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LOCAL STABILITY, THE EXISTENCE OF CHAOTIC BEHAVIOR AND BIFURCATIONS IN AN OPEN-ACCESS FISHERY MODEL

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Dedicated to the 65th birthday of the dear Professor Mehmed Nurkanović

ABSTRACT. In this paper, we investigate an open-access fishery model which is used to examine the dynamics of the resource and industry and to explain the current economic status of the anchovy fishery. We consider the local character of the interior and boundary equilibrium points. Also, we show that the considered system of difference equations exhibits Neimark-Sacker bifurcation under certain conditions. The existence of the repelling curve and invariant curve is demonstrated. We show that in a certain parameter region the corresponding map of the considered system is an area-preserving map, so the positive equilibrium point in that case is stable. Also, we produce numerical simulations to support our findings.

1. Introduction and Preliminaries

In this paper we investigate the following system of difference equations

$$\begin{aligned} x_{n+1} &= ax_n^b - d\alpha x_n^\nu y_n \\ y_{n+1} &= y_n\left(\eta\left(p\alpha x_n^\nu - c\right)+1\right), \end{aligned} \tag{1.1}$$

where all parameters $a,b,c,d,p,\alpha,\eta,\nu$ are positive due to biological significance with additional restriction $a>1$, $0<b<1$ and $0.5<d<1$ ($d\approx 0.75$). We investigate its local dynamics and bifurcations (Neimark-Sacker bifurcation).

System (1.1) represents an open-access or "bionomic" model which is used to examine the dynamics of the resource and industry and to explain the current economic status of the anchovy fishery. Given the complexity of factor influencing anchovy populations, including environmental variability and fishing pressure, multidisciplinary approaches that integrate biological, ecological, and socio-economic data are essential for effective modeling and management. Such strategies aim to

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balance the ecological role of anchovies in marine ecosystems with the economic interest of fisheries, ensuring the sustainability of both.

The dynamics of the northern anchovy have been studied by both biologists and economists in the papers such as Radovich and MacCall [24], Huppert et al. [9], MacCall et al. [18], and Methot [19], until J.D. Opsomier and J. M. Conrad [23]. J. D. Opsomier and J. M. Conrad in [23] have modeled the northern anchovy (Engraulis mordax) fishery using open access dynamics. They explored the dynamics of a simple open access model using estimates of growth and production functions from U.S. reduction fishery data. Also, they applied this model to forecast the likely future behavior of the northern anchovy fishery. They demonstrated numerical evidence for the existence of a stable long-term equilibrium point and limit cycles. The key components of the open-access model are the net growth function for the anchovy stock and the production function for the wetfish fleet. An open-access model can be described by (1.1), where x_n is the biomass of anchovy in the year n, y_n is the total harvest of anchovy in the year n, p is the exvessel price of anchovy, c is the cost per unit effort, d is the biological discount factor, η is the effort adjustment coefficient, and a,b are coefficients in the power function $F(x_n) = ax_n^b$ where we would expect $a > 1$ and $0 < b < 1$. Sensitivity analysis was performed in [23] to asses the impact of the biological discount factor d and given the results of that analysis we assume that $0.5 < d < 1$, i.e., $d \approx 0.75$. Below α and ν are positive parameters in the production function $H(x_n) = \alpha x_n^{\nu} y_n$.

Although the considered system depends on 8 parameters, the analysis of its local stability was successfully performed using the next lemma, which can be easily proved by the relations between roots and coefficients of the quadratic equation (see [1, 3, 13] and Lemma 2.2 in [17]).

Lemma 1.1. *Assume that* $\varphi(\lambda) = \lambda^2 - trA\lambda + detA$ *is a characteristic polynomial of the matrix A, and that* $\varphi(1) > 0$*. Then,*
(a) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *iff* $\varphi(-1) > 0$ *and* $\varphi(0) < 1$*,*
(b) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *iff* $\varphi(-1) > 0$ *and* $\varphi(0) > 1$*,*
(c) $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*) iff* $\varphi(-1) < 0$*,*
(d) λ_1 *and* λ_2 *are complex, and* $|\lambda_1| = |\lambda_2| = 1$ *iff* $(trA)^2 - 4\det A < 0$ *and* $\varphi(0) = 1$*,*
(e) $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *iff* $\varphi(-1) = 0$ *and* $trA \neq 0, -2$*.*

2. Local stability of equilibrium points

It does not make sense to study the anchovy yield per year if the anchovy biomass in a given year is zero. Yield refers to the amount of fish that can be harvested, and if there is no biomass, it means the population does not exist, so the yield cannot be defined. Therefore we assume that $x_0 > 0$.

Similarly, as J. D. Opsomer and M. D. Conrad concluded in [23] that the dynamics in the special case depends only on $\frac{c}{p}$ and η, we introduce a shift $\frac{c}{p} = s$ and

$t = p\eta$ so the corresponding map of System (1.1) is

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax^b - d\alpha x^\nu y \\ y\left(t\left(\alpha x^\nu - s\right) + 1\right) \end{pmatrix}. \tag{2.1}$$

The equilibrium points of System (1.1) are the solutions of system:

$$\begin{aligned} ax^b - d\alpha x^\nu y &= x \\ y\left(t\left(\alpha x^\nu - s\right) + 1\right) &= y. \end{aligned} \tag{2.2}$$

From (2.2) System (1.1) has zero equilibrium point $E_0 = (0,0)$, which is singular point as the Jacobian matrix in this equilibrium is undefined. In view of the fact that E_0 is not biologically feasible, we do not need to consider the stability of E_0. In addition, System (1.1) has a boundary equilibrium point $E_1 = \left(a^{\frac{1}{1-b}}, 0\right)$ for all considered parameter values, and the positive equilibrium point

$$E_+ = (\bar{x}, \bar{y}) = \left(\bar{x}, \frac{a\bar{x}^b - \bar{x}}{sd}\right), \quad \bar{x} = \left(\frac{s}{\alpha}\right)^{\frac{1}{\nu}}, \tag{2.3}$$

under the condition

$$a\bar{x}^b - \bar{x} > 0, \text{ i.e., } s < s_E = \alpha a^{\frac{\nu}{1-b}}. \tag{2.4}$$

The Jacobian matrix of the map T is:

$$J_T(x,y) = \begin{pmatrix} abx^{b-1} - dx^{\nu-1} y\alpha\nu & -dx^\nu \alpha \\ x^{\nu-1} y\alpha\nu t & x^\nu \alpha t - st + 1 \end{pmatrix},$$

so the characteristic polynomial of the $J_T(x,y)$ for $x \neq 0$ is:

$$\varphi(\lambda) = \lambda^2 - \frac{x(1-st) + abx^b + \alpha(tx - \nu y d)x^\nu}{x}\lambda + \frac{ab(1-st)x^b + \alpha\left(abtx^b + \nu d(st-1)y\right)x^\nu}{x}.$$

Using (2.3) we get that the characteristic polynomial at the equilibrium point $E_+ = (\bar{x}, \bar{y})$ is

$$\varphi(\lambda) = \lambda^2 - \left(1 + \nu + a(b-\nu)\bar{x}^{b-1}\right)\lambda + (1-st)\nu + a(b - \nu + st\nu)\bar{x}^{b-1}, \tag{2.5}$$

so the corresponding characteristic equation is

$$\lambda^2 - \left(1 + \nu + a(b-\nu)\bar{x}^{b-1}\right)\lambda + (1-st)\nu + a(b - \nu + st\nu)\bar{x}^{b-1} = 0. \tag{2.6}$$

2.1. **Local stability of the boundary equilibrium point**

The Jacobian matrix of the map T at the boundary equilibrium point $E_1 = \left(a^{\frac{1}{1-b}}, 0\right)$ is:

$$J_T(x,y) = \begin{pmatrix} b & -da^{\frac{\nu}{1-b}}\alpha \\ 0 & a^{\frac{\nu}{1-b}}\alpha t - st + 1 \end{pmatrix},$$

and its eigenvalues are $\lambda_1 = b \in (0,1)$ and $\lambda_2 = \alpha t a^{\frac{\nu}{1-b}} - st + 1 = t\left(\alpha a^{\frac{\nu}{1-b}} - s\right) + 1$. The following theorem holds.

Theorem 2.1. *If a,b,d,α,s,t are positive parameters such that $a>1$ and $b<1$, System (1.1) has the boundary the equilibrium point $E_1=\left(a^{\frac{1}{1-b}},0\right)$, which is:*

(1) locally asymptotically stable if $\alpha a^{\frac{\nu}{1-b}}<s<\alpha a^{\frac{\nu}{1-b}}+\frac{2}{t}$,
(2) a saddle point if $0<s<\alpha a^{\frac{\nu}{1-b}}$ or $s>\alpha a^{\frac{\nu}{1-b}}+\frac{2}{t}$,
(3) a non-hyperbolic equilibrium if $s=\alpha a^{\frac{\nu}{1-b}}$, with eigenvalues $\lambda_1=b<1$ and $\lambda_2=1$, or if $s=\alpha a^{\frac{\nu}{1-b}}+\frac{2}{t}$, with eigenvalues $\lambda_1=b<1$ and $\lambda_2=-1$.

For $x>0$ it holds

$$T\begin{pmatrix} x \\ 0 \end{pmatrix}=\begin{pmatrix} a^{1+b+b^2+\ldots+b^{n-1}}x^{b^n} \\ 0 \end{pmatrix}=\begin{pmatrix} a^{\frac{1-b^n}{1-b}}x^{b^n} \\ 0 \end{pmatrix},$$

so since $b<1$ it is satisfied:

$$T^n\begin{pmatrix} x \\ 0 \end{pmatrix}\longrightarrow\begin{pmatrix} a^{\frac{1}{1-b}} \\ 0 \end{pmatrix},\ n\longrightarrow\infty.$$

Therefore, the positive part of the x-axis is the stable manifold of the equilibrium point $E_1=\left(a^{\frac{1}{1-b}},0\right)$. In practice, this means that if the total harvest of anchovy in year is zero, it will remain zero in subsequent years, and over the years the biomass of anchovy will tend to the $a^{\frac{1}{1-b}}$.

2.2. Local stability of the positive equilibrium point

In applications, the examination of the local stability of the positive equilibrium is of particular importance. In order to apply Lemma 1.1 in examining its local stability, we calculate the value of the characteristic polynomial (2.5) for $\lambda=1$ at the equilibrium point $E_+=(\overline{x},\overline{y})$, so we get

$$\varphi(1)=\frac{st\nu}{\overline{x}}\left(a\overline{x}^b-\overline{x}\right)=\frac{s^2t\nu d\overline{y}}{\overline{x}}.$$

From the past equation we can notice that $\varphi(1)>0$ which implies that Lemma 1.1 is applicable. This means that local stability depends on the value of the expressions $\varphi(-1)$ and $\varphi(0)$. We break our further analysis into two cases, $0<\nu\le b<1$ and $0<b<1$, $b<\nu$, of course with condition $s<s_E$ which is related to the existence of the positive equilibrium.

