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CONVEX LATTICE HEPTAGONS WITH BOUNDARY QUADRILATERALS THAT ARE TRAPEZOIDS

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Dedicated to Professor Mehmed Nurkanović on the occasion of his 65th birthday

ABSTRACT. In this paper, we consider convex heptagons with the property that every four consecutive vertices of this heptagon determine a trapezoid in which the side that is a diagonal of the heptagon is also one of the bases of the trapezoid. The existence of such a heptagon embedded in the integer lattice has been proved.

1. INTRODUCTION

We consider a Cartesian coordinate system. A point in a plane whose both coordinates are integers is called a lattice point. A convex polygon that has lattice points for all its vertices is called a convex lattice polygon. A quadrilateral whose vertices are four consecutive vertices of a convex integer polygon is called a boundary quadrilateral of that polygon.

Rabinowitz has constructed a convex lattice polygon with n edges of minimum area, where $n \geq 6$ is an even number ([1]). All the boundary quadrilaterals of that polygon are trapezoids ([1]). Thus, it has been proved that for every even natural number n ($n \geq 6$) there is a convex lattice polygon with n edges, all of whose boundary quadrilaterals are trapezoids. The existence of a convex lattice polygon with n edges, where $n \geq 5$ is an odd number has shown to be a considerably more difficult problem. In [2] it has been proved that there does not exist a convex lattice pentagon such that all of its boundary quadrilaterals are trapezoids. Some results related to boundary lattice triangles, i.e. triangles whose vertices are three consecutive vertices of a convex lattice polygon, can be found in [3], [4] and [5].

In this paper, we consider the case of the existence of a convex lattice heptagon for which all of its boundary quadrilaterals are trapezoids. Section 2 deals with the properties of convex heptagons with trapezoids as boundary quadrilaterals, even when they are not drawn on an integer lattice. In other words, it discusses the

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relationships between coefficients associated with the considered trapezoids which are expressed in Lemma 2 and Lemma 3. In section 3 the following is proved.

Theorem 1.1. *There is a convex lattice heptagon such that all of its boundary quadrilaterals are trapezoids.*

The proof includes two examples of such heptagons constructed based on the relations between the coefficients obtained in Lemma 3.

2. PROPERTIES OF CONVEX HEPTAGONS WITH TRAPEZOIDS AS BOUNDARY QUADRILATERALS

Suppose $A_1A_2\dots A_7$ is a convex heptagon such that all of its boundary quadrilaterals are trapezoids. (Such a convex heptagon exists, it is a regular heptagon for example.) Denote $\mathbf{a_i} = \overrightarrow{A_iA_{i+1}}$, $\mathbf{d_i} = \overrightarrow{A_{i-1}A_{i+2}}$, $i = 1,2,\dots 7$ ($A_0 \equiv A_7$, $A_8 \equiv A_1$, $A_9 \equiv A_2$). Also, denote a_i the length of $\mathbf{a_i}$ and d_i the length of $\mathbf{d_i}$. Obviously, due to convexity, $\mathbf{d_i} \parallel \mathbf{a_i}$ and $d_i > a_i$ must hold.

Lemma 2.1. *Suppose $A_1A_2\dots A_7$ is a convex heptagon such that all of its boundary quadrilaterals are trapezoids. For each trapezoid $A_{i-1}A_iA_{i+1}A_{i+2}$ there exists a positive real number k_i such that*

$$\mathbf{a_{i-1}} + \mathbf{a_{i+1}} = k_i\mathbf{a_i},$$

where k_i is a positive real number.

Proof. Because $\mathbf{d_i} \parallel \mathbf{a_i}$ and $d_i > a_i$, there is a real number l_i ($l_i > 1$) such that $\mathbf{d_i} = l_i\,\mathbf{a_i}$, so we have

$$\mathbf{a_{i-1}} + \mathbf{a_i} + \mathbf{a_{i+1}} = l_i\mathbf{a_i} \;\Leftrightarrow\; \mathbf{a_{i-1}} + \mathbf{a_{i+1}} = k_i\mathbf{a_i},$$

where k_i is a positive real number ($k_i = l_i - 1$). □

The coefficient k_i from the previous lemma will be called the '$k-$coefficient' associated with the trapezoid $A_{i-1}A_iA_{i+1}A_{i+2}$.

