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EXTENDING THEOREMS FOR STRONGLY ANTI-COMPETITIVE MAPS TO APPLY TO WEAKLY ANTI-COMPETITIVE MAPS

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Dedicated to Professor Mehmed Nurkanović on the occasion of his 65th birthday

ABSTRACT. In the paper *Global Dynamics of Anti-Competitive Systems in the Plane* [4], the authors proved two major theorems that can be used to determine the global dynamics of anti-competitive systems of difference equations. These theorems require three hypotheses to be satisfied: (1) the corresponding map must be strongly anti-competitive, (2) the determinant of the Jacobian matrix of the map, evaluated at an interior fixed-point, does not equal zero, and (3) the only point mapped onto a fixed-point is the fixed-point itself.

In this paper, we prove theorems that obtain the same results as in [4], but do not require any of these hypotheses; furthermore, the new theorems use weaker hypotheses that extend the scope of the theorems to apply to many more cases. Finally, we demonstrate how to use the modified theorems to determine the global dynamics of a weakly anti-competitive system where hypothesis (1) is false in every region of the parameter space.

1. INTRODUCTION

In the paper *Global Dynamics of Anti-Competitive Systems in the Plane*, the authors proved two major theorems that can be used to determine the global dynamics of anti-competitive systems of difference equations, see [4]. These theorems require three hypotheses to be satisfied: (1) the corresponding map must be strongly anti-competitive, (2) the determinant of the Jacobian matrix of the map, evaluated at an interior fixed-point, does not equal zero, and (3) the only point mapped onto a fixed-point is the fixed-point itself.

In this paper, we prove theorems that obtain the same results as the theorems in [4], but do not require any of these hypotheses. The proofs in this paper use similar arguments as the proofs in [4]; however, the proofs have been modified to accommodate different hypotheses.

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The modified theorems have weaker hypotheses that extend the scope of the theorems to apply to more cases.

If you only consider the special case where the anti-competitive map has rational-linear equations, there are six systems whose corresponding map is homogeneous of order zero and hypotheses (2) and (3) are false in all regions of the parameter space. Also, non-homogeneous systems can fail to satisfy hypotheses (2) and (3) in some regions of the parameter space, see [2], [3], and [8].

There are 24 systems of rational-linear difference equations whose corresponding map is weakly anti-competitive and the second iterate of the map is strongly competitive. These systems fail to satisfy hypothesis (1) in all regions of the parameter space. In Section 4, we use the modified theorems to determine the global dynamics of one of these systems.

Anti-competitive maps are widely studied and there has been much research focused on anti-competitive maps with rational-linear equations, see [1] - [5] and [7] - [12]. While it was this type of anti-competitive system that motivated the creation of the theorems in this paper, the theorems are not restricted to maps that are rational-linear. The only restrictions on the map are that F is anti-competitive and F^2 is strongly competitive.

The theorems in this paper apply to situations where there is a unique interior fixed-point and possibly a unique pair of minimal period-two points. When analyzing strongly competitive maps with multiple fixed-points and multiple pairs of period-two points, it is often the case that the domain can be separated into invariant rectangular regions, each of which contains a unique fixed-point and possibly a unique pair of minimal period-two points. The theorems can be applied to each rectangular region.

2. PRELIMINARIES

In this paper, we use the symbol "$\subset$" to mean subset, not necessarily a proper subset. For $\mathcal{R} \subset \mathbb{R}^2$, we use the notation $\partial\mathcal{R}$, $\text{int}(\mathcal{R})$, and $\text{cl}(\mathcal{R})$ to mean the boundary of $\mathcal{R}$, the interior of $\mathcal{R}$, and the closure of $\mathcal{R}$, respectively. We use the notation J_F to mean the Jacobian matrix of F and the notation $J_F(\bar{\mathbf{x}})$ to mean the Jacobian matrix of F, evaluated at a fixed-point $\bar{\mathbf{x}}$.

The following definitions are written specifically to apply to planar competitive and anti-competitive maps.

Definition 2.1. *The symbol $\succ_{SE}$ denotes the south-east partial ordering on $\mathbb{R}^2$. For (x,y) and (a,b) in $\mathbb{R}^2$, we use the following notation.*

(a) $(x,y) \succeq_{SE} (a,b)$, if and only if, $x \geq a$ and $y \leq b$.
(b) $(x,y) \succ_{SE} (a,b)$, if and only if, $(x,y) \succeq_{SE} (a,b)$ and $(x,y) \neq (a,b)$.
(c) $(x,y) \gg_{SE} (a,b)$, if and only if, $x > a$ and $y < b$.

