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COMPACTNESS IN SINGULAR CARDINALS REVISITED

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ABSTRACT. This is the second combinatorial proof of the compactness theorem for singular from 1977. In fact it gives a somewhat stronger theorem.

1. INTRODUCTION

For a long time I have been interested in compactness in singular cardinals; i.e., whether if something occurs for "many" subsets of a singular λ of cardinality $< \lambda$, it occurs for λ. For the positive side in the seventies we have

Theorem 1.1. *Let λ be a singular cardinal, $\chi^* < \lambda$. Let $\mathscr{U}$ be a set, $\mathbf{F}$ a family of pairs (A,B) of subsets of $\mathscr{U}$, instead of $(A,B) \in \mathbf{F}$ we may write $A/B \in \mathbf{F}$ (formal quotient) or A/B is $\mathbf{F}$-free. Assume further that $\mathbf{F}$ is a nice freeness notion meaning it satisfies axioms II, III, IV,VI, VII from 1.1 below. Let $A^*, B^* \subseteq \mathscr{U}$ with $|B^*| = \lambda$. Then $B^*/A^* \in \mathbf{F}$ is free in a weak sense, that is: there is an increasing continuous sequence $\langle A_\alpha : \alpha < \delta \rangle$ of subsets of B_* of cardinality $< \lambda$ such that $A_0 = \emptyset$, $\bigcup\limits_{\alpha<\delta} A_\alpha = A_*$ and $A_{i+1}/A_i \cup A$ is $\mathbf{F}$-free for $i < \lambda$* when *(see Definition 1.2 below):*

- $(*)_0$ *for the $\mathscr{D}_{\chi^*}(B^*)$-majority of $B \in [B^*]^{<\lambda}$ we have $B/A^* \in \mathbf{F}$*
  or just
- $(*)_1$ *the set $\{\mu < \lambda : \{B \in [B^*]^\mu : B/A^* \in \mathbf{F}\} \in \mathscr{E}_\mu^{\mu^+}(B^*)\}$ contains a club of λ,*
  or at least
- $(*)_2$ *for some set C of cardinals $< \lambda$, unbounded in λ and closed (meaningful only if $\mathrm{cf}(\lambda) > \aleph_0$), for every $\mu \in C$, for an $\mathscr{E}_\mu^{\mu^+}(B^+)$ - positive set of $B \in [B^*]^\mu$ we have $B/A^* \in \mathbf{F}$.*

Where

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Research supported by the United States-Israel Binational Science Foundation. In a mimeographed from this was included in May 1977, and a lecture on it was given in Berlin 1977. References like [7, Th0.2=Ly5] means the label of Th.0.2 is y5. The reader should note that the version in my website is usually more updated than the one in the mathematical archive. Publication 266.

Definition 1.1. *For a set $\mathscr{U}$ and $\mathbf{F} \subseteq \{(A,B) : A,B \subseteq \mathscr{U}\}$ but we may write B/A instead (A,B), we say, $\mathbf{F}$ is a χ-nice freeness notion* if $\mathbf{F}$ *satisfies:*

- *Ax.II* $B/A \in \mathbf{F} \Leftrightarrow A \cup B/A \in \mathbf{F}$
- *Ax.III* *if* $A \subseteq B \subseteq C, B/A \in \mathbf{F}$ *and* $C/B \in \mathbf{F}$ *then* $C/A \in \mathbf{F}$,
- *Ax.IV* *if* $\langle A_i : i \leq \theta \rangle$ *is increasing continuous,* $\theta = \mathrm{cf}(\theta), A_{i+1}/A_i \in \mathbf{F}$ *then* $A_\theta/A_0 \in \mathbf{F}$,
- *Ax.VI* *if* $A/B \in \mathbf{F}$ *then for the* $\mathscr{D}_\chi$*-majority of* $A' \subseteq A$ *we have,* $A'/B \in \mathbf{F}$ *(see below),*
- *Ax.VII* *if* $A/B \in \mathbf{F}$ *then for the* $\mathscr{D}_\chi$*-majority of* $A' \subseteq A$ *we have,* $A/B \cup A' \in \mathbf{F}$.