Lemma 2.2. *For the $k-$coefficients of the boundary quadrilaterals $A_{i-1}A_iA_{i+1}A_{i+2}$*

$$(k_i+1)k_{i+1} = k_{i+3}(k_{i+4}+1),$$

$$(k_i+1)(k_{i+1}k_{i+2}-1) = k_{i+4},$$

hold, where $i = 1,2,\dots,7$ and $k_{j+7} = k_j$, for $j = 1,2,3,4$.

Proof. Let's express the vectors $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_5}$, $\mathbf{a_6}$ and $\mathbf{a_7}$ from the linearly independent vectors $\mathbf{a_3}$ and $\mathbf{a_4}$. Based on previous lemma there are positive real numbers k_i, $i = 1,2,\dots,7$, such that

$$\mathbf{a_{i-1}} + \mathbf{a_{i+1}} = k_i\mathbf{a_i}. \tag{2.1}$$

For $i = 3$ and $i = 4$ we respectively get that

$$\mathbf{a_2} = k_3\mathbf{a_3} - \mathbf{a_4},$$

$$\mathbf{a_5} = k_4\mathbf{a_4} - \mathbf{a_3}.$$

Using (2.1) and the previous two equations, we get that

$$\mathbf{a_1} = k_2\mathbf{a_2} - \mathbf{a_3} = k_2(k_3\mathbf{a_3} - \mathbf{a_4}) - \mathbf{a_3} = (k_2k_3 - 1)\mathbf{a_3} - k_2\mathbf{a_4},$$

$$\mathbf{a_6} = k_5\mathbf{a_5} - \mathbf{a_4} = k_5(k_4\mathbf{a_4} - \mathbf{a_3}) - \mathbf{a_4} = (k_4k_5 - 1)\mathbf{a_4} - k_5\mathbf{a_3},$$

$$\mathbf{a_7} = k_1\mathbf{a_1} - \mathbf{a_2} = (k_1k_2k_3 - k_1 - k_3)\mathbf{a_3} - (k_1k_2 - 1)\mathbf{a_4}.$$

Since

$$\mathbf{a_1} + \mathbf{a_2} + \mathbf{a_3} + \mathbf{a_4} + \mathbf{a_5} + \mathbf{a_6} + \mathbf{a_7} = \mathbf{0},$$

by including previously obtained values for $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_5}$, $\mathbf{a_6}$ and $\mathbf{a_7}$, we get

$$(k_1k_2k_3 + k_2k_3 - k_1 - k_5 - 1)\mathbf{a_3} + (k_4k_5 + k_4 - k_1k_2 - k_2)\mathbf{a_4} = \mathbf{0},$$

and therefore

$$k_1k_2k_3 + k_2k_3 - k_1 - k_5 = 1, \tag{2.2}$$

$$k_4k_5 + k_4 = k_1k_2 + k_2. \tag{2.3}$$

Due to symmetry,

$$(k_i + 1)k_{i+1} = k_{i+3}(k_{i+4} + 1) \tag{2.4}$$

and

$$(k_i + 1)(k_{i+1}k_{i+2} - 1) = k_{i+4}, \tag{2.5}$$

follow ($i = 1, 2, \ldots, 7$ and $k_{j+7} = k_j$, for $j = 1, 2, 3, 4$). □

Lemma 2.3. *For the $k-$coefficients of the boundary quadrilaterals $A_{i-1}A_iA_{i+1}A_{i+2}$*

$$k_1 = (ax-1)\frac{ax+a+x+\sqrt{D}}{2(a+1)x},$$

$$k_2 = \frac{-ax-a+x+\sqrt{D}}{2a(x+1)},$$

$$k_5 = \frac{-ax-x+a+\sqrt{D}}{2(a+1)x},$$

$$k_6 = (ax-1)\frac{ax+a+x+\sqrt{D}}{2a(x+1)},$$

$$k_7 = \frac{-ax-a-x-2+\sqrt{D}}{2},$$

hold, where $a = k_3$, $x = k_4$ and

$$D = \frac{(ax+1)(ax+a+x)^2 + 2(a^2x^2 - a^2 - x^2)}{ax-1}.$$

Proof. In order to solve this system, let's single out the following five equations