Definition 2.2. *Let $\mathcal{R}$ be a subset of $\mathbb{R}^2$ with nonempty interior and let $F : \mathcal{R} \to \mathcal{R}$ be a continuous map.*

(a) *F is **competitive** on $\mathcal{R}$, if $\mathbf{x} \succeq_{SE} \mathbf{y} \Rightarrow F(\mathbf{x}) \succeq_{SE} F(\mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathcal{R}$.*
(b) *F is **strongly competitive**, if $\mathbf{x} \succ_{SE} \mathbf{y} \Rightarrow F(\mathbf{x}) \gg_{SE} F(\mathbf{y})$.*
(c) *F is **anti-competitive**, if $\mathbf{x} \succeq_{SE} \mathbf{y} \Rightarrow F(\mathbf{y}) \succeq_{SE} F(\mathbf{x})$.*
(d) *F is **strongly anti-competitive**, if $\mathbf{x} \succ_{SE} \mathbf{y} \Rightarrow F(\mathbf{y}) \gg_{SE} F(\mathbf{x})$.*

Lemma 2.1. *Let $\mathcal{R}$ be a subset of $\mathbb{R}^2$ with nonempty interior and let $F : \mathcal{R} \to \mathcal{R}$ be a continuous map where $F(x,y) = (f(x,y), g(x,y))$. If F is differentiable, then a sufficient condition for F to be competitive, strongly competitive, anti-competitive, or strongly anti-competitive on $\mathcal{R}$, is for the Jacobian matrix of F to have the respective sign configurations below,*

$$\begin{bmatrix} \geq 0 & \leq 0 \\ \leq 0 & \geq 0 \end{bmatrix}, \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \begin{bmatrix} \leq 0 & \geq 0 \\ \geq 0 & \leq 0 \end{bmatrix}, \begin{bmatrix} - & + \\ + & - \end{bmatrix},$$

at every $\mathbf{x} \in \mathcal{R}$.

Remark 2.1. It follows from the Perron-Frobenius Theorem and a change of variables [13] that at each point, the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one being positive in absolute value, and the corresponding eigenvectors can be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that if the map is strongly competitive then no eigenvector is aligned with a coordinate axis. This remark is from [4].

Lemma 2.2. *Let F be a map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$ and let F^2 be the second iterate of the map F. If the map F is anti-competitive on $\mathcal{R}$, then the map F^2 is competitive on $\mathcal{R}$. If the map F is strongly anti-competitive on $\mathcal{R}$, then the map F^2 is strongly competitive on $\mathcal{R}$.*

For the system analyzed in Section 4, the corresponding map F is anti-competitive, and F^2, the second iterate of F, is strongly competitive. We will apply the next four theorems to the map F^2, so the next lemma is helpful.

Lemma 2.3. *Let F be a map on $\mathcal{R} \subset \mathbb{R}^2$ and let $\bar{\mathbf{x}}$ be a fixed-point of F in $\mathrm{int}(\mathcal{R})$. Suppose F is differentiable on a neighborhood of $\bar{\mathbf{x}}$. Let $J_F(\bar{\mathbf{x}})$ be the Jacobian matrix of F evaluated at $\bar{\mathbf{x}}$. Then, the following statements are true.*

(a) $J_{F^2}(\bar{\mathbf{x}}) = (J_F(\bar{\mathbf{x}}))^2$.
(b) $\det J_{F^2}(\bar{\mathbf{x}}) = (\det J_F(\bar{\mathbf{x}}))^2$.
(c) *If λ_1 and λ_2 are eigenvalues of $J_F(\bar{x})$, then λ_1^2 and λ_2^2 are eigenvalues of $J_{F^2}(\bar{\mathbf{x}})$.*

The next four theorems are from [6] and they are used in the proof of Theorem 3.1.

Definition 2.3. *Let $\bar{\mathbf{x}} = (\bar{x}, \bar{y})$ be a fixed-point of the map F. Define $Q_1(\bar{\mathbf{x}})$ to be Quadrant I relative to the fixed-point $\bar{\mathbf{x}}$. That is, $Q_1(\bar{\mathbf{x}}) = \{(x,y) : x \geq \bar{x} \text{ and } y \geq \bar{y}\}$.*

We can define $Q_2(\bar{\mathbf{x}})$, $Q_3(\bar{\mathbf{x}})$, and $Q_4(\bar{\mathbf{x}})$ in a similar way.

