}'^{*}_{SQ-AP_{\omega}}(X)$.*
c) *An ultradistribution T is said to be ($Q - AP_{\omega}$) if $T * \varphi \in Q - AP_{\omega}(I : X)$, for every $\varphi \in \mathcal{D}^{*}$. The space of all ($Q - AP_{\omega}$) ultradistributions is denoted by $\mathcal{D}'^{*}_{Q-AP_{\omega}}(X)$.*
d) *An ultradistribution T is said to be ($S^pQ - AP_{\omega}$) if $T * \varphi \in S^pQ - AP_{\omega}(I : X)$, for every $\varphi \in \mathcal{D}^{*}$. The space of all ($S^pQ - AP_{\omega}$) ultradistributions is denoted by $\mathcal{D}'^{*}_{S^pQ-AP_{\omega}}(X)$.*
e) *An ultradistribution T is said to be ($S^pQ - AP$) if $T * \varphi \in S^pQ - AP(I : X)$, for every $\varphi \in \mathcal{D}^{*}$. The space of all ($S^pQ - AP$) ultradistributions is denoted by $\mathcal{D}'^{*}_{S^pQ-AP}(X)$.*

Like in the case of distributions we can give the following theorems with the analogous proofs.

Theorem 3.1. *The following statements hold:*

i) $\mathcal{D}'^{*}_{Q-AP}(X) \cap \mathcal{D}'^{*}_{AAAut}(X) = \mathcal{D}'^{*}_{AAP}(X)$;
$[\mathcal{D}'^{*}_{AAAut}(X) \backslash \mathcal{D}'^{*}_{AAP}(X)] \cap \mathcal{D}'^{*}_{Q-AP}(X) = \emptyset$;
ii) $\mathcal{D}'^{*}_{AAut}(X) \cap \mathcal{D}'^{*}_{Q-AP}(X) = \mathcal{D}'^{*}_{AP}(X)$.

Theorem 3.2. *Let* $\omega \in I$. *It holds* $\mathcal{D}'^{*}_{S-AP_{\omega}}(X) \subseteq \mathcal{D}'^{*}_{Q-AP}(X)$.

Theorem 3.3. *Let* $\omega \in I$. *Then*

$$\mathcal{D}'^{*}_{S-AP_{\omega}}(X) \cap \mathcal{D}'^{*}_{AAAut}(X) \subseteq \mathcal{D}'^{*}_{AP_{\omega}}(X).$$

Theorem 3.4. *The following statements hold:*

i) $\mathcal{D}'^{*}_{S^p-AAAut}(X) \cap \mathcal{D}'^{*}_{S^pQ-AP}(X) = \mathcal{D}'^{*}_{S^p-AAP}(X)$
$[\mathcal{D}'^{*}_{S^p-AAAut}(X) \backslash \mathcal{D}'^{*}_{S^p-AAP}(X)] \cap \mathcal{D}'^{*}_{S^pQ-AP}(X) = \emptyset$;

ii) $\mathcal{D}'^{*}_{S^p-AAut}(X) \cap \mathcal{D}'^{*}_{S^pQ-AP}(X) = \mathcal{D}'^{*}_{S^p-AP}(X)$.

Theorem 3.5. *It holds that* $\mathcal{D}'^{*}_{S^p-AP_{\omega}}(X) \subseteq \mathcal{D}'^{*}_{S^pQ-AP}(X)$.

In order to span the investigation on the space of (Q-AP) ultradistributions $\mathcal{D}'^{*}_{Q-AP}(X)$, following the approach of B. Basit and H. Güenzler [1] (see also [24]), now we will focus to the case when $\mathrm{A} = Q - AP(\mathbb{R} : X)$. Then A is closed under the uniform convergence and addition, and we have $\mathrm{A} \subseteq \mathcal{D}'^{*}_{\mathrm{A}}(X)$ ([23]).

Theorem 3.6. *Let* (M_p) *satisfies the conditions* $(M.1)$, $(M.2)$ *and* $(M.3)'$ *and let* $T \in \mathcal{D}'^{*}_{L^1}(X)$. *Then the following holds:*

i) *Suppose that there exist an ultradifferential operator* $P(D) = \sum_{p=0}^{\infty} a_p D^p$ *of class* (M_p), *resp., of class* $\{M_p\}$ *and* $f, g \in \mathcal{D}'^{*}_{Q-AP}(X)$ *such that* $T = P(D)f + g$. *Then* $T \in \mathcal{D}'^{*}_{Q-AP}(X)$.

ii) *If* $T \in \mathcal{D}'^{(M_p)}_{Q-AP}(X)$ *then there exists* $h > 0$ *such that for each compact set* $K \subseteq \mathbb{R}$ *we have* $T * \varphi \in Q - AP(\mathbb{R} : X)$, *for all* $\varphi \in \mathcal{D}^{M_p,h}_{K}$.