Definition 1.2. *1) Let $\mathscr{D}$ be a function giving for any set B^* a filter $\mathscr{D}(B^*)$ on $\mathscr{P}(B^*)$ (or on $[B^*]^\mu$).*
Then to say "for the $\mathscr{D}$-majority of $B \subseteq B^*$ (or $B \in [B^*]^\mu$) we have $\varphi(B)$" means $\{B \subseteq B^* : \varphi(B)\} \in \mathscr{D}(B^*)$ (or $\{B \in [B^*]^\mu : \neg\, \varphi(B)\} = \emptyset \mod \mathscr{D}(B^*)$).

2) Let $\mathscr{D}_\mu(B^*)$ be the family of $Y \subseteq \mathscr{P}(B^*)$ such that for some algebra M with universe B^* and $\leq \mu$ functions,

$$Y \supseteq S_M = \{B \subseteq B^* : B \neq \emptyset \text{ is closed under the functions of } M\}.$$

2A) Let $\mathscr{D}_{=\mu}(B^*)$ be defined similarly considering only B's of cardinality $\leq \mu$.

3) $\mathscr{E}^\mu_\kappa(B^*)$ where $\mu \leq \kappa^+$ is the collection of all $Y \subseteq [B^*]^\kappa$ such that: for some χ, x satisfying $\{B^*, x\} \in \mathscr{H}(\chi)$, if $\bar{M} = \langle M_i : i < \mu \rangle$ is an increasing continuous sequence of elementary submodels of $(\mathscr{H}(\chi), \in)$ such that $x \in M_0$, $\kappa + 1 \subseteq M_0, \|M_i\| = \kappa$ and $i < \mu \Rightarrow \bar{M} \restriction (i+1) \in M_{i+1}$, then

(a) if $\mu \leq \kappa$ then $\bigcup_{i<\mu} M_i \cap B^* \in Y$

(b) if $\mu = \kappa^+$ then for some club C of μ^+ we have $i \in C \Rightarrow M_i \cap B^* \in Y$.

On $\mathscr{D}_\mu$ see Kueker [6], and on $\mathscr{E}^{\mu^+}_\mu$ see [9] repeated in §2 below, note that in [9] the axioms are phrased with elementary submodels rather then saying "majority". The theorem was proved in [9] but with two extra axioms, however it included the full case for varieties (i.e., including the non-Schreier ones). Later, the author eliminated those two extra axioms: Ax.V and Ax.I. Now Ax.V was used in one point only in [9, §1], and I eliminated it early (as presented in [1]). Axiom I is more interesting: it say that if $A' \subseteq A$ and A/B free then A'/B is $\mathbf{F}$-free"; this is like "every subgroup of a free group if free; (this was shown not to be necessary for varieties already in [9]).

In 77 Fleissner has asked for a simpler "combinatorial" proof and we find such proof circulateding it in mimeographed notes [10]. In May 77, and lecture on it in Berlin (summer 77 giving the full details only for the case close to Abelian groups). This proof eliminates the two extra axioms (as its assumptions holds by [9, Lemma 3.4,p.349], see §2 below).

Continuing this Hodges do [5] which contain a compactness result and new important applications. I have thought he just represent the theorem but looking at it lately it seems to me this is not exactly so; the main point in the proof appears but the frame is different so it is relative. This exemplifies the old maxim "if you want things done in the way you want it, you have to do them yourself".

Anyhow below in §1,§2 we repeat the mimeographed notes. Note that §2 repeats [9, 3.4] needed for deducing 1.1. Restricted to the needed case; note 3.3 give hypothesis I (the non $\mathscr{E}^{\lambda_i^+}_{\lambda_i^\gamma}$-non freeness is $(*)_2$ of 1.1 where hypothesis II is a weak form of Ax VII.

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2. A compactness theorem for singular

Here we somewhat improve and simplify the proof of [9] (and [1]). It may be considered an answer to question B2 of Fleissner [4].