Case 2.1 ($0<\nu\le b<1$).

Assume that $0<\nu\le b<1$, and $s<s_E=\alpha a^{\frac{\nu}{1-b}}$. Since $s<s_E$ we get $\varphi(-1)=0$ if

$$t=t_{-1}(s)=\frac{2(\nu+1)s^{\frac{1-b}{\nu}}-2a(\nu-b)\alpha^{\frac{1-b}{\nu}}}{s\nu\left(s^{\frac{1-b}{\nu}}-a\alpha^{\frac{1-b}{\nu}}\right)}, \tag{2.7}$$

and $\varphi(0) = 1$ if

$$t = t_0(s) = \frac{(\nu - 1)s^{\frac{1-b}{\nu}} - a(\nu - b)\alpha^{\frac{1-b}{\nu}}}{s\nu\left(s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}}\right)}.$$

Notice that

$$s < s_E \Longrightarrow s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}} < 0. \tag{2.8}$$

Using (2.8) and the condition $0 < \nu \leq b < 1$ we get $t = t_{-1}(s) < 0$ for all $s < s_E$, that is, its graph is below the s-axis in the st-plane. Also, if $0 < \nu \leq b < 1$ and $s < s_E$ we have $\varphi(-1) > 0$. Indeed, by using the inequality $a\bar{x}^{b-1} > 1$ we get the following estimate

$$\begin{aligned}\varphi(-1) &= 2 + 2\nu + a(b-\nu)\bar{x}^{b-1} - st\nu + a(b - \nu + st\nu)\bar{x}^{b-1} \\ &> 2 + 2\nu + (b - \nu) - st\nu + (b - \nu + st\nu) = 2b + 2 > 0.\end{aligned}$$

On the other hand $t_0(s) = 0$ for

$$s = s_0 = \alpha\left(\frac{a(b-\nu)}{1-\nu}\right)^{\frac{\nu}{1-b}}.$$

Notice that the point $(s_0, 0)$ is to the left of the point $(s_E, 0)$ on the s axis in the st-plane, i.e., $s_0 < s_E$ for $0 < \nu \leq b < 1$.

Using (2.8) and

$$s > s_0 \Longrightarrow (\nu - 1)s^{\frac{1-b}{\nu}} - a(\nu - b)\alpha^{\frac{1-b}{\nu}} < 0 \tag{2.9}$$

we can see that $t_0(s) > 0$ for $s \in (s_0, s_E)$ and $t_0(s) < 0$ for $0 < s < s_E$. Since $t_0(s)$ is a continuous function for $s < s_E$, the curve given by the equation $t = t_0(s)$ divides the domain $D(E_+) = \{(s,t) : s < s_E \wedge t > 0\}$ into two parts, one in which $\varphi(0) > 1$ and the other in which $\varphi(0) < 1$. The points $(s, 0)$, where $s \in (s_0, s_E)$ are in the st-plane below curve given by the equation $t = t_0(s)$. Let's show that $\varphi(0) < 1$ for $s \in (s_0, s_E)$ and $t = 0$. Indeed, the expression $\varphi(0)$ for $s \in (s_0, s_E)$ and $t = 0$ becomes

$$\begin{aligned}\varphi(0)\,|_{t=0} &= (1-0)\nu + a(b - \nu + 0)\left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}} \\ &= \nu + a(b-\nu)\left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}},\end{aligned}$$

so by

$$s > s_0 \Longrightarrow \left(\frac{s}{\alpha}\right)^{\frac{1-b}{\nu}} > \frac{a(b-\nu)}{1-\nu} \Longleftrightarrow \left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}} < \frac{1-\nu}{a(b-\nu)}$$

we have

$$\varphi(0)\,|_{t=0} < \nu + a(b-\nu)\frac{1-\nu}{a(b-\nu)} = 1.$$

Therefore,

$$\varphi(0) > 1 \text{ if } \left(0 < s < s_E \ \wedge \ t > \max\{t_0(s), 0\}\right),$$
$$\varphi(0) < 1 \text{ if } \left(s_0 < s < s_E \ \wedge \ 0 < t < t_0(s)\right).$$

If $\varphi(0) = 1$ we notice that $\det J_T(\bar{x}, \bar{y}) = (1 - st)\nu + a(b - \nu + st\nu)\bar{x}^{b-1} = 1$ which implies

$$\bar{x}^{b-1} = \frac{1 - \nu + st\nu}{a(b - \nu + st\nu)}. \tag{2.10}$$

and by using (2.3) we have

$$\left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}} = \frac{1 - \nu + st\nu}{a(b - \nu + st\nu)}.$$

The discriminant of the characteristic equation corresponding to the $J_T(\bar{x}, \bar{y})$ is

$$(trJ_T(\bar{x}, \bar{y}))^2 - 4\det J_T(\bar{x}, \bar{y}) = \left(-\left(1 + \nu + a(b - \nu)\bar{x}^{b-1}\right)\right)^2 - 4$$
$$\overset{(2.10)}{=} \left(1 + \nu + a(b - \nu)\left(\frac{1 - \nu + st\nu}{a(b - \nu + st\nu)}\right)\right)^2 - 4,$$

i.e.,

$$(trJ_T)^2 - 4\det J_T = \frac{st\nu(b-1)(4(b-\nu) + (b+3)st\nu)}{(b - \nu + st\nu)^2} \tag{2.11}$$

so $(trJ_T(\bar{x}, \bar{y}))^2 - 4\det J_T(\bar{x}, \bar{y}) < 0$ because $0 < \nu \leq b < 1$.

Using the previous computations and Lemma 1.1, we proved the following theorem.

Theorem 2.2. *If $a, b, d, \alpha, \nu, s, t$ are positive parameters such that $a > 1$, $\nu \leq b < 1$, and $s < s_E = \alpha a^{\frac{\nu}{1-b}}$, then System (1.1) has the unique positive the equilibrium point $E_+ = \left(\bar{x}, \frac{1}{ds}\left(a\bar{x}^b - \bar{x}\right)\right)$, where $\bar{x} = \left(\frac{s}{\alpha}\right)^{\frac{1}{\nu}}$ which is:*

(1) locally asymptotically stable if

$$s \in (s_0, s_E) \quad \textit{and} \quad t \in (0, t_0(s)),$$

(2) a repeller if

$$(s \in (0, s_0] \textit{ and } t > 0) \quad \textit{or} \quad (s \in (s_0, s_E) \textit{ and } t > t_0(s)),$$

(3) non-hyperbolic with complex conjugate eigenvalues if

$$s \in (s_0, s_E) \quad \textit{and} \quad t = t_0(s),$$

where

$$s_0 = \alpha\left(\frac{a(b-\nu)}{1-\nu}\right)^{\frac{\nu}{1-b}} \quad \textit{and} \quad t_0(s) = \frac{(\nu - 1)s^{\frac{1-b}{\nu}} + a(b-\nu)\alpha^{\frac{1-b}{\nu}}}{s\nu\left(s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}}\right)}.$$

Figure 1 represents areas of local stability in the st-plane for special parameter values that satisfy Case 1. In the blue area the equilibrium is a repeller, in the green area the equilibrium is locally asymptotically stable and on the red curve that separates them, the equilibrium is non-hyperbolic with eigenvalues that are complex conjugate numbers.

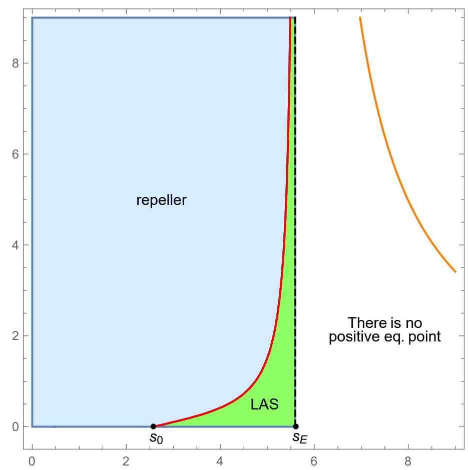


FIGURE 1. Parametric spaces of local dynamics of positive equilibrium E_+ for $a = 38.5132$, $b = 0.7365 > \nu = 0.3739$, $d = 0.75$, and $\alpha = 0.03147$. The red curve is $t = t_0(s)$ and the orange curve is $t = t_{-1}(s)$ in the st-plane.

Case 2.2 ($0 < b < 1$ and $b < \nu$).

Now suppose that $0 < b < 1$, $b < \nu$, and $0 < s < s_E$. Let us show that then it holds $t_0(s) > 0$. Using the conditions $a > 1$ and $b - \nu < 0$ we get

$$(\nu - 1)\,s^{\frac{1-b}{\nu}} + a(b-\nu)\,\alpha^{\frac{1-b}{\nu}} < (\nu - 1)\,s^{\frac{1-b}{\nu}} + (b-\nu)\,s^{\frac{1-b}{\nu}} = (b-1)\,s^{\frac{1-b}{\nu}},$$

and by $b < 1$ we have that the denominator of $t_0(s)$ is negative. This by (2.8) implies the above conclusion. Thus, since $t = t_0(s)$ is a continuous function for $s < s_E$ the curve $t = t_0(s)$ divides the domain $D(E_+)$ into two parts, one in which $\varphi(0) > 1$ and the other in which $\varphi(0) < 1$. Obviously the points $(s, 0)$ for $s < s_E$ are in the st-plane below curve given by the equation $t = t_0(s)$. Let's show that $\varphi(0) < 1$ for $s < s_E$ and $t = 0$. From $s = s_E < \alpha a^{\frac{\nu}{1-b}}$ we have $\left(\frac{s}{\alpha}\right)^{\frac{1-b}{\nu}} < a$, i.e., $1 < a\left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}}$ so by $b < \nu$ we have $a(b-\nu)\left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}} < (b-\nu)$ and for $t = 0$ it implies

$$\begin{aligned}\varphi(0)|_{t=0} &= (1-0)\nu + a(b-\nu+0)\left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}} \\ &= \nu + a(b-\nu)\left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}} < \nu + (b-\nu) = b < 1.\end{aligned}$$

Thus,

$$\begin{aligned}\varphi(0) > 1 \text{ if } (0 < s < s_E \ \wedge \ t > t_0(s)), \\ \varphi(0) < 1 \text{ if } (0 < s < s_E \ \wedge \ 0 < t < t_0(s)).\end{aligned}$$

On the other hand $t_-(s) = 0$ for $s = s_{-1}$, where

$$s_{-1} = \alpha\left(\frac{a(\nu-b)}{\nu+1}\right)^{\frac{\nu}{1-b}}.$$

Notice that $s_{-1} < s_E$, indeed

$$\alpha\left(\frac{a(\nu-b)}{\nu+1}\right)^{\frac{\nu}{1-b}} < \alpha a^{\frac{\nu}{1-b}} \Leftrightarrow \nu - b < \nu + 1 \Leftrightarrow b + 1 > 0$$

which is true.