Theorem 2.1. *Let F be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\bar{\mathbf{x}}$ be a fixed-point of F such that $\Delta := \operatorname{int}(Q_1(\bar{\mathbf{x}}) \cup Q_3(\bar{\mathbf{x}})) \cap \mathcal{R}$ is nonempty (i.e., $\bar{\mathbf{x}}$ is not the NW or SE corner of $\mathcal{R}$), and F is strongly competitive on Δ. Suppose the following statements are true.*

- (a) *The map F has a C^1 extension to a neighborhood of $\bar{\mathbf{x}}$.*
- (b) *The Jacobian matrix $J_F(\bar{\mathbf{x}})$ of F at $\bar{\mathbf{x}}$ has real eigenvalues, λ and μ, such that $0 < |\lambda| < \mu$, $|\lambda| < 1$, and the eigenspace E^{λ}, corresponding to λ, is not a coordinate axis.*

Then, there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\bar{\mathbf{x}}$ that is invariant under F and is a subset of the basin of attraction of $\bar{\mathbf{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^{λ} at $\bar{\mathbf{x}}$, and $\mathcal{C}$ is a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed-points or minimal period-two points of the map F.

Theorem 2.2. *For the curve $\mathcal{C}$ of Theorem 2.1 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

- (i) *The map F has no fixed-points nor minimal period-two points in Δ.*
- (ii) *The map F has no fixed-points in Δ, $\det J_F(\bar{\mathbf{x}}) > 0$, and $F(\mathbf{x}) = \bar{\mathbf{x}}$ has no solutions in Δ.*
- (iii) *The map F has no minimal period-two points in Δ, $\det J_F(\bar{\mathbf{x}}) < 0$, and $F(\mathbf{x}) = \bar{\mathbf{x}}$ has no solutions in Δ.*

Theorem 2.3.

- (a) *Assume the hypotheses in Theorem 2.1, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 2.1. If curve $\mathcal{C}$ has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{\mathbf{x} \in \mathcal{R} \setminus \mathcal{C} : \exists \mathbf{z} \in \mathcal{C} \text{ where } \mathbf{x} \prec_{SE} \mathbf{z}.\}$$

and

$$\mathcal{W}_+ := \{\mathbf{x} \in \mathcal{R} \setminus \mathcal{C} : \exists \mathbf{z} \in \mathcal{C} \text{ where } \mathbf{x} \succ_{SE} \mathbf{z}.\},$$

such that the following statements are true.

  - (i) *$\mathcal{W}_-$ is invariant, and $\operatorname{dist}(F^n(\mathbf{x}), Q_2(\bar{\mathbf{x}})) \to 0$ as $n \to \infty$ for every $\mathbf{x} \in \mathcal{W}_-$.*
  - (ii) *$\mathcal{W}_+$ is invariant, and $\operatorname{dist}(F^n(\mathbf{x}), Q_4(\bar{\mathbf{x}})) \to 0$ as $n \to \infty$ for every $\mathbf{x} \in \mathcal{W}_+$.*

- (b) *If, in addition to the hypotheses in part (a), $\bar{\mathbf{x}}$ is an interior point of $\mathcal{R}$ and F is C^2 and strongly competitive on a neighborhood of $\bar{\mathbf{x}}$, then the following statements are true.*

(*i*) *F has no periodic points in the boundary of $Q_1(\bar{\mathbf{x}}) \cup Q_3(\bar{\mathbf{x}})$, except for $\bar{\mathbf{x}}$.*
(*ii*) *For every $\mathbf{x} \in \mathcal{W}_-$, there exists $n_0 \in \mathbb{N}$ such that $F^n(\mathbf{x}) \in \operatorname{int}(Q_2(\bar{\mathbf{x}}))$ for $n \geq n_0$.*
(*iii*) *For every $\mathbf{x} \in \mathcal{W}_+$, there exists $n_0 \in \mathbb{N}$ such that $F^n(\mathbf{x}) \in \operatorname{int}(Q_4(\bar{\mathbf{x}}))$ for $n \geq n_0$.*

Theorem 2.4. *In addition to the hypotheses in Theorem 2.3 part (b), suppose that $\mu > 1$ and that the eigenspace E^μ, corresponding to μ, is not a coordinate axis. If the curve $\mathcal{C}$ from Theorem 2.1 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\bar{\mathbf{x}})$ of $\bar{\mathbf{x}}$, and the unstable set $\mathcal{W}^u(\bar{\mathbf{x}})$ of $\bar{\mathbf{x}}$ is a curve in $\mathcal{R}$ whose graph is a strictly decreasing function of the first coordinate on an interval and is tangential to the eigenspace E^μ at $\bar{\mathbf{x}}$. Any endpoints of $\mathcal{W}^u(\bar{\mathbf{x}})$ in $\mathcal{R}$ are fixed-points of the map F.*

The next theorem is from [4] and it is used when analyzing a region of the parameter space for an anti-competitive system in Section 4.1.