Theorem 2.1. *Assume*

(a) *λ is a singular cardinal, $\lambda_i (i < \kappa)$ an increasing and continuous sequence of cardinals (we let $\lambda(i) = \lambda_i$) and*
$$\lambda_0 = 0, \kappa = \mathrm{cf}(\lambda), \kappa \leq \lambda_1, \lambda = \sum_{i<\kappa} \lambda_i.$$
(b) *Let $S_i = \{A \subseteq \lambda : |A| = \lambda_i\}$ and $S'_i = S_i \cup \{\emptyset\}$*

(c) *$\mathbf{F}$ is a family*[1] *of pairs $(A,B), \lambda \supseteq A \supseteq B$; we may write "$A/B$ belong to $\mathbf{F}$"*

(d) hypothesis I*: for each $i, i < \kappa, i$ a successor, there is a function g_i, two-place, from S'_i to S'_i, such that: if $A_1 \subseteq A_2$ are from $S'_i, A_1 \in \{\emptyset\} \cup \mathrm{Range}(g_i)$,* then *$A_2 \subseteq g_i(A_1, A_2)$ and $[g_i(A_1,A_2)/A_1] \in \mathbf{F}$*

(e) hypothesis II*: if $i < \kappa, A, B \in S'_{i+1}, A \subseteq B$ and $B/A \in \mathbf{F}$ and $B \in \mathrm{Rang}(g_{i+1})$,* then *player II has a winning strategy in the following game $Gm_i[A,B]$. In the n-th move $(n < \omega)$ player I choose $A_n \in S_i$, such that $B_{n-1} \subseteq A_n$, and then player II choose B_n, such that $A_n \subseteq B_n \in S_i$ (where we stipulate $B_{-1} = \emptyset$). Player II wins in the play if $(B \cup \bigcup_{n<\omega} B_n, A \cup \bigcup_{n<\omega} B_n) \in \mathbf{F}$ (for $i = 0$ this is an empty demand as $S'_i = \{\emptyset\}$).*

Then we can find an increasing and continuous chain $A_\alpha (\alpha < \omega\kappa)$, such that *$A_0 = \emptyset, \lambda = \bigcup_\alpha A_\alpha$ and $A_{\alpha+1}/A_\alpha \in \mathbf{F}$ for each α.*

Proof. Let in Hypothesis II the winning strategy of player II in the game Gm_i be given by the functions $h^n_i(A_0, \dots, A_n; A, B)$. We define by induction on $i < \omega$ sets A^n_i, B^n_i (for $i < \kappa$) such that:

(1) $A^n_i (i < \kappa)$ is increasing and continuous in i and $A^n_i, B^n_i \in S_i$

(2) $A^n_i \subseteq B^n_i \subseteq A^{n+1}_i$

[1] Note that none of the axioms of 1.1 is assumed.

(3) $(B^n_i/B^{n-1}_i) \in \mathbf{F}$ where we stipulate $B^{-1}_i = \emptyset$ and $B^n_i \in \mathrm{Rang}(g_i)$ for i successor
(4) for $i < \kappa, 0 \le m < n, i < \omega$ we have

$$h^{n-m}_i(A^{m+1}_i, A^{m+2}_i, \dots, A^n_i; B^{m-1}_{i+1}, B^m_{i+1}) \subseteq A^{n+1}_i.$$

For $n = 0$

Let $A^0_i = \lambda_i, B^0_i = g_i(\emptyset, \lambda_i)$; clearly condition (1) holds, (2) and (4) say nothing and condition (3) holds by Hypothesis I.

For $n+1$ assuming that for n we have defined.

Let

$$C^n_i = \bigcup_{m<n} h^{n-m-1}_i(A^{m+1}_i, A^{m+2}_i, \dots, A^n_i, B^{m-1}_{i+1}, B^m_{i+1}) \cup B^n_i$$

clearly $|C^n_i| = \lambda_i$, hence we can let $C^n_i = \{{}^{\alpha}b^n_i : \alpha < \lambda_i\}$.

Now we define $A^{n+1}_i = \{{}^{\alpha}b^n_j : j < \kappa, \alpha < \min\{\lambda_i, \lambda_j\}\}$.