$$(k_3+1)k_4 = k_6(k_7+1), \tag{2.6}$$

$$(k_5+1)k_6 = k_1(k_2+1), \tag{2.7}$$

$$(k_7+1)k_1 = k_3(k_4+1), \tag{2.8}$$

$$k_2k_3k_4 + k_3k_4 - k_2 - k_6 = 1, \tag{2.9}$$

$$k_3k_4k_5 + k_4k_5 - k_3 - k_7 = 1. \tag{2.10}$$

Suppose $k_3 = a$, $k_4 = x$. Let's express the other unknowns from the system of equations (2.6)–(2.10) using a and x. The equations (2.6) and (2.8) get the form of $(a+1)x = k_6(k_7+1)$ and $(k_7+1)k_1 = a(x+1)$. From this, as well as from (2.7), we get that

$$\frac{k_6}{k_1} = \frac{(a+1)x}{a(x+1)} = \frac{k_2+1}{k_5+1}, \tag{2.11}$$

whereby

$$k_2+1 = \frac{(a+1)x(k_5+1)}{a(x+1)}. \tag{2.12}$$

Let's express the unknowns k_1, k_6 and k_7 in a, x and k_5. From equation (2.10), it follows that

$$k_7+1 = axk_5 + xk_5 - a,$$

$$k_7 = (a+1)(xk_5-1). \tag{2.13}$$

By substituting the obtained value for k_7 in (2.8), we get that

$$(axk_5 + xk_5 - a)k_1 = a(x+1),$$

$$k_1 = \frac{a(x+1)}{(a+1)xk_5 - a}.$$

From this, as well as from (2.11) we get that

$$k_6 = \frac{(a+1)x}{(a+1)xk_5 - a}. \tag{2.14}$$

Equation (2.9) can be written in the form

$$(k_2+1)(ax-1) = k_6. \tag{2.15}$$

By including the results for k_2+1 from (2.12) we respectively get the following

$$\frac{(a+1)x(k_5+1)}{a(x+1)}(ax-1) = \frac{(a+1)x}{(a+1)xk_5 - a},$$

$$((a+1)xk_5 - a)(k_5+1) = \frac{a(x+1)}{ax-1},$$

$$(a+1)xk_5^2 + (ax+x-a)k_5 = \frac{a(a+1)x}{ax-1}. \tag{2.16}$$

Similarly, we can get the quadratic equation for k_2. Since from (2.11)

$$(a+1)xk_5 = a(x+1)k_2 + a - x,$$

by making a substitution in (2.14), we get that

$$k_6 = \frac{(a+1)x}{a(x+1)k_2 - x}.$$

By substituting this value in (2.15), we get that

$$(k_2+1)(ax-1) = \frac{(a+1)x}{a(x+1)k_2 - x},$$

Whence by multiplying by $a(x+1)k_2 - x$ we get the quadratic equation for k_2

$$a(x+1)k_2^2 + (ax+a-x)k_2 = \frac{ax(x+1)}{ax-1}. \tag{2.17}$$

The discriminants of the quadratic equations (2.16) and (2.17) are as follows

$$D_5 = (ax+x-a)^2 + \frac{4a(a+1)^2x^2}{ax-1},$$

$$D_2 = (ax+a-x)^2 + \frac{4a^2x(x+1)^2}{ax-1}.$$

Denote by

$$D = \frac{(ax+1)(ax+a+x)^2 + 2(a^2x^2 - a^2 - x^2)}{ax-1}. \tag{2.18}$$

It is easy to check that the identities $D_5 = D_2 = D$ are valid, from which it follows that D_2 and D_5 are symmetric functions of the variables a and x. Since k_2 and k_5 are the positive solutions of equations (2.17) and (2.16) respectively, it follows that