Theorem 2.5. *Let I_1 and I_2 be intervals in $\mathbb{R}$ with endpoints a_1, a_2, b_1, and b_2 respectively, where $-\infty < a_1 < a_2 < \infty$ and $-\infty < b_1 < b_2 < \infty$. Let F be an anti-competitive map on $\mathcal{R} = I_1 \times I_2$. Suppose F has a unique fixed-point $\bar{\mathbf{x}}$ in $\mathcal{R}$ and does not have any minimal period-two points.*

Then, $\bar{\mathbf{x}}$ is globally asymptotically stable on $\mathcal{R}$.

3. Theorems for Anti-Competitive Maps

In this section, we extend two theorems and a lemma from [4] to apply to maps that are not strongly anti-competitive.

The next two theorems (3.1 and 3.2) are similar to Theorem 9 and 10 in [4]; however, there are two differences. First, the hypothesis that F is strongly anti-competitive was replaced with the hypothesis that F^2 is strongly competitive.

If F is strongly anti-competitive, then by Lemma 2.2, F^2 is strongly competitive. However, there are maps where F^2 is strongly competitive, but F is not strongly anti-competitive, see Section 4. This shows that the scope of Theorems 9 and 10 in [4] is contained in the scope of Theorems 3.1 and 3.2.

The second difference is that the alternative hypothesis (d) part (*ii*) has been added to Theorem 3.1.

Adding the alternative hypothesis allows us to determine the global dynamics of many more systems of difference equations. For systems with maps that are homogeneous of order zero, in all regions of parameter space, hypothesis (d) part (*i*) is false and hypothesis (d) part (*ii*) is true. Also, there are many systems for which these hypotheses act as a toggle switch. That is, in one region of the parameter space, hypothesis (d) part (*i*) is true, and in the compliment of that region, part (*ii*) is true, see [2], [3], and [8]. When hypothesis (d) part (*i*) is false, the Jacobian matrix, evaluated at the fixed-point, has an eigenvalue of zero and there is a super-attracting manifold.

Theorem 3.1. *Let I_1 and I_2 be intervals in $\mathbb{R}$ with endpoints a_1, a_2, b_1, and b_2 respectively, where $-\infty < a_1 < a_2 \leq \infty$ and $-\infty < b_1 < b_2 \leq \infty$. Let $\mathcal{R} = I_1 \times I_2$ and let $F : \mathcal{R} \to \mathcal{R}$ be an anti-competitive map with an interior saddle fixed-point $\bar{\mathbf{x}}$. Let $\Delta := \mathrm{int}\,(Q_1(\bar{\mathbf{x}}) \cup Q_3(\bar{\mathbf{x}})) \cap \mathcal{R}$. Suppose the following statements are true.*

- (a) *$F(\mathrm{int}(\mathcal{R})) \subset \mathrm{int}(\mathcal{R})$ and F^2 is a strongly competitive map on $\mathrm{int}(\mathcal{R})$.*
- (b) *There are no fixed-points or minimal period-two points of F in Δ.*
- (c) *F is a continuously differentiable map in some neighborhood of $\bar{\mathbf{x}}$.*
- (d) *One of the following is true:*
  - (i) *$\det J_F(\bar{\mathbf{x}}) \neq 0$ and if $\mathbf{x} \neq \bar{\mathbf{x}}$, then $F(\mathbf{x}) \neq \bar{\mathbf{x}}$.*
  - (ii) *There is a curve $\mathcal{C} \subset \mathcal{R}$ that is the graph of a strictly increasing continuous function of the first coordinate on an interval, that passes through $\bar{\mathbf{x}}$, and has endpoints in $\partial\mathcal{R}$. Also, $F(\mathcal{C}) = \bar{\mathbf{x}}$ and if $\mathbf{x} \in (\partial\mathcal{R} \cap Q_3(\bar{\mathbf{x}}))\backslash\mathcal{C}$, then $F(\mathbf{x}) \neq \bar{\mathbf{x}}$.*

Then, the following statements are true.