iii) *Suppose that* $T \in \mathcal{D}'^{(M_p)}(X)$ *and there exists* $h > 0$ *such that for each compact set* $K \subseteq \mathbb{R}$ *we have* $T * \varphi \in Q - AP(\mathbb{R} : X)$, *for all* $\varphi \in \mathcal{D}^{M_p,h}_{K}$. *If* (M_p) *additionally satisfies* $(M.3)$, *then there exist* $l > 0$ *and two elements* $f, g \in Q - AP(\mathbb{R} : X)$ *such that* $T = P(D)f + g$.

As a consequence of the Theorem 3.6, we immediately have the following:

Corollary 3.1. *Let* (M_p) *satisfies the conditions* $(M.1)$, $(M.2)$ *and* $(M.3)'$ *and let* $T \in \mathcal{D}'^{*}_{L^1}(X)$. *Consider the following statements:*

i) $T \in \mathcal{D}'^{*}_{Q-AP}(X)$.

ii) *There exist a number* $l > 0$, *resp., a positive increasing sequence* (r_p) *and a two functions* $f, g \in Q - AP(\mathbb{R} : X)$ *such that* $T = P_l(D)f + g$, *resp.,* $T = P_{r_p}(D)f + g$.

iii) *There exist an ultradifferential operator* $P(D) = \sum_{p=0}^{\infty} a_p D^p$ *of class* (M_p), *resp.,* $\{M_p\}$ *and two functions* $f, g \in Q - AP(\mathbb{R} : X)$ *such that* $T = P(D)f + g$.

iv) *There exists* $h > 0$ *such that for each compact set* $K \subseteq \mathbb{R}$, *resp., for each* $h > 0$ *and for each compact set* $K \subseteq \mathbb{R}$, *we have* $T * \varphi \in Q - AP(\mathbb{R} : X)$, *for all* $\varphi \in \mathcal{D}^{M_p,h}_{K}$.

Then we have $ii) \Rightarrow iii) \Rightarrow i) \Leftrightarrow iv)$*. Furthermore, if* (M_p) *additionally satisfies the condition* $(M.3)$*, then the assertions* $i) - iv)$ *are equivalent for the Beurling class.*

Similarly, like in distribution case we have the following theorem:

Theorem 3.7. *Let* $T \in \mathcal{D}'^{*}_{L^1}(X)$*. The then following holds:*

- *i) Let* $T \in \mathcal{D}'^{*}_{Q-AP}(X)$*, resp.,* $T \in \mathcal{D}'^{*}_{S^pQ-AP}(X)$*. Then* cT *is a (Q-AP) ultradistribution, resp., (*S^pQ-AP*) ultradistribution, for any* $c \in \mathbb{C}$*;*
- *ii) If* (T_n) *is a sequence in* $\mathcal{D}'^{*}_{Q-AP}(X)$*, resp.,* $\mathcal{D}'^{*}_{S^pQ-AP}(X)$ *and* $T_n \to T$ *uniformly in* $\mathcal{D}'^{*}_{Q-AP}(X)$*, resp.,* $\mathcal{D}'^{*}_{S^pQ-AP}(X)$*, then* $T \in \mathcal{D}'^{*}_{Q-AP}(X)$*, resp.,* $T \in \mathcal{D}'^{*}_{S^pQ-AP}(X)$*;*
- *iii) Any translation* $T_h = \langle T, \varphi(\cdot - h)\rangle$ *of* $T \in \mathcal{D}'^{*}_{Q-AP}(X)$ *(*$T \in \mathcal{D}'^{*}_{S^pQ-AP}(X)$*) is again in* $\mathcal{D}'^{*}_{Q-AP}(X)$ *(*$\mathcal{D}'^{*}_{S^pQ-AP}(X)$*).*

Let we introduce the following space

$$\mathcal{E}^{*}_{Q-AP} = \left\{\varphi \in \mathcal{E}^{*}(X) : \varphi^{(i)} \in Q-AP(\mathbb{R} : X) \text{ for all } i \in \mathbb{N}_0\right\}.$$

We have that $\mathcal{E}^{*}_{Q-AP}(X) \subseteq \mathcal{D}'^{*}_{L^1}(X)$, $\mathcal{E}^{*}_{Q-AP}(X) = \mathcal{E}^{*}(X) \cap Q-AP(\mathbb{R} : X)$ and $\mathcal{E}^{*}_{Q-AP} * L^1(\mathbb{R}) \subseteq \mathcal{E}^{*}_{Q-AP}(X)$. Furthermore, the space of $\mathcal{E}^{*}_{Q-AP}(X)$ is the space of all mappings f from $\mathcal{E}^{*}(X)$ for which $f * \varphi \in Q-AP(\mathbb{R} : X)$ for all $\varphi \in \mathcal{D}^{*}$. The proof of the following theorem is the same like [24, Lemma 1].