Clearly condition (4) and the relevant parts of conditions (1) and (2) hold. We have to choose B^{n+1}_i such that

$$A^{n+1}_i \subseteq B^{n+1}_i \text{ and } |B^{n+1}_i| = \lambda_i, \text{ and } i \text{ successor } \Rightarrow B^{n+1}_i/B^n_i \in \mathbf{F}.$$

So we let $B^{n+1}_i = g_i(B^n_i, A^{n+1}_i)$ except that $B^{n+1}_0 = \emptyset$. By Hypothesis I this is O.K.

Now we can prove the conclusion of the theorem.

We let $D_{\omega i+k} = (B^{k-1}_{i+1} \cap \bigcup_{m<\omega} A^m_i) \cup \bigcup_{j<i, m<\omega} A^m_j$ for $i < \kappa$. Clearly $D_0 = \emptyset$, (in fact A^n_i, B^n_i are $\emptyset$ for $i = 0$); $\lambda = \bigcup_{i<\omega\kappa} D_i$ as $\lambda_i = A^0_i \subseteq D_{\omega(i+1)} \subseteq \lambda$. The sequence is increasing and continuous.

[that is e.g., if $\delta = \omega i + \omega$ so $\delta = \omega(i+1)+0$ then $D_\delta \subseteq \bigcup_{\alpha<\delta} D_\alpha$ as $B^{-1}_{i+1} = \emptyset$,

so $D_\delta = \bigcup_{j<i+1, m<\omega} A^m_j = (\bigcup_{j<i, m<\omega} A^m_j) \cup \bigcup_m A^m_i$ but $A^m_i \subseteq A^m_{i+1} \subseteq B^m_{i+1}$.

so $D_\delta \subseteq \bigcup_k \big[\bigcup_{\binom{j<i}{m<\omega}} A^m_j \cup (B^{k-1}_{i+1} \cap \bigcup_{m<\omega} A^m_i) = \bigcup_k D_{\omega i+k} \subseteq \bigcup_{\alpha<\delta} D_\alpha \big]\big]$.

Now $D_{\omega i+k+1}/D_{\omega i+k} \in \mathbf{F}$ as $B^k_{i+1}/B^{k-1}_{i+1} \in \mathbf{F}$ by condition (3), and then use condition (4)) and the choice of the $h^i_n - s'$ [that is, player II wins the play

$$\langle A^{k+\ell}_i, h^{k+\ell}_i(A^{k+1}_i, A^{k+2}_i \dots, A^{k+\ell}_i, B^{k-1}_{i+1}, B^k_{i+1}) : \ell < \omega\rangle$$

of the game $Gm_i[B^{k-1}_{i+1}, B^k_{i+1}]\big]$. □

Remark 2.1. 1) In the context of [8], [1] Hypothesis I holds quite straightforwardly whereas Hypothesis II is proved separately, see [9, Lemma 3.4 p. 344].

2) Usually the choice of the λ_i's is not important, and then Hypothesis I, Hypothesis II should speak on $\mu < \lambda, \mu < \mu' < \lambda$.

3) In the construction proving the Theorem we can continue $\chi < \lambda_1$ steps instead of ω steps. We succeed if: in Hypothesis II the game has length χ and we add to hypothesis II: if $A_i/A_0 \in \mathbf{F}$ for $i < \chi, A_i$ increasing continuous then $\bigcup_{i<\chi} A_i/A_0 \in \mathbf{F}$.

An example is: G is a group with universe λ and

$$\mathbf{F} = \{(A,B) : \mathrm{Ext}(A/B, C^*) = \emptyset\}$$

where $A \subseteq B$ are subgroup of G, $\mathrm{cf}(\lambda) < \chi < \lambda, \chi$ measurable (C^* a fixed group of cardinality $< \chi$) and e.g. G.C.H. (see below).