Let us show that $t_{-1}(s) > 0$ for $0 < s < s_{-1}$ and $t_{-1}(s) < 0$ for $s_{-1} < s < s_E$. Indeed, from (2.8) and

$$s < s_{-1} \Longrightarrow s < \alpha\left(\frac{a(\nu-b)}{\nu+1}\right)^{\frac{\nu}{1-b}} \Longrightarrow (\nu+1)s^{\frac{1-b}{\nu}} - a(\nu-b)\alpha^{\frac{1-b}{\nu}} < 0,$$

follows the above conclusion. Thus, since $t = t_{-1}(s)$ is a continuous function for $s < s_E$ the curve given by the equation $t = t_{-1}(s)$ divides the domain $D(E_+)$ into two parts, one in which $\varphi(-1) > 0$ and the other in which $\varphi(-1) < 0$. Obviously the points $(s,0)$ for $s < s_{-1}$ are in the st-plane below curve given by the equation $t = t_{-1}(s)$. Let us show that $\varphi(-1) < 0$ for $s < s_{-1}$ and $t = 0$. From $s = s_{-1} < \alpha\left(\frac{a(\nu-b)}{\nu+1}\right)^{\frac{\nu}{1-b}}$ we have $\left(\frac{s}{\alpha}\right)^{\frac{1-b}{\nu}} < \frac{a(\nu-b)}{\nu+1}$, i.e., $\left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}} > \frac{\nu+1}{a(\nu-b)}$ so by $b < \nu$ we have $(b-\nu)\left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}} < (b-\nu)\frac{\nu+1}{a(\nu-b)} = -\frac{\nu+1}{a}$, i.e., $a(b-\nu)\left(\frac{s}{\alpha}\right)^{\frac{b-1}{\nu}} < -(\nu+1)$ and for $s < s_{-1}$ and $t = 0$ it implies

$$\begin{aligned}\varphi(-1)|_{t=0} &= 2 + \nu + a(b-\nu)\bar{x}^{b-1} + \nu + a(b-\nu)\bar{x}^{b-1} \\ &= 2\left(1 + \nu + a(b-\nu)\bar{x}^{b-1}\right) \\ &< 2(1 + \nu - (\nu+1)) = 0.\end{aligned}$$

Thus,

$$\varphi(-1) > 0 \text{ if } (0 < s < s_E \wedge t > \max\{t_{-1}(s), 0\}),$$
$$\varphi(-1) < 0 \text{ if } (0 < s < s_{-1} \wedge 0 < t < t_{-1}(s)).$$

Let us now fix the intersection point of curves given by the equations $t = t_{-1}(s)$ and $t = t_0(s)$. The equation $t_{-1}(s) = t_0(s)$ implies

$$2(\nu+1)s^{\frac{1-b}{\nu}} + 2a(b-\nu)\alpha^{\frac{1-b}{\nu}} = (\nu-1)s^{\frac{1-b}{\nu}} + a(b-\nu)\alpha^{\frac{1-b}{\nu}},$$

i.e.,

$$(\nu+3)s^{\frac{1-b}{\nu}} = a(\nu-b)\alpha^{\frac{1-b}{\nu}},$$

from which we get

$$s = s_p = \alpha\left(\frac{a(\nu-b)}{\nu+3}\right)^{\frac{\nu}{1-b}}.$$

Notice that $s_p < s_{-1}$ for the considered parameter values in this case. The ordinate of the intersection point of the curves is

$$t_p = t_{-1}(s_p) = \frac{2\left((\nu+1)\frac{a(\nu-b)}{\nu+3} + a(b-\nu)\right)\alpha^{\frac{1-b}{\nu}}}{s_p\nu\left(\frac{a(\nu-b)}{\nu+3} - a\right)\alpha^{\frac{1-b}{\nu}}} = \frac{4(\nu-b)}{(b+3)\nu s_p}.$$

Notice that if $s = s_p$ then $\bar{x} = \left(\frac{s}{\alpha}\right)^{\frac{1}{\nu}} = \left(\frac{a(\nu-b)}{\nu+3}\right)^{\frac{1}{1-b}}$, so the characteristic polynomial for $s = s_p$ and $t = t_p$ at the equilibrium point E_+ is

$$\varphi(\lambda) = -\bar{x}(\lambda+1)^2(b+3),$$

i.e., if $s = s_p$ and $t = t_p$ then $\lambda_{1,2} = -1$, i.e., we have 1:2 resonance.

Also we have that $t_{-1}(s) - t_0(s) > 0$ for $s < s_p$ and $t_{-1}(s) - t_0(s) < 0$ for $s_{-1} > s > s_p$. Namely,

$$t_{-1}(s) - t_0(s) = \frac{2(\nu+1)s^{\frac{1-b}{\nu}} + 2a(b-\nu)\alpha^{\frac{1-b}{\nu}}}{s\nu\left(s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}}\right)} - \frac{(\nu-1)s^{\frac{1-b}{\nu}} + a(b-\nu)\alpha^{\frac{1-b}{\nu}}}{s\nu\left(s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}}\right)}$$
$$= \frac{(\nu+3)s^{\frac{1-b}{\nu}} - a(\nu-b)\alpha^{\frac{1-b}{\nu}}}{s\nu\left(s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}}\right)},$$

so from (2.8) and

$$s < s_p \Longrightarrow (\nu+3)s^{\frac{1-b}{\nu}} - a(\nu-b)\alpha^{\frac{1-b}{\nu}} < 0$$

the conclusion follows.

Let us now show that for $s_p < s < s_E$ and $t = t_0(s)$ the characteristic values $\lambda_{1,2}$ corresponding to polynomial (2.5) are complex conjugate numbers. Notice,

$$s_p < s < s_E \Longleftrightarrow \alpha^{\frac{1-b}{\nu}} \left(\frac{a(\nu - b)}{\nu + 3} \right) < s^{\frac{1-b}{\nu}} < a\alpha^{\frac{1-b}{\nu}}.$$

By $0 < b < 1$ and $b < \nu$ we see that the discriminant of the characteristic equation (2.6) corresponding to the $J_T(\overline{x}, \overline{y})$ is negative if

$$4(b-\nu) + (b+3)st\nu > 0,$$

which is satisfied for $t = t_0(s)$. Namely, if $t = t_0(s)$ we have

$$4(b-\nu) + (b+3)st\nu = 4(b-\nu) + (b+3)s \left(\frac{(\nu-1)s^{\frac{1-b}{\nu}} - a(\nu-b)\alpha^{\frac{1-b}{\nu}}}{s\nu\left(s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}}\right)} \right)\nu$$

$$= \frac{(b-1)\left((\nu+3)s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}}(\nu-b)\right)}{s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}}}$$

which is positive because of (2.8) and

$$s > s_p \Longrightarrow (\nu+3)s^{\frac{1-b}{\nu}} - a(\nu-b)\alpha^{\frac{1-b}{\nu}} > 0.$$

Notice that

$$trJ_T = \nu - 1 + a(b-\nu)\overline{x}^{b-1} < 0$$

for $b < \nu < 1$, so $trJ_T \neq 0$ and $trJ_T \neq 2$.

If $\nu > 1$ and $b < \nu$ then $trJ_T = -1 + \nu + a(b-\nu)\overline{x}^{b-1} = 0$ if and only if $\overline{x}^{b-1} = \frac{\nu-1}{a(\nu-b)}$ which is impossible because

$$s < s_p \Longrightarrow \overline{x}^{1-b} < \frac{a(\nu-b)}{\nu+3} \Longleftrightarrow \overline{x}^{b-1} > \frac{\nu+3}{a(\nu-b)}.$$

Analogously we get that $trJ_T \neq 2$. Thus, if $0 < b < 1$ and $b < \nu$ it is always satisfied $trJ_T \neq 0$ and $trJ_T \neq 2$ for $s < s_p$.

In view of the previous consideration, we have proven the following theorem.

Theorem 2.3. *If $a, b, d, \alpha, \nu, s, t$ are positive parameters such that $a > 1$, $b < \nu$, $b < 1$, and $s < s_E = \alpha a^{\frac{\nu}{1-b}}$, then System (1.1) has unique positive equilibrium point $E_+ = \left(\overline{x}, \frac{1}{sd}\left(a\overline{x}^b - \overline{x}\right)\right)$, where $\overline{x} = \left(\frac{s}{\alpha}\right)^{\frac{1}{\nu}}$, which is:*

(1) locally asymptotically stable if

$$(s \in (s_p, s_{-1}) \text{ and } t \in (t_{-1}(s), t_0(s))) \text{ or } (s \in [s_{-1}, s_E) \text{ and } t \in (0, t_0(s))),$$

(2) a repeller if

$$(s \in (0, s_p] \text{ and } t > t_{-1}(s)) \text{ or } (s \in (s_p, s_E) \text{ and } t > t_0(s)),$$

(3) *a saddle point if*

$$s \in (0, s_{-1}) \quad \textit{and} \quad t \in (0, t_{-1}(s)),$$

(4) *a non-hyperbolic with complex conjugate eigenvalues if*

$$s \in (s_p, s_E) \quad \textit{and} \quad t = t_0(s),$$

(5) *non-hyperbolic with both eigenvalues equal to* -1 *if*

$$s = s_p \quad \textit{and} \quad t = t_p,$$

and then is $\bar{x} = \left(\frac{a(\nu-b)}{\nu+3}\right)^{\frac{1}{1-b}}$.