1. *There is a curve $\mathcal{C} \subset \mathcal{R}$ through $\bar{\mathbf{x}}$ that is invariant under F^2 and is a subset of the basin of attraction of $\bar{\mathbf{x}}$. The curve $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval and it has endpoints in $\partial\mathcal{R}$.*
2. *The curve $\mathcal{C}$ separates $\mathcal{R}$ into two connected components.*
   Define

$$\mathcal{W}_- := \{\mathbf{x} \in \mathcal{R}\backslash\mathcal{C} : \exists \mathbf{z} \in \mathcal{C} \text{ where } \mathbf{x} \prec_{SE} \mathbf{z}.\};$$
$$\mathcal{W}_+ := \{\mathbf{x} \in \mathcal{R}\backslash\mathcal{C} : \exists \mathbf{z} \in \mathcal{C} \text{ where } \mathbf{x} \succ_{SE} \mathbf{z}.\}.$$

   - (i) *If $\mathbf{x} \in \mathcal{W}_-$, then $\{F^{2n}(\mathbf{x})\}$ eventually enters $\mathrm{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$ and $\{F^{2n+1}(\mathbf{x})\}$ eventually enters $\mathrm{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$.*
   - (ii) *If $\mathbf{x} \in \mathcal{W}_+$, then $\{F^{2n}(\mathbf{x})\}$ eventually enters $\mathrm{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$ and $\{F^{2n+1}(\mathbf{x})\}$ eventually enters $\mathrm{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$.*
   - (iii) *Both $\mathrm{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$ and $\mathrm{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$ are invariant under F^2.*

Proof.

Assume hypothesis (d) part (*i*).

By Lemma 2.2, F^2 is competitive, but not necessarily strongly competitive, on $\partial\mathcal{R}$. So, in order to apply Theorem 2.1, we need to use the rectangular region $\mathcal{K} = \mathrm{int}(\mathcal{R})$. Note that F^2 is strongly competitive on $\mathcal{K}$ and that $F^2(\mathcal{K}) \subset \mathcal{K}$.

By Lemma 2.3, $\det J_{F^2}(\bar{\mathbf{x}}) > 0$. Also, $J_{F^2}(\bar{\mathbf{x}})$ has eigenvalues λ and μ such that $0 < \lambda < 1 < \mu$. By Remark 1, the eigenvectors of $J_{F^2}(\bar{\mathbf{x}})$ are not aligned with a coordinate axis. In view of hypotheses (a) and (c), we can apply Theorem 2.1 using the rectangular region $\mathcal{K}$, so there is a curve $\mathcal{C} \subset \mathcal{K}$ through $\bar{\mathbf{x}}$ that is invariant under F^2 and is a subset of the basin of attraction of $\bar{\mathbf{x}}$. The curve $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval.

A local injectivity implies that $F^2(\mathbf{x}) \neq \bar{\mathbf{x}}$ if $\mathbf{x} \neq \bar{\mathbf{x}}$. In view of hypothesis (b) and $\det J_{F^2}(\bar{\mathbf{x}}) > 0$, we can apply Theorem 2.2 part (*ii*), so curve $\mathcal{C}$ has endpoints in $\partial\mathcal{K}$.

Since $\bar{\mathbf{x}}$ is an interior fixed-point, we can apply Theorem 2.3 part (b), so $\mathcal{W}_-$ and $\mathcal{W}_+$ (restricted to $\mathcal{K}$) are invariant under F^2. Also, for every $\mathbf{x} \in \mathcal{W}_-$, $\{F^{2n}(\mathbf{x})\}$ eventually enters $\operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{K}$ and for every $\mathbf{x} \in \mathcal{W}_+$, $\{F^{2n}(\mathbf{x})\}$ eventually enters $\operatorname{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{K}$.

By Theorem 2.4, C is the global stable manifold $\mathcal{W}^s(\bar{\mathbf{x}})$ of $\bar{\mathbf{x}}$ for F^2.

Now consider $\mathbf{x} \in (\partial\mathcal{R} \cap Q_3(\bar{\mathbf{x}})) \setminus \mathcal{C}$. Let $\mathbf{y} = \mathcal{C} \cap \partial\mathcal{R} \cap Q_3(\bar{\mathbf{x}})$.

Suppose $\mathbf{x} \in \mathcal{W}_-$. Then $\mathbf{y} \succ_{SE} \mathbf{x}$. In view of hypothesis (b), neither $\mathbf{x}$ nor $\mathbf{y}$ are fixed-points of F^2. Since F^2 is competitive on $\partial\mathcal{R}$ and $F^2(\mathbf{x}) \notin \mathcal{C}$, we have $F^2(\mathbf{y}) \succ_{SE} F^2(\mathbf{x})$ and $F^2(\mathbf{y}) \in \operatorname{int}(\mathcal{R})$. So, there is a $\mathbf{z} \in \operatorname{int}(\mathcal{R}) \cap \mathcal{W}_-$ such that $\mathbf{z} \succ_{SE} F^2(\mathbf{x})$. Since $\{F^{2n}(\mathbf{z})\}$ eventually enters $\operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$ and F^2 preserves the south-east ordering, $\{F^{2n}(\mathbf{x})\}$ must eventually enter $\operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$.