4) We can improve a little Eklof's results on compactness [3] where "A free" is replace by "$\mathrm{Ext}(A,\mathbb{Z}) = 0$".
Note that in his proofs $\diamondsuit_S$ can be replaced by "S not small" e.g. (see [2]), and instead "$\diamondsuit_S$ for stationary S" by the above "S not small for all stationary S such that $(\forall\delta \in S)\mathrm{cf}(\delta) = \aleph_0$" suffice but if $\sup(S) = \lambda^+, \lambda^{\aleph_0} = \lambda, 2^\lambda = \lambda^+$, this holds. So we can get compactness for $\beth_{\alpha+\omega}$ assuming G.C.H.

4A) Hypothesis I can be rephrased similary to Hypothesis II, as the existence of a winning strategy (to player II) in appropriate game.

5) For the Whithead problem we need only "any λ-free abelian group is λ^+-free" for singular λ. So suppose G is a $\lambda-$free group with universe λ and $\mathbf{F} = \{(A,B) : A/B$ is free$\}$. There we do not need Hypothesis I, and can represent the proof somewhat differently.

In the construction we choose *pure* subgroups A^n_i, B^n_i and choose a free basis I^n_i of A^n_i and demand satisfying

(a) (1) + (2)
(b) for $m < n$, $A^m_{i+1} \cap B^n_i$ is generated by a subset of I^m_{i+1}
(c) for each $m < n$ and integer a,
$(\forall x \in B^n_i \cap A^{m+1}_{i+1})[(\exists y \in A^{m+1}_{i+1})[ay + x \in A^m_{i+1}] \to (\exists y \in A^{m+1}_{i+1} \cap B^n_i)ay + x \in A^m_{i+1})]$

By (c) we shall get $A^m_{i+1}/\bigcup\limits_{m<\omega} B^n_i$ hence it is known (Hill) that

$$\bigcup_m A^m_{i+1}/\bigcup_m B^m_i = \bigcup_n A^m_i$$

is free thus finishing.

3. On the hypothesis

Context 3.1. $\mathscr{U}, \mathbf{F}$ is as in Definition 1.1.

Notation 3.2. 0) A,B,C,D denote subsets of $\mathscr{U}$.

1) $\mathscr{S}_\kappa(A) = \{B \subseteq A : |B| < \kappa\}$.

2) A/B is free mean $(A,B) \in \mathbf{F}$.

3) A,B,D denote subsets of $\mathscr{U}$.

4) $\mathscr{M} = (\mathscr{H}(\chi), \in, <^*_\chi)$ where χ is large enough such that $\mathscr{P}(\mathscr{U}) \in \mathscr{H}(\chi)$ and $<^*_\chi$ a well ordering of $\mathscr{M}$. We say $\mathscr{M}^*$ is a κ-expansion of $\mathscr{M}$ if we expand $\mathscr{M}$ by $\leq \kappa$ additional relations and functions.

5) $\mathscr{E}^{\mathrm{ub}}_\kappa(A)$ is the following filter or $\mathscr{S}_\kappa(A) : Y \in \mathscr{E}^{\mathrm{ub}}_\kappa(A)$ iff $Y \supseteq Y_C = Y_C(A)$ for some $Y \subseteq A, C \in \mathscr{S}_\kappa(A)$ where $Y_C = \{B \in \mathscr{S}_\kappa(A) : C \subseteq B\}$ we call $Y_C[A]$ a generator.

Definition 3.1. *1) The pair A/B is $\mathscr{E}$-free (where $\mathscr{E}$, or $\mathscr{E}(A)$, is a filter over a family of subsets of A so $C \in X \in \mathscr{E} \Rightarrow C \subseteq A$) if:*

$$\{C : C \in \cup \mathscr{E} \text{ and } C/B \quad \text{is free}\} \in \mathscr{E}.$$

2) We can replace "free" by any other property.

Remark 3.1. Obvious monotonicity results hold.