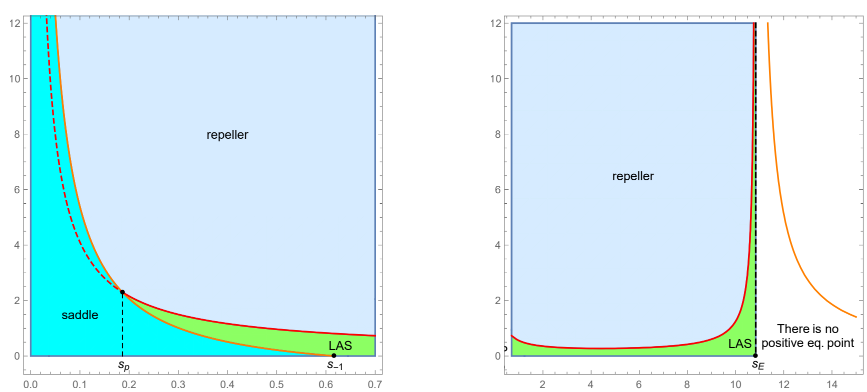
(6) *a non-hyperbolic with eigenvalues* $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *iff*

$$s \in (0, s_p) \quad \textit{and} \quad t = t_{-1}(s),$$

where

$$s_p = \alpha\left(\frac{a(\nu-b)}{\nu+3}\right)^{\frac{\nu}{1-b}}, \quad s_{-1} = \alpha\left(\frac{a(\nu-b)}{\nu+1}\right)^{\frac{\nu}{1-b}},$$

$$t_0(s) = \frac{(\nu-1)s^{\frac{1-b}{\nu}} + a(b-\nu)\alpha^{\frac{1-b}{\nu}}}{s\nu\left(s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}}\right)}, \quad t_{-1}(s) = \frac{2(\nu+1)s^{\frac{1-b}{\nu}} - 2a(\nu-b)\alpha^{\frac{1-b}{\nu}}}{s\nu\left(s^{\frac{1-b}{\nu}} - a\alpha^{\frac{1-b}{\nu}}\right)}.$$



(A) (B)

FIGURE 2. Parametric spaces of local dynamics of positive equilibrium E_+ for $a = 38.5132$, $b = 0.5 < \nu = 0.8$, $d = 0.75$, and $\alpha = 0.03147$. The red curve is $t = t_0(s)$ and the orange curve is $t = t_{-1}(s)$ in the st-plane.

Figure 2 represents areas of local stability in the st-plane for special parameter values that satisfy Case 2. In the blue area the equilibrium is a repeller, in the green area the equilibrium is locally asymptotically stable, in the cyan area the equilibrium is a saddle, on the red not dashed curve the equilibrium is non-hyperbolic with eigenvalues that are complex conjugate numbers, and on the orange not dashed curve (for $s < s_p$) the equilibrium is non-hyperbolic with both eigenvalues equal to -1. Figure 2(A) is a magnified view of the area near the origin, and Figure 2(B) contains the rest of the area showing local stability.

3. Neimark-Sacker bifurcation

In this section we discuss the existence of Neimark-Sacker bifurcation for the unique positive equilibrium and compute asymptotic approximation of the invariant curve near the positive equilibrium point E_+ of System (1.1). We follow the algorithm from Theorem 1 and Corollary 1 in [11,16] (see also [2,6,7,14,15,20,22]). We assume that $0 < \nu < b < 1$ and $s_0 < s < s_E$. If we make a change of variable $u_n = x_n - \bar{x}$ and $w_n = y_n - \bar{y}$ in (2.1), we will shift the equilibrium point to the origin. Then, the transformed system is given by

$$\begin{aligned} u_{n+1} + \bar{x} &= a\left(u_n + \bar{x}\right)^b - d\alpha\left(u_n + \bar{x}\right)^\nu\left(w_n + \bar{y}\right), \\ w_{n+1} + \bar{y} &= \left(w_n + \bar{y}\right)\left(t\left(\alpha\left(u_n + \bar{x}\right)^\nu - s\right) + 1\right), \end{aligned}$$

i.e.,

$$\begin{aligned} u_{n+1} &= a\left(u_n + \bar{x}\right)^b - d\alpha\left(u_n + \bar{x}\right)^\nu\left(w_n + \bar{y}\right) - \bar{x}, \\ w_{n+1} &= \left(w_n + \bar{y}\right)\left(t\left(\alpha\left(u_n + \bar{x}\right)^\nu - s\right) + 1\right) - \bar{y}. \end{aligned} \tag{3.1}$$

Let K denote the corresponding map defined by

$$K\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} a\left(u + \bar{x}\right)^b - d\alpha\left(u + \bar{x}\right)^\nu\left(w + \bar{y}\right) - \bar{x} \\ \left(w + \bar{y}\right)\left(t\left(\alpha\left(u + \bar{x}\right)^\nu - s\right) + 1\right) - \bar{y} \end{pmatrix}.$$

Then, the Jacobian matrix of K is given by

$$Jac_K(u,w) = \begin{pmatrix} ab\left(u + \bar{x}\right)^{b-1} - d\alpha\nu\left(u + \bar{x}\right)^{\nu-1}\left(w + \bar{y}\right) & -d\alpha\left(u + \bar{x}\right)^\nu \\ \left(w + \bar{y}\right)t\alpha\nu\left(u + \bar{x}\right)^{\nu-1} & t\left(\alpha\left(u + \bar{x}\right)^\nu - s\right) + 1 \end{pmatrix},$$

and the characteristic equation of the corresponding characteristic polynomial of $Jac_K(0,0)$ is:

$$\lambda^2 - \left(1 + \nu + a\left(b - \nu\right)\bar{x}^{b-1}\right)\lambda + \left(1 - st\right)\nu + a\left(b - \nu + st\nu\right)\bar{x}^{b-1} = 0. \tag{3.2}$$

Solutions $\lambda(t)$, $\overline{\lambda(t)}$ of (3.2) are

$$\lambda(t) = \frac{1 + \nu + a\left(b - \nu\right)\bar{x}^{b-1} + i\sqrt{\Delta_{\lambda(t)}}}{2},$$

where

$$\Delta_{\lambda(t)} = -\left(a\left(b - \nu\right)\bar{x}^{b-1}\right)^2 - \left(\nu - 1\right)^2 + 2\left(1 - \nu\right)\left(b - \nu\right)a\bar{x}^{b-1} + 4st\nu\left(a\bar{x}^{b-1} - 1\right),$$

and

$$|\lambda(t)| = \sqrt{s\nu t\left(a\overline{x}^{b-1}-1\right)+a\overline{x}^{b-1}(b-\nu)+\nu}. \tag{3.3}$$

By using $s > s_0$ and $s < s_E$, or equivalently $a(b-\nu)\overline{x}^{b-1} < 1-\nu$ and $a\overline{x}^{b-1} > 1$ the following estimate is valid

$$\begin{aligned}\Delta_{\lambda(t)} &= -\left(a(b-\nu)\overline{x}^{b-1}\right)^2-(\nu-1)^2+2(1-\nu)(b-\nu)a\overline{x}^{b-1}+4st\nu\left(a\overline{x}^{b-1}-1\right)\\ &> -(1-\nu)^2-(\nu-1)^2+2(1-\nu)(b-\nu)\\ &= 2(1-\nu)(b-\nu)>0,\end{aligned}$$

i.e., the discriminant $\Delta_{\lambda(t)}$ is positive for all $t > 0$.

Lemma 3.1. *Let $a > 0$, $\alpha > 0$, $0 < \nu \le b < 1$, $s_0 = \alpha\left(\frac{a(b-\nu)}{1-\nu}\right)^{\frac{\nu}{1-b}}$, $s_E = \alpha a^{\frac{\nu}{1-b}}$, $\overline{x} = \left(\frac{s}{\alpha}\right)^{\frac{1}{\nu}}$ and $s_0 < s < s_E$. If $t = t_0 = \frac{1-\nu-a(b-\nu)\overline{x}^{b-1}}{s\nu\left(a\overline{x}^{b-1}-1\right)}$ then K has equilibrium point at $(0,0)$ and eigenvalues of Jacobian matrix of K at $(0,0)$ are $\lambda(t_0)$ and $\overline{\lambda(t_0)}$, where*

$$\lambda(t_0) = \frac{1+\nu+a(b-\nu)\overline{x}^{b-1}+i\sqrt{\left(3+\nu+a(b-\nu)\overline{x}^{b-1}\right)\left(1-\nu-a(b-\nu)\overline{x}^{b-1}\right)}}{2}.$$

Moreover, $\lambda(t_0)$ satisfies the following

(i) $\lambda^k(t_0) \neq 1$ *for* $k = 1,2,3,4$.
(ii) $\left.\frac{d(|\lambda(t)|)}{dt}\right|_{t=t_0} = \frac{1}{2}s\nu\left(a\overline{x}^{b-1}-1\right) > 0.$
(iii) *Eigenvectors associated to the $\lambda(t_0)$ are*

$$\mathbf{p}(t_0) = \begin{pmatrix} \frac{-i\sqrt{\left(1-\nu-a(b-\nu)\overline{x}^{b-1}\right)\left(3+\nu+a(b-\nu)\overline{x}^{b-1}\right)}}{ds\left(3+\nu+a(b-\nu)\overline{x}^{b-1}\right)} \\ \frac{\left(3+\nu+a(b-\nu)\overline{x}^{b-1}\right)-i\sqrt{\left(1-\nu-a(b-\nu)\overline{x}^{b-1}\right)\left(3+\nu+a(b-\nu)\overline{x}^{b-1}\right)}}{2\left(3+\nu+a(b-\nu)\overline{x}^{b-1}\right)}\end{pmatrix}^T, \tag{3.4}$$

$$\mathbf{q}(t_0) = \begin{pmatrix} \frac{sd\left(-\left(1-\nu-a(b-\nu)\overline{x}^{b-1}\right)+i\sqrt{\left(1-\nu-a(b-\nu)\overline{x}^{b-1}\right)\left(3+\nu+a(b-\nu)\overline{x}^{b-1}\right)}\right)}{2\left(1-\nu-a(b-\nu)\overline{x}^{b-1}\right)} \\ 1\end{pmatrix}, \tag{3.5}$$

such that $\mathbf{p}A = \lambda\mathbf{p}$, $A\mathbf{q} = \lambda\mathbf{q}$, *and* $\mathbf{pq} = 1$, *where* $A = Jac_K(0,0)|_{t=t_0}$.

Proof. Let $1-\nu-a(b-\nu)\overline{x}^{b-1} = k^2$. The goal of introducing this substitution is to facilitate the notation and calculation in the following. Notice that $s > s_0$ implies $k^2 > 0$, and $s < s_E$ and $b < 1$ imply $k^2 < 1$. Thus $0 < k^2 < 1$. Then, for

$t=t_0=\frac{1-\nu-a(b-\nu)\bar{x}^{b-1}}{s\nu\left(a\bar{x}^{b-1}-1\right)}=\frac{k^2}{s\nu\left(a\bar{x}^{b-1}-1\right)}$ we obtain

$$Jac_K(0,0)|_{t=t_0}=A=\begin{pmatrix} 1-k^2 & -ds \\ \dfrac{k^2}{sd} & 1 \end{pmatrix}.$$

Also, for $t=t_0$ we get

$$\Delta_{\lambda(t_0)}=\left(3+\nu+a(b-\nu)\bar{x}^{b-1}\right)\left(1-\nu-a(b-\nu)\bar{x}^{b-1}\right),$$

and

$$\lambda(t_0)=\frac{2-k^2+ik\sqrt{4-k^2}}{2}$$

and $|\lambda(t_0)|=1$. Notice $\lambda(t_0)\neq 1$ (because $0<k^2<1$). After straightforward calculation for $t=t_0$ we obtain

$$\lambda^2(t_0)=\frac{1}{2}\left(k^4-4k^2+2-ik\left(k^2-2\right)\sqrt{4-k^2}\right),$$
$$\lambda^3(t_0)=\frac{1}{2}\left(\left(2-k^2\right)\left(k^4-4k^2+1\right)+ik\left(3-k^2\right)\left(1-k^2\right)\sqrt{4-k^2}\right),$$
$$\lambda^4(t_0)=\frac{1}{2}\left(k^8-8k^6+20k^4-16k^2+2+ik\left(2-k^2\right)\left(k^4-4k^2+2\right)\sqrt{4-k^2}\right).$$

Notice that $\lambda^i(t_0)\neq 1$ for $i=2,3$ because their imaginary part is non-zero due to $0<k^2<1$. The imaginary part of $\lambda^4(t_0)$ is equal to zero only for $k^2=2-\sqrt{2}$, in which case it is $\lambda^4(t_0)=-1\neq 1$.