Similarly, if $\mathbf{x} \in \mathcal{W}_+$, then $\{F^{2n}(\mathbf{x})\}$ eventually enters $\operatorname{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$.

Assume hypothesis (d) part (*ii*).

Then, conclusion (1) holds. Note: in many cases where $\det J_F(\bar{\mathbf{x}}) = 0$, we are able to find a linear super-attracting manifold.

Suppose $\mathbf{x} \in \mathcal{W}_-$.

If $\mathbf{x} \in \operatorname{int}(\mathcal{R})$, then, there exists $\mathbf{y} \in \mathcal{C}$ such that $\mathbf{x} \prec_{SE} \mathbf{y}$. Since F^2 is strongly competitive and $F^2(\mathbf{y}) = \bar{\mathbf{x}}$, we have $F^2(\mathbf{x}) \ll_{SE} \bar{\mathbf{x}}$, so $F^2(\mathbf{x}) \in \operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$.

If $\mathbf{x} \in (\partial\mathcal{R} \cap Q_3(\bar{\mathbf{x}})) \setminus \mathcal{C}$, then by assumption, $F(\mathbf{x}) \neq \bar{\mathbf{x}}$. Let $\mathbf{y} = \mathcal{C} \cap \partial\mathcal{R} \cap Q_3(\bar{\mathbf{x}})$. Then, $\mathbf{x} \prec_{SE} \mathbf{y}$. Since F^2 is competitive on $\partial\mathcal{R}$ and $F^2(\mathbf{y}) = \bar{\mathbf{x}}$, we have $F^2(\mathbf{x}) \prec_{SE} \bar{\mathbf{x}}$. There is a $\mathbf{z} \in \operatorname{int}(\mathcal{R}) \cap \mathcal{W}_-$ such that $F^2(\mathbf{x}) \prec_{SE} \mathbf{z}$. So, $F^4(\mathbf{x}) \preceq_{SE} F^2(\mathbf{z}) \ll_{SE} \bar{\mathbf{x}}$. Thus, $F^4(\mathbf{x}) \in \operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$.

Similarly, if $\mathbf{x} \in \mathcal{W}_+$, then $\{F^{2n}(\mathbf{x})\}$ eventually enters $\operatorname{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$.

So, using either hypothesis (d) part (*i*) or part (*ii*), we see that for $\mathbf{x} \notin \mathcal{C}$, $\{F^{2n}(\mathbf{x})\}$ eventually enters $\operatorname{int}(Q_2(\bar{\mathbf{x}}))$ or $\operatorname{int}(Q_4(\bar{\mathbf{x}}))$. We now show that $\operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$ and $\operatorname{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$ are invariant under F^2.

If $\mathbf{y} \in \operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$, then there exists $\mathbf{z} \in \operatorname{int}(\mathcal{R})$ such that $\mathbf{y} \prec_{SE} \mathbf{z} \ll_{SE} \bar{\mathbf{x}}$. So, $F^2(\mathbf{y}) \preceq_{SE} F^2(\mathbf{z}) \ll_{SE} \bar{\mathbf{x}}$, thus $\operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$ is invariant under F^2.

Similarly, $\operatorname{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$ is invariant under F^2.

We now show that if $\mathbf{x} \notin \mathcal{C}$, then $\{F^n(\mathbf{x})\}$ eventually bounces back and forth between $\operatorname{int}(Q_2(\bar{\mathbf{x}}))$ and $\operatorname{int}(Q_4(\bar{\mathbf{x}}))$.

Let $\mathbf{x} \in \mathcal{W}_-$. Since $\{F^{2n}(\mathbf{x})\}$ eventually enters $\operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap R$, we can choose $\mathbf{x} \in \operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$. Since $F^2(\mathbf{x}) \in \operatorname{int}(Q_2(\bar{\mathbf{x}}))$, $F(\mathbf{x}) \neq \bar{\mathbf{x}}$. Since F is anti-competitive, we have $F(\mathbf{x}) \succ_{SE} \bar{\mathbf{x}}$. There is a $\mathbf{y} \in \operatorname{int}(\mathcal{R}) \cap \mathcal{W}_+$ such that $\mathbf{y} \prec_{SE} F(\mathbf{x})$. Since

$\{F^{2n}(\mathbf{y})\}$ eventually enters $\text{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$ and F^2 preserves the south-east ordering, $\{F^{2n+1}(\mathbf{x})\}$ eventually enters $\text{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$.