$$k_2 = \frac{-ax-a+x+\sqrt{D}}{2a(x+1)}, \tag{2.19}$$

$$k_5 = \frac{-ax-x+a+\sqrt{D}}{2(a+1)x}. \tag{2.20}$$

From (2.15), we get that

$$k_6 = (ax-1)\frac{ax+a+x+\sqrt{D}}{2a(x+1)}, \tag{2.21}$$

while from (2.13) we get that

$$k_7 = \frac{-ax-a-x-2+\sqrt{D}}{2}. \tag{2.22}$$

From (2.11) and (2.21) we get that

$$k_1 = (ax-1)\frac{ax+a+x+\sqrt{D}}{2(a+1)x}. \tag{2.23}$$

□

By checking, we conclude that the ordered septuple $(k_1, k_2, k_3, k_4, k_5, k_6, k_7)$, where $k_3 = a$ and $k_4 = x$ are arbitrary positive real numbers, where $ax > 1$ and k_2, k_5, k_6, k_7 and k_1 are expressed via a and x in formulas (2.19)–(2.23), is the solution of the system of equations (2.4)–(2.5). From equation (2.5), it follows that $k_i k_{i+1} > 1$ for each i. Specifically, when $i = 3$ we get that $ax > 1$.

For each pair (a, x) of positive real numbers where $ax > 1$, it is easy to construct a convex heptagon $A_1A_2 \dots A_7$ such that all of its boundary quadrilaterals are trapezoids for which the relations of parallel sides, respectively k_1, k_2, $k_3 = a$, $k_4 = x$, k_5, k_6, k_7 are obtained from the above formulas.

Example 2.1. *Let's say, for example, that we have $A_3(0,1)$, $A_4(0,0)$ and $A_5(1,0)$. Let's construct the point A_2 so that $\overrightarrow{A_2A_5} = (1+k_3)\overrightarrow{A_3A_4}$, and then let's do the same for the points A_6, A_7, A_1 so that $\overrightarrow{A_3A_6} = (1+k_4)\overrightarrow{A_4A_5}$, $\overrightarrow{A_4A_7} = (1+k_5)\overrightarrow{A_5A_6}$ and $\overrightarrow{A_1A_4} = (1+k_2)\overrightarrow{A_2A_3}$. We respectively get $A_2(1, 1+k_3)$, $A_6(1+k_4, 1)$, $A_7(k_4 + k_4k_5, 1+k_5)$ and $A_1(1+k_2, k_3+k_2k_3)$. Let's check if the remaining three equations also apply to the points thus determined i.e. $\overrightarrow{A_5A_1} = (1+k_6)\overrightarrow{A_6A_7}$, $\overrightarrow{A_6A_2} = (1+k_7)\overrightarrow{A_7A_1}$ and $\overrightarrow{A_7A_3} = (1+k_1)\overrightarrow{A_1A_2}$.*

The equation $\overrightarrow{A_5A_1} = (1+k_6)\overrightarrow{A_6A_7}$ comes down to two equations

$$k_4k_5k_6 + k_4k_5 - k_6 - 1 = k_2,$$

$$k_3 + k_2k_3 = k_5 + k_5k_6.$$

The first equation $k_4k_5(k_6+1) - k_2 - k_6 = 1$ based on (2.4) *when $i = 2$ reduces to*

$$k_4(k_2+1)k_3 - k_2 - k_6 = 1,$$

and that is equation (2.9)*. The second equation is equation* (2.4) *when $i = 2$.*

The equation $\overrightarrow{A_6A_2} = (1+k_7)\overrightarrow{A_7A_1}$ reduces to the following equations

$$1 + k_2 - k_4k_5 + k_7 + k_2k_7 - k_4k_7 - k_4k_5k_7 = 0,$$

$$k_2k_3 - 1 - k_5 + k_3k_7 + k_2k_3k_7 - k_7 - k_5k_7 = 0.$$

By applying (2.4) *when $i = 1$ and* (2.5) *when $i = 7$, we once again reduce the first equation to* (2.4) *when $i = 1$ in the following way*

$$1 + k_2 - k_4k_5 + k_7 + k_2k_7 - k_7k_4(k_5+1) = 0,$$

$$1 + k_2 - k_4k_5 + k_7 + k_2k_7 - k_7(k_1+1)k_2 = 0,$$

$$k_2 - k_4k_5 = k_7k_1k_2 - k_7 - 1,$$

$$k_2 - k_4k_5 = k_4 - k_1k_2,$$

$$k_1k_2 + k_2 = k_4k_5 + k_4.$$

By applying (2.4) *when* $i = 2$ *and* (2.5) *when* $i = 5$ *, we reduce the second equation to* (2.4) *when* $i = 6$*, in the following way*