Furthermore, from (3.3) we get

$$\frac{d(|\lambda(t)|)}{dt}=\frac{s\nu\left(a\bar{x}^{b-1}-1\right)}{2\sqrt{s\nu t\left(a\bar{x}^{b-1}-1\right)+a\bar{x}^{b-1}(b-\nu)+\nu}}$$

and

$$\left.\frac{d(|\lambda(t)|)}{dt}\right|_{t=t_0}=\frac{s\nu\left(a\bar{x}^{b-1}-1\right)}{2}>0.$$

Notice

$$\mathbf{p}(t_0)=\begin{pmatrix} \frac{ik\sqrt{4-k^2}}{ds(k-2)(k+2)} \\ \frac{4-k^2-ik\sqrt{4-k^2}}{2(4-k^2)} \end{pmatrix}^T \text{ and } \mathbf{q}(t_0)=\begin{pmatrix} \frac{sd\left(-k+i\sqrt{4-k^2}\right)}{2k} \\ 1 \end{pmatrix}. \tag{3.6}$$

It is easy to see that $\mathbf{p}A=\lambda\mathbf{p}$, $A\mathbf{q}=\lambda\mathbf{q}$, and $\mathbf{pq}=1$. And, we get (3.4) and (3.5) from (3.6) by using $1-\nu-a(b-\nu)\bar{x}^{b-1}=k^2$. □

Let $t = t_0 + \varepsilon$, where ε is a sufficiently small positive parameter and $dt = d\varepsilon$. From Lemma 3.1, we can transform System (3.1) into the normal form

$$K(t,\mathbf{x}) = \mathcal{K}(t,\mathbf{x}) + O\left(\|\mathbf{x}\|^5\right),$$

and there are smooth functions $a(t)$, $b(t)$ and $\omega(t)$ so that in polar coordinates, the function $\mathcal{K}(t,\mathbf{x})$ is given by

$$\begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} |\lambda(t)| - a(t) r^3 \\ \theta + \omega(t) + b(t) r^2 \end{pmatrix}.$$

Now, we compute $a(t_0)$ following the procedure in [16]. Notice that $t = t_0$ if and only if $\varepsilon = 0$. First, we compute K_{20}, K_{11} and K_{02} defined in [16]. For $t = t_0$, we have

$$K\begin{pmatrix} u \\ w \end{pmatrix} = A\begin{pmatrix} u \\ w \end{pmatrix} + H\begin{pmatrix} u \\ w \end{pmatrix},$$

where

$$H\begin{pmatrix} u \\ w \end{pmatrix} = \begin{pmatrix} a(u+\bar{x})^b - d\alpha(u+\bar{x})^\nu (w+\bar{y}) - \bar{x} - \left(1-k^2\right)u + dsw \\ (w+\bar{y})\left(t_0\left(\alpha(u+\bar{x})^\nu - s\right)+1\right) - \bar{y} - \frac{k^2}{sd}u - w \end{pmatrix}.$$

Hence, for $b \neq \nu$ and $t = t_0 = \frac{k^2}{s\nu\left(a\bar{x}^{b-1}-1\right)} = \frac{(b-\nu)k^2}{s\nu(1-b-k^2)}$, System (3.1) is equivalent to

$$K\begin{pmatrix} u_n \\ w_n \end{pmatrix} = A\begin{pmatrix} u_n \\ w_n \end{pmatrix} + H\begin{pmatrix} u_n \\ w_n \end{pmatrix}.$$

Define the basis of $\mathbb{R}^2$ by $\Phi = (\mathbf{q}, \overline{\mathbf{q}})$, where $\mathbf{q}(t_0) = \begin{pmatrix} \frac{sd\left(-k+i\sqrt{4-k^2}\right)}{2k} & 1 \end{pmatrix}^T$.
If we denote $M = i\sqrt{4-k^2}$, then $\mathbf{q}(t_0) = \begin{pmatrix} \frac{sd(-k+M)}{2k} & 1 \end{pmatrix}^T$, and we can represent

$$\begin{aligned}\begin{pmatrix} u \\ w \end{pmatrix} &= \Phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathbf{q}, \overline{\mathbf{q}})\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \mathbf{q}z + \overline{\mathbf{q}}\bar{z} \\ &= \begin{pmatrix} \frac{sd(-k+M)}{2k} \\ 1 \end{pmatrix} z + \begin{pmatrix} \frac{sd(-k-M)}{2k} \\ 1 \end{pmatrix} \bar{z} = \begin{pmatrix} \frac{sd((-k+M)z+(-k-M)\bar{z})}{2k} \\ z+\bar{z} \end{pmatrix}.\end{aligned}$$

Let $H\left(\Phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \frac{1}{2}\left(g_{20}z^2 + 2g_{11}z\bar{z} + g_{02}^2\bar{z}\right) + O\left(|z|^3\right)$. We have

$$H\left(\Phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = H\begin{pmatrix} \frac{sd\left((-k+M)z+(-k-M)\bar{z}\right)}{2k} \\ z+\bar{z} \end{pmatrix},$$

so

$$H\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)=\begin{pmatrix} h_1(u,w) \\ h_2(u,w) \end{pmatrix},$$

where

$$h_1(u,w)=a\left(\frac{sd((-k+M)z+(-k-M)\overline{z})}{2k}+\overline{x}\right)^b-d\alpha\left(\frac{sd((-k+M)z+(-k-M)\overline{z})}{2k}+\overline{x}\right)^\nu\cdot$$
$$\cdot\left(z+\overline{z}+\frac{\left(1-k^2-b\right)\overline{x}}{sd(b-\nu)}\right)-\overline{x}-\frac{sd((-k+M)z+(-k-M)\overline{z})\left(1-k^2\right)}{2k}+ds\left(z+\overline{z}\right),$$

and

$$h_2(u,w)=\left(z+\overline{z}+\frac{\left(1-k^2-b\right)\overline{x}}{sd(b-\nu)}\right)\left(\frac{k^2(b-\nu)\left(\alpha\left(\frac{sd((-k+M)z+(-k-M)\overline{z})}{2k}+\overline{x}\right)^\nu-s\right)}{s\nu(1-k^2-b)}+1\right)$$
$$-\frac{\left(1-k^2-b\right)\overline{x}}{sd(b-\nu)}-\frac{k((-k+M)z+(-k-M)\overline{z})}{2}-z-\overline{z}.$$

Since

$$g_{20}=\frac{\partial^2}{\partial z^2}H\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)\Bigg|_{z=0},\quad K_{20}=\left(\lambda^2 I-A\right)^{-1}g_{20},$$

$$g_{11}=\frac{\partial^2}{\partial z\partial\overline{z}}H\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)\Bigg|_{z=0},\quad K_{11}=\left(I-A\right)^{-1}g_{11},$$

$$g_{02}=\frac{\partial^2}{\partial \overline{z}^2}H\left(\Phi\begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)\Bigg|_{z=0},\quad K_{02}=\left(\overline{\lambda}^2 I-A\right)^{-1}g_{02},$$

we get

$$g_{20}=\begin{pmatrix} \frac{d^2s^2\left((1-b)\left(1-k^2\right)(kM-k^2+2)+\nu\left(b(kM-k^2+2)+k^2(kM-k^2+5)-3kM-2\right)\right)}{2k^2\overline{x}} \\ \frac{ds}{\overline{x}}\left(1-\nu+\frac{k(M-k)\left(\nu\left(b+k^2-3\right)+b-k^2+1\right)}{2(1-b-k^2)}\right) \end{pmatrix},$$

$$g_{11}=\begin{pmatrix} \dfrac{d^2s^2(1-b)\left(k^2+\nu-1\right)}{k^2\overline{x}} \\ \dfrac{ds(1-b)\left(k^2+\nu-1\right)}{\overline{x}\left(1-b-k^2\right)} \end{pmatrix},$$

$$g_{02}=\begin{pmatrix} \frac{d^2s^2\left((1-b)\left(1-k^2\right)(2-k(M+k))-\nu\left((b-3)Mk+(b-5)k^2-2b+Mk^3+k^4+2\right)\right)}{2k^2\overline{x}} \\ \frac{ds\left(b(k(\nu+1)(M+k)-2\nu+2)+\left(1-k^2\right)\left(Mk+k^2-2\right)+\nu\left(M\left(k^2-3\right)k+k^4-5k^2+2\right)\right)}{2\overline{x}(b+k^2-1)} \end{pmatrix},$$

$$K_{20} = \begin{pmatrix} \dfrac{d^2s^2}{2k^3(k^2-3)\bar{x}(b+k^2-1)}k_{201} \\ \dfrac{ds}{2k^2(k^2-3)\bar{x}(b+k^2-1)}k_{202} \end{pmatrix}, \tag{3.7}$$

where

$$\begin{aligned} k_{201} &= M\left(2b^2\left(k^2+\nu-1\right)+b\left(k^2(\nu-3)-4\nu+4\right)+\left(k^4-3k^2+2\right)(\nu-1)\right) \\ &\quad + bk\left(-2k^4+k^2(7-3\nu)+10\nu-2\right)+k\left(2-k^2\right)\left(k^2(\nu-1)-5\nu+1\right), \end{aligned}$$

$$\begin{aligned} k_{202} &= Mk\left(b^2\left(1-k^2-\nu\right)+b\left(k^2-2\right)\left(k^2+\nu\right)-k^2(\nu-1)+3\nu-1\right) \\ &\quad + b\left(2-k^2\right)\left(bk^2+b\nu-b-k^2\nu\right)+\nu\left(2-4b+k^4-5k^2\right) \\ &\quad + b\left(2-k^2\right)\left(2-k^4+2k^2\right)+\left(k^2-2\right)\left(1-k^2\right), \end{aligned}$$

$$K_{11} = \begin{pmatrix} \dfrac{d^2s^2(1-b)\left(1-k^2-\nu\right)}{k^2\bar{x}(1-b-k^2)} \\ \dfrac{ds(b-1)^2\left(k^2+\nu-1\right)}{k^2\bar{x}(1-b-k^2)} \end{pmatrix}, \tag{3.8}$$

$$K_{02} = \begin{pmatrix} -\dfrac{d^2s^2}{2k^3(k^2-3)\bar{x}(b+k^2-1)}k_{021} \\ \dfrac{ds}{2k^2(k^2-3)\bar{x}(b+k^2-1)}k_{022} \end{pmatrix},$$

where

$$\begin{aligned} k_{021} &= M\left(2b^2\left(k^2+\nu-1\right)+b\left(k^2(\nu-3)-4\nu+4\right)+\left(k^4-3k^2+2\right)(\nu-1)\right) \\ &\quad + bk\left(2k^4+k^2(3\nu-7)-10\nu+2\right)+\left(k^2-2\right)k\left(k^2(\nu-1)-5\nu+1\right), \end{aligned}$$

$$\begin{aligned} k_{022} &= -Mk\left(\left(k^2-1\right)-\nu\left(k^2-3\right)+b\left(k^2-\nu\right)\left(k^2-2\right)-b^2\left(k^2+\nu-1\right)\right) \\ &\quad + \nu\left(k^4-5k^2+2\right)+\left(k^2-2\right)\left(b\left(k^4-2k^2-2\right)-k^2+1\right) \\ &\quad + b\left(k^2-2\right)\left(k\nu(k-4)-b\left(k^2+\nu-1\right)\right). \end{aligned}$$