Theorem 3.8. *Let* (M_p) *satisfy the conditions* $(M.1)$*,* $(M.2)$ *and* $(M.3)'$ *and let* $T \in \mathcal{D}'^{*}_{L^1}(X)$*. We have the following statements are equivalent:*

- *i)* $T \in \mathcal{D}'^{*}_{Q-AP}(X)$*.*
- *ii) There exists a sequence* (T_n) *in* $\mathcal{E}^{*}_{Q-AP}(X)$ *such that* $\lim_{n\to\infty} T_n = T$ *in the topology of* $\mathcal{D}'^{*}_{L^1}(X)$*.*

Likewise in distribution case, we have the following theorem:

Theorem 3.9. *Let* (M_p) *satisfy the conditions* $(M.1)$*,* $(M.2)$ *and* $(M.3)'$ *and let* $T \in \mathcal{D}'^{*}_{L^1}(X)$*. We have the following statements:*

- *i)* $T \in \mathcal{D}'^{*}_{Q-AP}(X)$*.*
- *ii) There exist an element* $S \in \mathcal{D}'^{*}_{L^1}(X)$*, a number* $l > 0$ *in the Beurling case (a sequence* $(r_p) \in \mathrm{R}$ *in the Roumieu case), and bounded functions* $f,\ g \in Q-AP(\mathbb{R} : X)$ *such that* $S = P_l(D)f + g$*, resp.* $S = P_{r_p}(D)f + g$*, and* $S = T$ *on* $[0,\infty)$*.*
- *iii) There exist a number* $l > 0$*, resp. a sequence* $(r_p) \in \mathrm{R}$*, and bounded functions* $f,\ g \in Q-AP(\mathbb{R} : X)$ *such that* $T = P_l(D)f + g$*, resp.* $T = P_{r_p}(D)f + g$*, on* $[0,\infty)$*.*
- *iv) There exist an ultradifferential operator* $P(D) = \sum_{p=0}^{\infty} a_p D^p$ *of* $*$*-class and bounded functions* $f_1,\ f_2 \in Q-AP(\mathbb{R} : X)$ *such that* $T = P(D)f_1 + f_2$ *on* $[0,\infty)$*.*
- *v) There exist an element* $S \in \mathcal{D}'^{*}_{L^1}(X)$*, an ultradifferential operator* $P(D) = \sum_{p=0}^{\infty} a_p D^p$ *of* $*$*-class and bounded functions* $f_1,\ f_2 \in Q-AP(\mathbb{R} : X)$ *such that* $S = P(D)f_1 + f_2$ *and* $S = T$ *on* $[0,\infty)$*.*

(vi) $T \in D'^{*}_{Q-AP}(X)$ *and there exists an element* $S \in \mathcal{D}'^{*}_{L^1}(M_p : X)$ *such that* $S = T$ *on* $[0,\infty)$ *and* $S * \varphi \in Q - AP(\mathbb{R} : X)$, $\varphi \in \mathcal{D}^*$.

Then we have $ii) \Rightarrow iii) \Rightarrow iv) \Rightarrow v) \Rightarrow vi) \Rightarrow i)$. *If* (M_p) *additionally satisfies the condition* $(M.3)$, *then the upper statements are equivalent for the Beurling case.*

Proof. The implications ii) $\Rightarrow$ iii) $\Rightarrow$ iv) $\Rightarrow$ v) are trivial. We are going to prove that v) $\Rightarrow$ vi). Let $S := P(D)g_1 + g_2$ is a (Q-AP) ultradistribution of Beurling class, for $f_1,\ f_2, g_1,\ g_2 \in Q - AP(\mathbb{R} : X)$ We need to show is

$$T_2 := P(D)(f_1 - g_1) + (f_2 - g_2) \in \mathcal{D}'^{(M_p)}_{+,0}(X), \text{ i.e.,}$$
$$\lim_{h\to+\infty} \Big\langle P(D)\big(f_1 - g_1\big) + \big(f_2 - g_2\big), \varphi(\cdot - h)\Big\rangle = 0, \quad \varphi \in \mathcal{D}^{(M_p)}.$$