$$k_2k_3 - 1 - k_5 + (k_2+1)k_3k_7 - k_7 - k_5k_7 = 0,$$

$$k_2k_3 - 1 - k_5 + k_5(k_6+1)k_7 - k_7 - k_5k_7 = 0,$$

$$k_5k_6k_7 - k_5 - 1 = k_7 - k_2k_3,$$

$$k_2 - k_6k_7 = k_7 - k_2k_3,$$

$$k_2 + k_2k_3 = k_6k_7 + k_7.$$

The equation $\overrightarrow{A_7A_3} = (1+k_1)\overrightarrow{A_1A_2}$ *is equivalent to the following equations*

$$k_4 + k_4k_5 = k_2 + k_1k_2,$$

$$1 + k_1 + k_5 - k_2k_3 - k_1k_2k_3 = 0,$$

which are valid based on (2.4) *when* $i = 1$ *and* (2.5) *when* $i = 1$.

By including the values for k_1*,* k_2*,* k_3*,* k_4 *and* k_5 *in the function of* a *and* x *we get that the coordinates of the heptagon are* $A_1A_2\ldots A_7$ *given by*

$$A_1\left(\frac{ax+a+x+\sqrt{D}}{2a(x+1)}, \frac{ax+a+x+\sqrt{D}}{2(x+1)}\right),\ A_2(1,1+a),\ A_3(0,1), \tag{2.24}$$

$$A_4(0,0),\ A_5(1,0),\ A_6(1+x,1),\ A_7\left(\frac{ax+a+x+\sqrt{D}}{2(a+1)}, \frac{ax+a+x+\sqrt{D}}{2(a+1)x}\right).$$

In this heptagon the seven boundary quadrilaterals are trapezoids for which the relations of the parallel sides are $d_i/a_i = 1+k_i$*,* $i = 1,2,\ldots,7$. □

Example 2.2. *Specifically, when* $a = 1$ *and* $x = 2$ *in* (2.18) $D = 73$*, from* (2.24) *we get one such heptagon whose vertices are* $A_2(1,2)$*,* $A_3(0,1)$*,* $A_4(0,0)$*,* $A_5(1,0)$*,* $A_6(3,1)$*,*

$$A_7\left(\frac{5+\sqrt{73}}{4}, \frac{5+\sqrt{73}}{8}\right) \text{ and } A_1\left(\frac{5+\sqrt{73}}{6}, \frac{5+\sqrt{73}}{6}\right).$$ □

3. Proof of Theorem 1.1

We shall now deal with the problem of the existence of a convex lattice heptagon such that its boundary quadrilaterals are all trapezoids. It is obvious that in a convex lattice heptagon the $k-$coefficients are rational numbers.

Proof. Suppose there is a convex lattice heptagon such that all of its boundary quadrilaterals are trapezoids. If we multiply its coordinates by an arbitrary positive rational number, all k_i in the obtained convex heptagon will also be rational numbers.

Suppose therefore that there are positive rational numbers a, x such that that $\sqrt{D}$ is a rational number. Let's now consider the heptagon from the Example 2.1. Taking into account (2.24) we get a convex heptagon whose vertices have rational coordinates. If we multiply these coordinates by the common denominator of their denominators, we get a convex lattice heptagon for which all of its boundary quadrilaterals are trapezoids. Thus, if there are $a, x \in \mathbb{Q}^+$, $ax > 1$, such that D is the square of a rational number, then there is also a convex lattice heptagon such that all of its boundary quadrilaterals are trapezoids. □

Example 3.1. *For* $(a,x) = \left(\frac{5}{2}, \frac{3}{4}\right)$ *from* (2.18) *we know that* $D = (71/8)^2$*, so based on* (2.24)*, we get a convex heptagon with vertices whose coordinates are rational numbers:*

$$C_1\left(\frac{8}{5}, 4\right),\ C_2\left(1, \frac{7}{2}\right),\ C_3(0,1),\ C_4(0,0),\ C_5(1,0),\ C_6\left(\frac{7}{4}, 1\right),\ C_7\left(2, \frac{8}{3}\right).$$

Multiplying the coordinates of these 7 *vertices by* 60*, that is, with the least common denominator of their denominators, we get a convex integer heptagon* $B_1B_2\ldots B_7$ *such that all of its boundary quadrilaterals are trapezoids, with vertices*

$$B_1(96,240), B_2(60,210), B_3(0,60), B_4(0,0), B_5(60,0), B_6(105,60), B_7(120,160)$$

(Figure 1).