Similarly, if $\mathbf{x} \in \mathcal{W}_+$, then $\{F^{2n+1}(\mathbf{x})\}$ eventually enters $\text{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$. □

Note: If $\mathcal{C}$ came from part (i) of hypothesis (d), then we need to prove that $\mathcal{C}$ is invariant under F.

Corollary 3.1. *Suppose the hypotheses of Theorem 3.1 are true. Then, the following statements are true.*

(i) *If* $\mathbf{x} \in \mathcal{W}_+$, *then* $F(\mathbf{x}) \in \mathcal{W}_-$.
(ii) *If* $\mathbf{x} \in \mathcal{W}_-$, *then* $F(\mathbf{x}) \in \mathcal{W}_+$.
(iii) *If* $\mathbf{x} \in \mathcal{C}$, *then* $F(\mathbf{x}) \in \mathcal{C}$.

Proof.

(i) Let $\mathbf{x} \in \mathcal{W}_+$. Suppose $F(\mathbf{x}) \in \mathcal{W}_+$. By Theorem 3.1, there is an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, both $F^{2n}(\mathbf{x}) \in \text{int}(Q_4(\bar{\mathbf{x}}))$ and $F^{2n+1}(\mathbf{x}) \in \text{int}(Q_4(\bar{\mathbf{x}}))$. So, for $n \geq n_0$, $F^{2n}(\mathbf{x}) \gg_{SE} \bar{\mathbf{x}}$ and $F^{2n+1}(\mathbf{x}) \gg_{SE} \bar{\mathbf{x}}$. This contradicts the fact that F reverses the south-east ordering.

Suppose $F(\mathbf{x}) \in \mathcal{C}$. Then for all n, $F^{2n+1}(\mathbf{x}) \in \mathcal{C}$ and there is an $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $F^{2n}(\mathbf{x}) \in \text{int}(Q_4(\bar{\mathbf{x}}))$. So, for $n \geq n_0$, $F^{2n}(\mathbf{x}) \gg_{SE} \bar{\mathbf{x}}$. Since F reverses the south-east ordering, $F\left(F^{2n}(\mathbf{x})\right) \preceq_{SE} \bar{\mathbf{x}}$. However, $F\left(F^{2n}(\mathbf{x})\right) \neq \bar{\mathbf{x}}$ because $F^{2n+2}(\mathbf{x}) \gg_{SE} \bar{\mathbf{x}}$. Thus, $F^{2n+1}(\mathbf{x}) \in \mathcal{W}_-$, which is a contradiction.

(ii) The proof is similar to part (i) and will be omitted.

(iii) Let $\mathbf{x} \in \mathcal{C}$. Suppose $F(\mathbf{x}) \notin \mathcal{C}$. Then $F^2(\mathbf{x}) \in \mathcal{C}$ and either $F(\mathbf{x}) \in \mathcal{W}_-$ or $F(\mathbf{x}) \in \mathcal{W}_+$. Without loss of generality, assume $F(\mathbf{x}) \in \mathcal{W}_+$. By part (i), we have $F^2(\mathbf{x}) \in \mathcal{W}_-$, which is a contradiction.

□

Before we prove Theorem 3.2, it is helpful to prove the next lemma. The lemma and its proof are a modified version of Theorem 6 in [4].

Lemma 3.1. *Let F be an anti-competitive map on $\mathcal{R} \subset \mathbb{R}^2$. Suppose the following statements are true.*

(a) *F^2 is strongly competitive on* $\text{int}(\mathcal{R})$.
(b) *$\bar{\mathbf{x}}$ is the unique fixed-point of F in* $\text{int}(\mathcal{R})$ *and it is a saddle point.*
(c) *F is continuously differentiable on some neighborhood of* $\bar{\mathbf{x}}$.

Then, for all $\delta > 0$, there is a point $\mathbf{t} \in \text{int}(Q_2(\bar{\mathbf{x}}) \cap \mathcal{R})$ such that $||\bar{\mathbf{x}} - \mathbf{t}|| < \delta$ and $F^2(\mathbf{t}) \ll_{SE} \mathbf{t}$. Also, there is a point $\mathbf{u} \in \text{int}(Q_4(\bar{\mathbf{x}}) \cap \mathcal{R})$ such that $||\bar{\mathbf{x}} - \mathbf{u}|| < \delta$ and $F^2(\mathbf{u}) \gg_{SE} \mathbf{u}$.

Proof.

Let $\delta > 0$.

Since $\bar{\mathbf{x}}$, is a saddle point of F, by Lemma 2.3, $\bar{\mathbf{x}}$ is a saddle point of F^2 and $J_{F^2}(\bar{\mathbf{x}})$ has eigenvalues λ and μ such that $0 \leq \lambda < 1 < \mu$. Let v be the eigenvector of $J_{F^2}(\bar{\mathbf{x}})$ corresponding to μ. By Remark 1, v can be chosen so that it points in the direction of Quadrant II.