Definition 3.2. *1) For every $\mu \leqq \kappa < \lambda, C \in \mathscr{S}_\kappa(A), A \subseteq \mathscr{U}$ such that $|A| = \lambda$, and $B \subseteq \mathscr{U}$ and filter $\mathscr{E}$ over $\mathscr{S}_\kappa(A)$, we define the rank $R(C,\mathscr{E}) R^\mu_\kappa(C,\mathscr{E};A/B)$ as an ordinal or ∞, so that*

- *(a) $R(C,\mathscr{E}) \geqq \alpha + 1$ iff C/B is free and $\{D \in \mathscr{S}_\kappa(A) : C \subseteq D$ and $D/C \cup B$ is free and $R(D,\mathscr{E}) \geqq \alpha\} \neq \emptyset \mod \mathscr{E}$*
- *(b) $R(C,\mathscr{E}) \geqq \delta\, (\delta = 0$ or δ limit)* iff *C/B is free and $\alpha < \delta$ implies $R^\mu_\kappa(C,\mathscr{E}) \geqq \alpha$.*

2) $R(A/B,\mathscr{E}) = \sup\{R^\mu_\kappa(C,\mathscr{E}) : C \in S_\kappa(A)\}$.
3) Writing $R^\mu_\kappa(C) = R^\mu_\kappa(C;A/B)$ means $R(C,\mathscr{E}^\mu_\kappa;A/B)$ and writing $R^{\mathrm{ub}}_\kappa(C) = R^{\mathrm{ub}}_\kappa(C;A/B)$ means $R(C,\mathscr{E}^{\mathrm{ub}}_\kappa;A/B)$. Similarly $R^\mu_\kappa(A/B)$ means $R(A/B,\mathscr{E}^\mu_\kappa)$ and $R^{\mathrm{ub}}_\kappa(A/B)$ means $R(A/B,\mathscr{E}^{\mathrm{ub}}_\kappa)$.

Remark 3.2. Note that omitting A/B is reasonable because mostly they are clear from the content.

Lemma 3.3. *Suppose $\kappa^+ < \lambda, \mu \leqq \kappa, A/B$ is not $\mathscr{E}^{\kappa^+}_{\kappa^+}$-non-free and $S_1 \in \mathscr{E}^{\kappa^+}_{\kappa^+}(A)$. Then $R^{\mathrm{ub}}_\kappa = \infty$, [moreover for every $S_1 \in \mathscr{E}^{\kappa^+}_{\kappa^+}(A)$ and κ-expansion $\mathscr{M}^*$ of $\mathscr{M}$ there are $C \in S_2$ and $D \in S_1$ and $N \prec \mathscr{M}^*, \{A,B\} \in N, \|N\| = \kappa$ such that $D \in N$, $C = D \cap N$ and $R^\mu_k(C) = \infty$.]*

Proof. Let $S_1 \supseteq \mathscr{S}_\kappa(\mathscr{M}^*)$ if $C \in \mathscr{S}_\kappa(A), 0 \leqq R^{\mathrm{ub}}_\kappa(C) < \infty$, then there is a generator $S(C) \in \mathscr{E}^{\mathrm{ub}}_\kappa(A), S(C) = \mathscr{S}^{\mathrm{ub}}_\kappa(\mathscr{M}^*_C)$, such that for $D \in S(C), D/C \cup B$ is not free or $R^{\mathrm{ub}}_\kappa(D) < R^{\mathrm{ub}}_\kappa(C)$. If C B is not free or $R^{\mathrm{ub}}_\kappa(C) = \infty$, let $\mathscr{M}^*_C$ be any κ-expansion of $\mathscr{M}$, and let $S_2 = S^{\mathrm{ub}}_\kappa(\mathscr{M}^2)$. Let $\mathscr{M}^+$ be a κ-expansion of $\mathscr{M}$,expanding $\mathscr{M}^*, \mathscr{M}^2$ and having the relations P, P_2 where

$$P = \{(C,N) : C \in \mathscr{S}_\kappa(A), N \prec \mathscr{M}^*_C, \|N\| < \chi_2\}$$
$$P_2 = \{N : N < \mathscr{M}^2, \|N\| < \chi_2\}.$$

As

$$\{D \in S_{\kappa^+}(A) : D/B \text{ is free }\} \neq \emptyset \quad \mod \mathscr{E}^{\kappa^+}_{\kappa^+}(A)$$

and $S_1 \in \mathscr{E}^{\kappa^+}_{\kappa^+}(A)$ and (by 3.1 $\mathscr{S}_{\kappa^+}(A)$); there are D, $\bar{N}$ such that:

(1) S/B is free
(2) $D \in S_1$
(3) $N_i (i < \kappa^+)$ is an $\mathscr{M}^+$-sequence and $\|N_i\| \leqq \kappa$, so
(4) $D = A \cap \bigcup_{i<\kappa^+} N_i$, without loss of generality $\|N_i\| = \kappa,\ \kappa \subseteq N_i$.