Towards this end, assume that $-\infty < a < b < +\infty$ and $\text{supp}(\varphi) \subseteq [a,b]$. Let $\varepsilon > 0$ be given. Then there exist a sufficiently large finite number $h_0(\varepsilon,h) > 0$ and a sufficiently large finite number $c_\varphi >, 0$ independent of ε, such that, for every $|t| \geq h_0(\varepsilon,h)$, we have the following (cf. also the proof of [24, Theorem 1]):

$$\begin{aligned}
&\left\|\Big\langle P(D)\big(f_1 - g_1\big) + \big(f_2 - g_2\big), \varphi(\cdot - h)\Big\rangle\right\| \\
&= \left\|\sum_{p=0}^{\infty}(-1)^p a_p \int_{-\infty}^{\infty}\big(f_1(t) - g_1(t)\big)\varphi^{(p)}(t-h)\,dt + \int_{-\infty}^{\infty}\big(f_2(t) - g_2(t)\big)\varphi(t-h)\,dt\right\| \\
&= \left\|\sum_{p=0}^{\infty}(-1)^p a_p \int_{a}^{b}\big(f_1(t+h) - g_1(t+h)\big)\varphi^{(p)}(t)\,dt + \int_{a}^{b}\big(f_2(t) - g_2(t)\big)\varphi(t)\,dt\right\| \\
&= \left\|\sum_{p=0}^{\infty}(-1)^p a_p \int_{a}^{b} q_1(t+h)\varphi^{(p)}(t)\,dt + \int_{a}^{b} q_2(t)\varphi(t)\,dt\right\| \\
&\leq \varepsilon\left[\sum_{p=0}^{\infty}\big|a_p\big|\big\|\varphi^{(p)}\big\|_{L^1} + \big\|\varphi\big\|_{L^1}\right] \leq \varepsilon c_\varphi.
\end{aligned}$$

This yields vi). By Theorem 3.6 i), we have that vi) $\Rightarrow$ i). If (M_p) additionally satisfies the condition $(M.3)$, in Beurling case, holds i) $\Rightarrow$ ii), so all the upper statements are equivalent in Beurling case. □

4. Applications

Let $n \in \mathbb{N}$, and let $A = [a_{ij}]_{1\leq i,\ j\leq n}$ be a given complex matrix such that $\sigma(A) \subseteq \{z \in \mathbb{C} : \Re z < 0\}$. In this section, we analyze the existence of (Q-AP) (ultra) distribution solutions of the equation

$$T' = AT + F, \quad T \in \mathcal{D}'(X^n) \text{ on } [0,\infty) \tag{4.1}$$

and the equation

$$T' = AT + F, \quad T \in \mathcal{D}'^{*}(X^n) \text{ on } [0,\infty), \tag{4.2}$$

where F is a (Q-AP) X^n-valued distribution in (4.1) and F is a (Q-AP) X^n-valued ultradistribution of $*$-class in (4.2). By a solution of (4.1), resp. (4.2), we mean any distribution $T \in \mathcal{D}'(X^n)$, resp. $\mathcal{D}'^*(X^n)$, such that (4.1), resp. (4.2), holds in distributional, resp. ultradistributional, sense on $[0,\infty)$.

Now, we can state the following theorem, as a consequence [25, Theorem 4.1]:

Theorem 4.1. *(i) Let $F = [F_1\ F_2 \cdots F_n]^T \in \mathcal{D}'_{Q-AP}(X^n)$. Then there exists a solution $T = [T_1\ T_2 \cdots T_n]^T \in \mathcal{D}'_{Q-AP}(X^n)$ of* (4.1). *Furthermore, any distributional solution T of* (4.1) *belongs to the space $\mathcal{D}'_{Q-AP}(X^n)$,*

(ii) Let (M_p) satisfy the conditions (M.1), (M.2) *and* (M.3)', *and let $F = [F_1\ F_2 \cdots F_n]^T \in \mathcal{D}'^*(X^n)$ be such that, for every $i \in \mathbb{N}_n$, there exist an ultradifferential operator $P_i(D) = \sum_{p=0}^{\infty} a_{i,p} D^p$ of $*$-class and bounded functions $f_{1,i},\ f_{2,i} \in Q-AP(\mathbb{R}:X)$ such that $F_i = P_i(D) f_{1,i} + f_{2,i}$ on $[0,\infty)$. Then there exist ultradifferential operators $P_{ij}(D)$ of $*$-class, bounded functions $h_{ij} \in Q-AP(\mathbb{R}:X)$ and bounded functions $h_{2,i} \in Q-AP(\mathbb{R}:X)$ $(1 \leq i \leq n)$, such that $T = [T_1\ T_2 \cdots T_n]^T \in \mathcal{D}'^*_{Q-AP}(X^n)$ is a solution of* (4.2), *where $T_i = \sum_{j=1}^{n} P_{ij}(D) h_{ij} + h_{2,i}$ for $i \in \mathbb{N}_n$. Furthermore, any ultradistributional solution T of $*$-class to the equation* (4.2) *has such a form.*

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