By using formulas (2.19)–(2.23)*, where* $k_3 = a$ *and* $k_4 = x$*, we get that the coefficients of parallelism for this heptagon are*

$$(k_1, k_2, k_3, k_4, k_5, k_6, k_7) = \left(\frac{7}{3}, \frac{3}{5}, \frac{5}{2}, \frac{3}{4}, \frac{5}{3}, \frac{7}{5}, \frac{7}{8}\right).$$

□

Example 3.2. *For* $(a,x) = \left(\frac{5}{2}, \frac{3}{5}\right)$ *from* (2.18) *we know that* $D = (47/5)^2$*, so based on* (2.24) *we get a convex heptagon with rational vertices:*

$$C_1'\left(\frac{7}{4}, \frac{35}{8}\right),\ C_2'\left(1, \frac{7}{2}\right),\ C_3'(0,1),\ C_4'(0,0),\ C_5'(1,0),\ C_6'\left(\frac{8}{5}, 1\right),\ C_7'\left(2, \frac{10}{3}\right).$$

Multiplying the coordinates of these 7 *vertices by* 120*, we get a convex integer heptagon* $B_1'B_2'\ldots B_7'$ *such that all of its boundary quadrilaterals are trapezoids, with vertices*

$$B_1'(210,525),\ B_2'(120,420),\ B_3'(0,120),\ B_4'(0,0),\ B_5'(120,0),\ B_6'(192,120)$$

and $B_7'(240,400)$ *(Figure 2).*

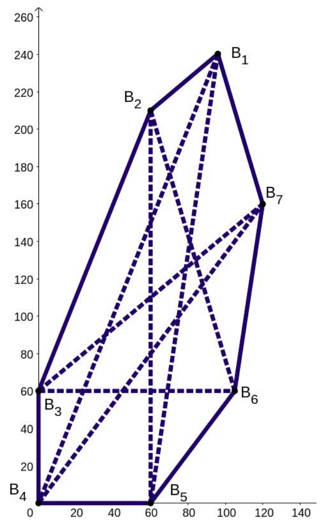

FIGURE 1

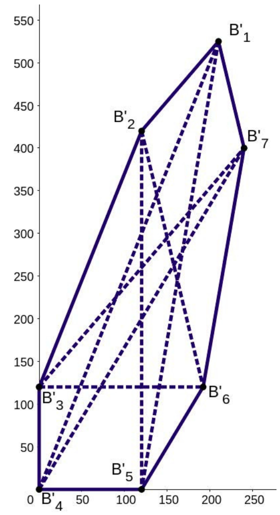

FIGURE 2

The coefficients of parallelism for this heptagon are

$$(k_1,k_2,k_3,k_4,k_5,k_6,k_7) = \left(\frac{5}{3},\frac{3}{4},\frac{5}{2},\frac{3}{5},\frac{7}{3},\frac{7}{8},\frac{7}{5}\right).$$

□

The existence of the polygon $B_1B_2\dots B_7$ from Example 3.1 and the polygon $B'_1B'_2\dots B'_7$ from Example 3.2 confirm the statements of Theorem 1.1.

Remark 3.1. Since the parallelism of the line A_iA_{i+1} and $A_{i-1}A_{i+2}$ is equivalent to the equality of the surface areas of the triangles $A_{i-1}A_iA_{i+1}$ and $A_iA_{i+1}A_{i+2}$, it follows that in an arbitrary heptagon for which all of its boundary quadrilaterals are trapezoids, the surface areas of all boundary triangles are equal to each other. For example, in the heptagon $A_1A_2\dots A_7$ this value is $1/2$, while in the heptagon $B_1B_2\dots B_7$ this value is 1800.

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