Let $A_i^* = D \cap N_i$, so $A_i^* \in N_{i+1}$ and let $N = \bigcup_{i<\kappa^+} N_i$. Clearly $\langle N_i : i < \kappa^+\rangle$ is also an $\mathscr{M}^2$-sequence hence for each $\delta < \kappa^+, \langle N_i : i < \delta\rangle$ is an $\mathscr{M}^2$-sequence, hence, if κ divides $\delta, \mathrm{cf}(\delta) = \mu$, then $A_\delta^* \in S_2$. If $C \in N_i, C \in \mathscr{S}_\kappa(A)$, then for every $j > i, j < \kappa^+$ there is a model $N_i^j \prec \mathscr{M}_C^*, \|N_i^j\| = \kappa, |N_i^j|$ and $N_j^i \in N_{j+1}$, hence $N_j^i \subseteq N_{j+1}$.

Hence, for any limit ordinal $\delta, i < \delta < \kappa^+$ implies $N_\delta \prec \mathscr{M}_C^*$.

Clearly $\langle N_j : i < j < \kappa^+, j \text{ limit } \rangle$ is an $\mathscr{M}^+$-sequence, hence it is an $\mathscr{M}_C^*$ sequence, hence, is $i < \delta < \kappa^+, \delta$ is limit, κ^2 divides $\delta, \mathrm{cf}(\delta) = \mu$, then $A_\delta^* \in S(C)$. As S/B is free, by [9],1.2(7) there is a closed unbounded subset of κ^+, W, such that for $i, j \in W,\ i < j,\ A_j^*/A_i^* \cup B$ is free and A_i^*/B is free. We can assume that such $i \in W$ is divisible by κ^2. Hence, if $i,\ j \in W, i < j,\ \mathrm{cf}(j) = \mu, R_\kappa^{\mathrm{ub}}(A_i^*) < \infty$, then $R_\kappa^\mu(A_j^*) < R_\kappa^\mu(A_i^*) < \infty$ (by the definition of $S(C)$).

So, if for some $i \in W, R_\kappa^\mu(A_i^*) < \infty, \mathrm{cf}(i_n) = \mu, i_n \in W, i < i_n < i_{n+1}$ then $R_\kappa^\mu(A_{i_n}^*)$ is an infinite decreasing sequence of ordinals, a contradiction.

Hence, $i \in W$ implies $R_\kappa^\mu(A_i^*) = \infty$.

Let $D = \bigcup_{i<\kappa^+} A_i^*$, and choose $N \prec \mathscr{M}^*,\ D \in N,\ N \cap \bigcup_{i<\kappa^+} A_i^* = A_\delta^*,\ \delta \in W,\ \mathrm{cf}(\delta) = \mu$, and $C = A_\delta^*$. So we are finished. □

Lemma 3.4. *1) If $\mu \leqq \kappa < \lambda,\ C \in \mathscr{S}_\kappa(A),\ R_\kappa^\mu(C) = \infty,\ S \in \mathscr{E}_\kappa^\mu(A)$, then for some $D \in S,\ C \subseteq D,\ R_\kappa^\mu(D) = \infty$ and $D/C \cup B$ is free.*
2) The same holds for any filter over $S_\kappa(A)$.

Proof. 1) As $\mathscr{S}_\kappa(A)$ is a set, for some ordinal $\alpha_0 < |\mathscr{S}_\kappa(A)|^+$, for no $C \in \mathscr{S}_\kappa(A)$ is $R_\kappa^\mu(C) = \alpha_0$. We can easily prove that $R_\kappa^\mu(C) \geqq \alpha_0$ iff $R_\kappa^\mu(C) = \infty$. Using the definition we get our assertion.
2) The same proof. □

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