By using K_{20}, K_{11} and K_{02} and formula

$$g_{21} = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \frac{\partial^3}{\partial z^2 \partial \bar{z}} H\left(\Phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}\left(K_{20}z^2+2K_{11}z\bar{z}+K_{02}\bar{z}^2\right)\right)\Bigg|_{z=0}$$

we get

$$M_1 = \frac{d^3s^3}{2k^4(k^2-3)\bar{x}^2(b+k^2-1)}m_1,$$

where

$$\begin{aligned} m_1 = & \, b^3\left(k^2+\nu-1\right)\left(2k^4+k^2(\nu-8)-4\nu+4\right) \\ & -2b^2\left(k^2+\nu-1\right)\left(k^6+k^4(2\nu-1)-k^2(4\nu+9)-6\nu+6\right) \\ & +Mk\left(k^2+\nu-1\right)\left(\nu\left(bk^4+2((b-3)b+1)k^2+(b-6)(b-1)^2\right)\right) \\ & -bk^8(\nu-2)-bk^6((\nu-15)\nu+4)-bk^4(\nu(35-12\nu)+14) \\ & -bk^2(\nu(23\nu+9)-28)-b12(\nu-1)^2+2bMk\left(k^2+\nu-1\right)(b-1)^2 \\ & -2\left(k^6(2\nu+1)+k^4(3(\nu-2)\nu-4)+k^2\left(5-7\nu^2\right)-2(\nu-1)^2\right), \end{aligned}$$

$$M_2 = \frac{d^2 s^2}{2k^2\left(k^2-3\right)\overline{x}^2\left(b+k^2-1\right)^2} m_2,$$

where

$$\begin{aligned} m_2 = & -b^3k^2(\nu-10)\left(k^2+\nu-1\right)-4b^3\left(k^2+\nu-1\right)+4b^3\nu\left(k^2+\nu-1\right) \\ & -3b^3k^4\left(k^2+\nu-1\right)+b^2k^8+b^2k^6(5\nu+4)+b^2k^4(\nu(4\nu-7)-33) \\ & +b^2\left(k^2(38-2\nu(4\nu+17))-12(\nu-1)^2\right) \\ & +bk^8(\nu-2)+bk^6((\nu-15)\nu+2) \\ & +bk^4(\nu(31-12\nu)+22)+bk^2(\nu(23\nu+19)-34)+12b(\nu-1)^2 \\ & +6k^4\nu^2-14k^2\nu^2+4\left(k^2-3\right)k^4\nu+2\left(k^4-4k^2+5\right)k^2-4\nu^2+8\nu-4 \\ & +M[b^2k\left(k^2+\nu-1\right)\left(k^4-k^2(2\nu+5)+8\nu+12\right) \\ & -bk\left(k^2+\nu-1\right)\left(k^2\left(\left(k^2-6\right)\nu-2\right)+13\nu+6\right)-2k\left(k^2-3\right)\nu^2 \\ & +2k\left(k^2-3\right)\nu-2k^3\left(k^2-3\right)\nu+b^3k\left(k^2-\nu-6\right)\left(k^2+\nu-1\right)]. \end{aligned}$$

Finally, we get

$$\begin{aligned} a(t_0) &= \frac{1}{2}\Re\left(\mathbf{p}q_{21}\overline{\lambda}\right) \\ &= \frac{(1-b)d^2s^2\left(b^2-b\left(k^2+3\right)+2\right)\left(k^2+\nu-1\right)\left(b\left(k^2+\nu\right)-\nu\right)}{4k^2\overline{x}^2\left(b+k^2-1\right)^2}. \end{aligned}$$

By replacing k^2 with $1-\nu-a(b-\nu)\overline{x}^{b-1}$ we get

$$a(t_0) = \frac{ad^2s^2\overline{x}^{b-1}\left(ab\overline{x}^{b-1}-1\right)(1-b)\left(2-4b+b\nu+b^2+ab(b-\nu)\overline{x}^{b-1}\right)}{4\overline{x}^2\left(a\overline{x}^{b-1}-1\right)^2\left(1-\nu-a(b-\nu)\overline{x}^{b-1}\right)}.$$

Notice that $2-4b+b\nu+b^2+ab(b-\nu)x^{b-1}>0$. Indeed, by using $\nu\le b<1$ and $s<s_E$ or equivalently $a\overline{x}^{b-1}>1$ we get

$$\begin{aligned}2-4b+b\nu+b^2+ab(b-\nu)\overline{x}^{b-1} &> 2-4b+b\nu+b^2+b(b-\nu)\\ &= 2(b-1)^2>0.\end{aligned}$$

Also, $a(t_0)=0$ if $ab\overline{x}^{b-1}-1=0$ and $s<s_E$. It implies that $a\overline{x}^{b-1}=\frac{1}{b}>1$, which is true since $b<1$. A degenerate case is also possible, for $\overline{x}=\left(\frac{s}{\alpha}\right)^{\frac{1}{\nu}}=(ab)^{\frac{1}{1-b}}$, i.e., when $s=\alpha(ab)^{\frac{\nu}{1-b}}$. Therefore, the following inequalities hold

$$a(t_0)\begin{cases}<0 & \text{for} \quad s>\alpha(ab)^{\frac{\nu}{1-b}},\\ =0 & \text{for} \quad s=\alpha(ab)^{\frac{\nu}{1-b}},\\ >0 & \text{for} \quad s<\alpha(ab)^{\frac{\nu}{1-b}}.\end{cases}$$

If $(0,0)$ is fixed point of K, then $(\overline{x},\overline{y})$ is fixed point of T and invariant or repelling curve can be approximated by

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}\approx\begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix}+2\rho_0\Re\left(\mathbf{q}e^{i\theta}\right)+\rho_0^2\left(\Re\left(K_{20}e^{2i\theta}\right)+K_{11}\right),$$

where

$$d=\left.\frac{d(|\lambda(t)|)}{dt}\right|_{t=t_0},$$

$$\rho_0=\sqrt{-\frac{d}{a}\varepsilon},\ \theta\in\mathbb{R}$$

We considered the case $b=\nu$ separately and carried out all the previous procedure as for the case $\nu<b$ and found that all coefficients can be obtained from the procedure carried out for $\nu<b$, specifically setting $b=\nu$. Thus for $b=\nu$ and $\overline{x}=\left(\frac{s}{\alpha}\right)^{\frac{1}{\nu}}$ we have that

$$a(t_0)=\frac{1}{2}\Re\left(\mathbf{p}q_{21}\overline{\lambda}\right)=\frac{ad^2s^3(\nu-1)^2(as\nu-\overline{x}\alpha)}{2\overline{x}^2(as-\overline{x}\alpha)^2},$$

and

$$a(t_0)\begin{cases}<0 & \text{for} \quad s>\alpha(a\nu)^{\frac{\nu}{1-\nu}},\\ =0 & \text{for} \quad s=\alpha(a\nu)^{\frac{\nu}{1-\nu}},\\ >0 & \text{for} \quad s<\alpha(a\nu)^{\frac{\nu}{1-\nu}},\end{cases}$$

$$\left.\frac{d(|\lambda(t)|)}{dt}\right|_{t=t_0}=\frac{s\nu(as-\overline{x}\alpha)}{2\overline{x}\alpha}>0.$$

Thus, we prove the following result.

Theorem 3.1. *Let* $a>0$, $\alpha>0$, $0<\nu\le b<1$, $s_0=\alpha\left(\frac{a(b-\nu)}{1-\nu}\right)^{\frac{\nu}{1-b}}$, $s_E=\alpha a^{\frac{\nu}{1-b}}$, $s_\Gamma=\alpha(ab)^{\frac{\nu}{1-b}}$, $\bar{x}=\left(\frac{s}{\alpha}\right)^{\frac{1}{\nu}}$, $t_0=\frac{1-\nu-a(b-\nu)\bar{x}^{b-1}}{s\nu\left(a\bar{x}^{b-1}-1\right)}$, $s_0<s<s_E$ *and* $E_+=\left(\bar{x},\frac{a\bar{x}^b-\bar{x}}{sd}\right)$.

If $s>s_\Gamma$, *i.e.,* $a(t_0)<0$, *then there is a neighborhood* U *of the equilibrium point* E_+ *and* $\varepsilon>0$ *such that for* $|t-t_0|<\varepsilon$ *and* $(x_{-1},x_0)\in U$, *then* α*-limit set of the solution of System (1.1), with initial condition* (x_{-1},x_0) *is the equilibrium point* E_+ *if* $t>t_0$ *and belongs to a closed invariant* C^1 *curve* $\Gamma(t)$ *encircling the equilibrium point* E_+ *if* $t<t_0$.

If $s<s_\Gamma$, *i.e.,* $a(t_0)>0$, *then there is a neighborhood* U *of the equilibrium point* E_+ *and* $\varepsilon>0$ *such that for* $|t-t_0|<\varepsilon$ *and* $(x_{-1},x_0)\in U$, *then* ω*-limit set of the solution of System (1.1), with initial condition* (x_{-1},x_0) *is the equilibrium point* E_+ *if* $t<t_0$ *and belongs to a closed invariant* C^1 *curve* $\Gamma(t)$ *encircling the equilibrium point* E_+ *if* $t>t_0$.

Furthermore, $\Gamma(t_0)=0$ *and invariant curve* $\Gamma(t)$ *can be approximated by*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \frac{a\bar{x}^b-\bar{x}}{sd} \end{pmatrix} + 2\rho_0\Re\left(\mathbf{q}e^{i\theta}\right)+\rho_0^2\left(\Re\left(K_{20}e^{2i\theta}\right)+K_{11}\right), \tag{3.9}$$

where

$$\rho_0=\frac{\bar{x}}{ds}\sqrt{\frac{2\left(a\bar{x}^{b-1}-1\right)^2\left(a(b-\nu)\bar{x}^{b-1}+\nu-1\right)\left(s\nu t\left(1-a\bar{x}^{b-1}\right)-(\nu-1)-a(b-\nu)\bar{x}^{b-1}\right)}{a\bar{x}^{b-1}\left(ab\bar{x}^{b-1}-1\right)(b-1)\left(2-4b+b\nu+b^2+ab(b-\nu)\bar{x}^{b-1}\right)}},$$

and K_{20}, K_{11} *are given by (3.7) and (3.8).*

The approximative curve $\Gamma(t)$ for $0<\nu=b<1$ has a simpler notation compared to the general form (3.9), which we had for $0<\nu<b<1$ and has the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} \bar{x}+\frac{P}{\alpha(as\nu-\bar{x}\alpha)(1-\nu)^2}-\frac{\bar{x}(as-\bar{x}\alpha)S((1-\nu)\cos\theta+L\sin\theta)}{1-\nu} \\ \frac{(as-\bar{x}\alpha)((as\nu-2\bar{x}\alpha)(1-\nu)+st\nu(as-\bar{x}\alpha))}{ds\alpha(1-\nu)(as\nu-\bar{x}\alpha)}+\frac{2\bar{x}(as-\bar{x}\alpha)S\cos\theta}{ds} \end{pmatrix}$$
$$+\begin{pmatrix} \frac{\left(3\bar{x}\alpha+as-6as\nu+4\bar{x}\alpha\nu-3as\nu^2+\bar{x}\alpha\nu^2\right)P\cos 2\theta}{2as\alpha(as\nu-\bar{x}\alpha)(1-\nu)^3(\nu+2)}+\frac{LP(as-\bar{x}\alpha-as\nu-\bar{x}\alpha\nu)\sin 2\theta}{2as\alpha(as\nu-\bar{x}\alpha)(1-\nu)^3(\nu+2)} \\ \frac{\left(3\bar{x}\alpha-as+3as\nu+\bar{x}\alpha\nu+2as\nu^2\right)P\cos 2\theta}{2a\alpha s^2d(as\nu-\bar{x}\alpha)(1-\nu)^2(\nu+2)}+\frac{LP(\nu+1)(as+\bar{x}\alpha-2as\nu)\sin 2\theta}{2a\alpha s^2d(as\nu-\bar{x}\alpha)(1-\nu)^3(\nu+2)} \end{pmatrix},$$

where

$$L=\sqrt{(1-\nu)(\nu+3)},\quad P=(as-\bar{x}\alpha)(\bar{x}\alpha(1-\nu)-st\nu(as-\bar{x}\alpha)),$$

$$S=\sqrt{\frac{\bar{x}\alpha(1-\nu)-st\nu(as-\bar{x}\alpha)}{\bar{x}as\alpha(as\nu-\bar{x}\alpha)(1-\nu)^2}}.$$

Example 3.1. *For* $a=38.5132$, $b=0.7365$, $d=0.75$, $\alpha=0.03147$, $\nu=0.3739$, $s=3$, $s_\Gamma=3.6254$, $s_E=5.5955$ *(*$s<s_\Gamma$*) we obtain* $\bar{x}=1.9654\times10^5$, $\bar{y}=48185.0$,

$t_0 = 0.1026$ *and* $a(t_0) = 7319 \times 10^{-11}$. *Since* $a(t_0) > 0$, *by charging the value of the parameter* t *from* $t < t_0$ *to* $t > t_0$, *supercritical Neimark-Sacker bifurcation occurs of the critical value (see Figure 3). To compute the largest Lyapunov exponents (see Figure 3(B)) we used numerical calculation of Lyapunov exponents given by M. Sandri (see [25]) with* 7500 *iterations and* $(x_0, y_0) = (196000.0, 48000.0)$. *Namely, if* $t = 0.1 < t_0$ *(see Figure 4) a unique closed invariant curve* Γ *(unstable) encireling the equilibrium point, Figure 4(A) which is a repelling curve, Figure 4(B) for same values of parameters also shows trajectories (red and green) with initial values* $(x_0, y_0) = (108000, 50000)$, $(x_0, y_0) = (120000, 50000)$, *respectively. In cases when* $t = 0.105 > t_0$ *then the equilibrium point* E_+ *is a repeller with* $a(t_0) > 0$. *Figure 5(A) shows trajectory (blue) with initial value* $(x_0, y_0) = (200000, 45000)$ *for* $a = 38.5132$, $b = 0.7365$, $d = 0.75$, $\alpha = 0.03147$, $\nu = 0.3739$, $s = 3 < s_\Gamma$, $t = 0.105 > t_0$.

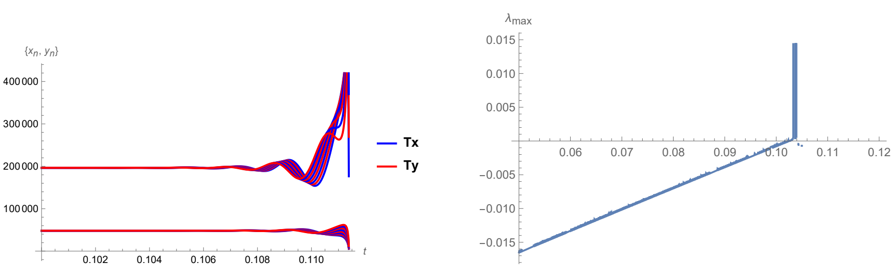
Figure 6 shows a family of repelling curves for $t \in (0, 0.1026)$ *that form a paraboloid. The interior of the paraboloid is the area of attraction of the positive equilibrium* E_+ *and we see that it decreases as* t *increases and approaches* t_0.

If we start further from the equilibrium we have fewer points. For example, if the starting point is $(x_0, y_0) = (120000, 77000)$ *then it is*

$$\begin{aligned}(x_1, y_1) &= (67980.58361373155`, 72913.91961471214`),\\ (x_2, y_2) &= (29222.407260023043`, 65388.804713458354`),\\ (x_3, y_3) &= (2779.275126993598`, 54891.31437202725`),\\ (x_4, y_4) &= (-11882.001736524737`, 41118.387042887865`),\end{aligned}$$

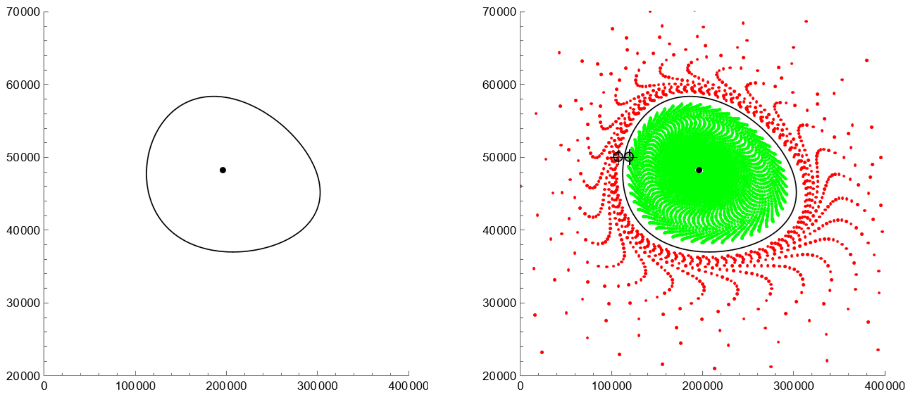
(see Figure 5(B)). All the following points have complex coordinates:

$$\begin{aligned}x_5 &= -38627.02662982175` - 1455.421992851152`i,\\ y_5 &= 29916.621562215558` + 4185.1784967756075`i.\end{aligned}$$



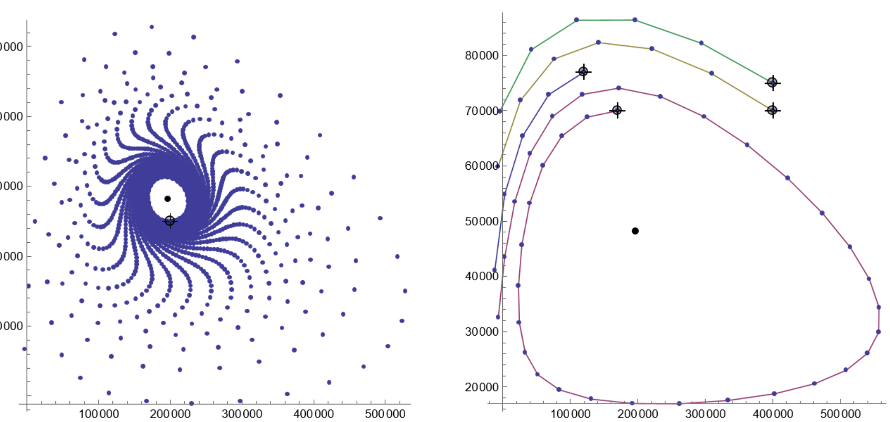
(A) (B)

FIGURE 3. (A) Bifurcation diagrams in the (t, x_n)-plane and (t, y_n)-plane (B) corresponding largest Lyapunov exponents for $t \in (0.1, 120)$.



(A) (B)

FIGURE 4. (A) Repelling curve (B) repelling curve and trajectories (red and green) for $a = 38.5132$, $b = 0.7365$, $d = 0.75$, $\alpha = 0.03147$, $\nu = 0.3739$, $s = 3 < s_\Gamma$, $t = 0.1 < t_0 = 0.1026$, and $a(t_0) > 0$.



(A) (B)

FIGURE 5. Trajectories for $a = 38.5132$, $b = 0.7365$, $d = 0.75$, $\alpha = 0.03147$, $\nu = 0.3739$, $s = 3 < s_\Gamma$, $t = 0.105 > t_0 = 0.1026$, and $a(t_0) > 0$. The equilibrium point E_+ is black point.

For $b = \nu = \frac{1}{2}$ we have

$$\det J_T(x,y) = \frac{1}{2}\left(at\alpha - \frac{(st-1)(a-dy\alpha)}{\sqrt{x}}\right), \tag{3.10}$$

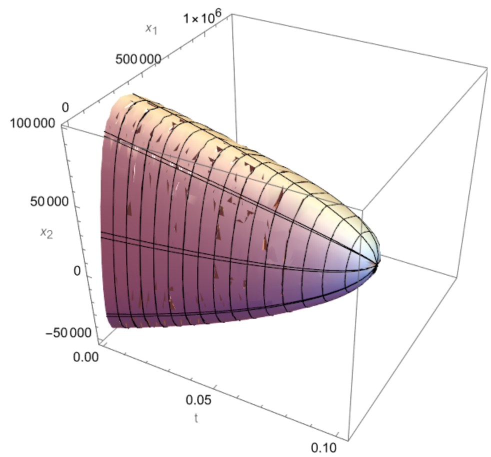


FIGURE 6. Repelling curves for $a = 38.5132$, $b = 0.7365$, $d = 0.75$, $\alpha = 0.03147$, $\nu = 0.3739$, $s = 3 < s_\Gamma$, $t \in (0, t_0] = (0, 0.1026]$ and $a(t_0) > 0$.

and $\det J_T(x, y)$ does not depend on x and y if $st = 1$. So, for $b = \nu = \frac{1}{2}$, $t = \frac{1}{s}$ and $s = s_\Gamma = \frac{a\alpha}{2}$ it holds

$$\det J_T(x, y) = \frac{a\alpha}{2s} = 1, \tag{3.11}$$

i.e., the map T is an area-preserving map.

Example 3.2. *For $a = 8$, $b = \nu = \frac{1}{2}$, $d = \frac{3}{5}$, $\alpha = \frac{1}{4}$, $s = \frac{1}{2}$, $s_\Gamma = 1$, $s_E = 2$ ($s < s_\Gamma$) we obtain $\bar{x} = 4$, $\bar{y} = 40$, $t_0 = \frac{2}{3} \approx 0.66667$ and $a(t_0) = \frac{1}{3200} > 0$. Since $a(t_0) > 0$, by charging the value of the parameter t from $t < t_0$ to $t > t_0$, supercritical Neimark-Sacker bifurcation occurs of the critical value. If $t = 0.6666 < t_0$ a unique closed invariant curve Γ (see Figure 7(A)) encircling the equilibrium point, which is a repelling curve. Figure 7(B) for same values of parameters also shows trajectories (red and green) with initial values $(x_0, y_0) = (4, 39.35)$, $(x_0, y_0) = (4.24, 39.83)$, 90000 and 21100 point respectively.*

Example 3.3. *For $a = 8$, $b = \nu = \frac{1}{2}$, $d = \frac{3}{5}$, $\alpha = \frac{1}{4}$, $s = 1$, $s_\Gamma = 1$, $s_E = 2$ ($s = s_\Gamma$) we obtain $\bar{x} = 16$, $\bar{y} = \frac{80}{3} \approx 26.667$, $t_0 = 1$ and $a(t_0) = 0$. For $t = t_0 = 1$, $a = 8$ and $\alpha = \frac{1}{4}$, we have $\det J_T(x, y) = 1$ for each x and y (see (3.11)). By applying the Kolmogorov-Arnold-Moser (KAM) theory (see [4, 5, 10, 12, 21], we conclude that equilibrium $(\bar{x}, \bar{y}) = \left(16, \frac{80}{3}\right)$ is a stable fixed point (see Figure 8).*

Example 3.4. *For $a = 8$, $b = \nu = \frac{1}{2}$, $d = \frac{3}{5}$, $\alpha = \frac{1}{4}$, $s = \frac{3}{2}$, $s_\Gamma = 1$, $s_E = 2$ ($s > s_\Gamma$) we obtain $\bar{x} = 36$, $\bar{y} = \frac{40}{3} \approx 13.333$, $t_0 = 2$ and $a(t_0) = -\frac{1}{3200}$. Since $a(t_0) < 0$, by charging the value of the parameter t from $t < t_0$ to $t > t_0$, subcritical Neimark-Sacker bifurcation occurs of the critical value. If $t = 2.01 > t_0$ (see Figure 9(A))*

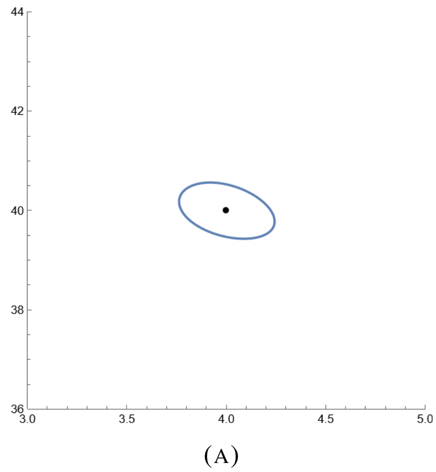


FIGURE 7. (A) Repelling curve (blue), (B) repelling curve (blue) and trajectories (red and green) for $a = 8$, $b = \nu = \frac{1}{2}$, $d = \frac{3}{5}$, $\alpha = \frac{1}{4}$, $s = \frac{1}{2} < s_\Gamma = 1$, $t = 0.6666 < t_0 = \frac{2}{3}$ and $a(t_0) > 0$.

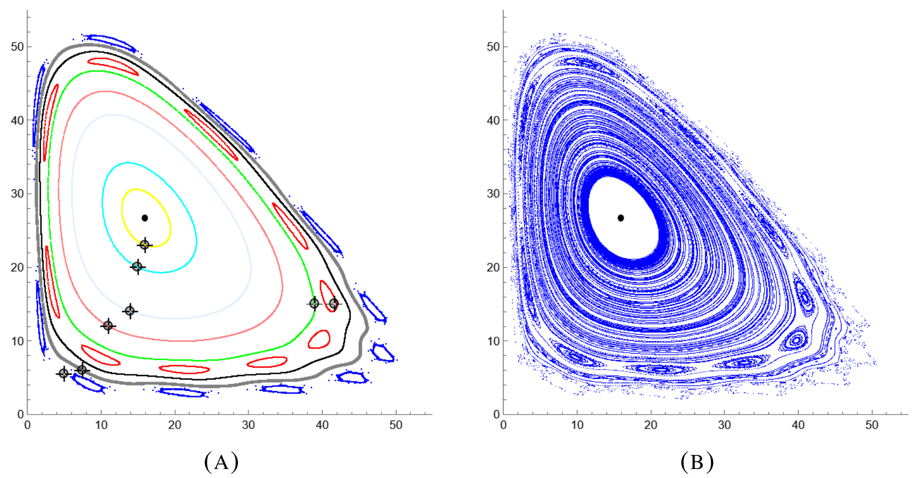
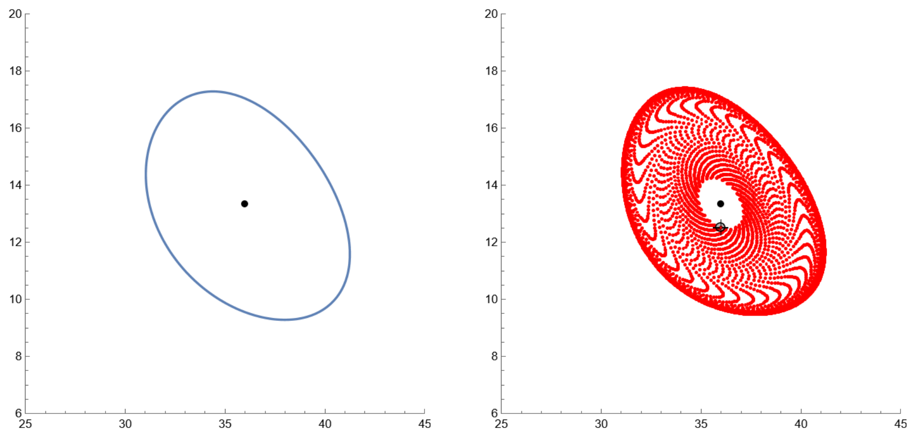


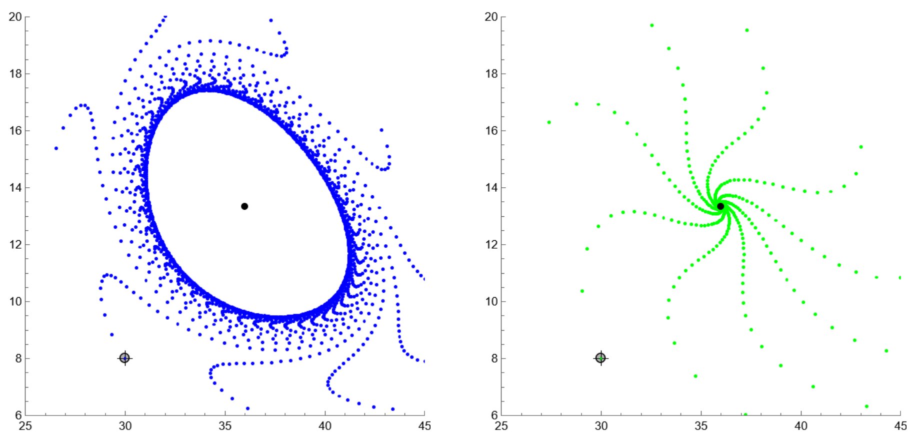
FIGURE 8. (A) selected iterations, (B) randomly selected iterations for $a = 8$, $b = \nu = \frac{1}{2}$, $d = \frac{3}{5}$, $\alpha = \frac{1}{4}$, $s = s_\Gamma = 1 = t = t_0 = 1$, $a(t_0) = 0$ the equilibrium E_+ (black point) is stable.

a unique closed invariant curve Γ *encircling the equilibrium point, which is a stable invariant curve (blue). Figure 9(B) shows trajectory (red) with initial value* $(x_0, y_0) = (30, 13)$ *and* $t = 2.01 > t_0 = 2$*, and Figure 10 shows trajectories (blue and green) with the same initial value* $(x_0, y_0) = (30, 8)$ *for* $t = 2.01 > t_0$ *and for*



(A) (B)

FIGURE 9. (A) Stable curve for $a = 8$, $b = \nu = \frac{1}{2}$, $d = \frac{3}{5}$, $\alpha = \frac{1}{4}$, $s = \frac{3}{2} > s_\Gamma = 1$, $t_0 = 2$, $a(t_0) < 0$ and,
(B) 4500 trajectories (red) for $(x_0, y_0) = (36, 12.5)$, $t = 2.01 > t_0$.



(A) (B)

FIGURE 10. For $a = 8$, $b = \nu = \frac{1}{2}$, $d = \frac{3}{5}$, $\alpha = \frac{1}{4}$, $s = \frac{3}{2} > s_\Gamma = 1$, $t_0 = 2$, $a(t_0) > 0$ and $(x_0, y_0) = (30, 8)$:
(A) 2000 trajectories (blue), $t = 2.01 > t_0$,
(B) 600 trajectories (green), $t = 1.95 < t_0$.

$t = 1.95 < t_0$. *This means that the anchovy biomass and the total anchovy catch will eventually form a cycle (as described in [8]).*

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