There is an $r_0 > 0$ such that $\bar{\mathbf{x}} + r_0 v \in \operatorname{int}(Q_2(\bar{\mathbf{x}}) \cap \mathcal{R})$, $\bar{\mathbf{x}} - r_0 v \in \operatorname{int}(Q_4(\bar{\mathbf{x}}) \cap \mathcal{R})$, and $||r_0 v|| < \delta$.

There is an $r_1 > 0$ such that F is continuously differentiable at $\mathbf{x}$ if $||\mathbf{x} - \bar{\mathbf{x}}|| < r_1$. Let $r < r_1$. Since F^2 is differentiable,

$$F^2(\bar{\mathbf{x}} + rv) - (\bar{\mathbf{x}} + rv) = F^2(rv) - rv + o(r) = r(\mu - 1)v + o(r). \tag{3.1}$$

Since $\mu > 1$, there is an r_2 such that for $0 < r < r_2$, both expressions on the left-hand side of Equation (3.1) are of constant sign.

Also,

$$F^2(\bar{\mathbf{x}} - rv) - (\bar{\mathbf{x}} - rv) = -F^2(rv) + rv + o(r) = r(1 - \mu)v + o(r). \tag{3.2}$$

Since $\mu > 1$, there is an r_3 such that for $0 < r < r_3$, both expressions on the left-hand side of Equation (3.2) are of constant sign.

Choose r such that $0 < r < \min\{r_0, r_1, r_2, r_3\}$. So, for r sufficiently small, $F^2(\bar{\mathbf{x}} + rv) - (\bar{\mathbf{x}} + rv)$ points in the direction of Quadrant II. Thus, $F^2(\bar{\mathbf{x}} + rv) \ll_{SE} (\bar{\mathbf{x}} + rv)$. Also, $F^2(\bar{\mathbf{x}} - rv) - (\bar{\mathbf{x}} - rv)$ points in the direction of Quadrant IV. Thus, $F^2(\bar{\mathbf{x}} - rv) \gg_{SE} (\bar{\mathbf{x}} - rv)$. □

Theorem 3.2. *Suppose the hypotheses in Theorem 3.1 are true. Then, the following statements are true.*

Set $F^{2n}(\mathbf{x}) = (x_{2n}, y_{2n})$ *and* $F^{2n+1}(\mathbf{x}) = (x_{2n+1}, y_{2n+1})$.

(*a*) *If there exists a unique minimal period-two solution* (v_1, w_1) *and* (v_2, w_2) *in* $\operatorname{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$ *and* $\operatorname{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$ *respectively,* $F^2(a_1, b_2) \gg_{SE} (a_1, b_2)$, *and* $F^2(a_2, b_1) \ll_{SE} (a_2, b_1)$, *then the following statements are true.*
   - (*i*) *If* $\mathbf{x} \in \mathcal{W}_+$, *then* $\{x_{2n}\} \to v_2$, $\{y_{2n}\} \to w_2$ *and* $\{x_{2n+1}\} \to v_1$, $\{y_{2n+1}\} \to w_1$.
   - (*ii*) *If* $\mathbf{x} \in \mathcal{W}_-$, *then* $\{x_{2n}\} \to v_1$, $\{y_{2n}\} \to w_1$ *and* $\{x_{2n+1}\} \to v_2$, $\{y_{2n+1}\} \to w_2$.
   - (*iii*) *If* $\mathbf{x} \in \mathcal{C}$, *then* $\{(x_n, y_n)\} \to \bar{\mathbf{x}}$.

(*b*) *If there are no minimal period-two points of* F, *then* $a_2 = b_2 = \infty$ *and the following statements are true.*
   - (*i*) *If* $\mathbf{x} \in \mathcal{W}_+$, *then* $\{x_{2n}\} \to \infty$, $\{y_{2n+1}\} \to \infty$, *and* $\{x_{2n+1}\}$ *and* $\{y_{2n}\}$ *are bounded.*
   - (*ii*) *If* $\mathbf{x} \in \mathcal{W}_-$, *then* $\{x_{2n+1}\} \to \infty$, $\{y_{2n}\} \to \infty$, *and* $\{x_{2n}\}$ *and* $\{y_{2n+1}\}$ *are bounded.*
   - (*iii*) *If* $\mathbf{x} \in \mathcal{C}$, *then* $\{(x_n, y_n)\} \to \bar{\mathbf{x}}$.

Proof.

(a) Let (v_1,w_1) and (v_2,w_2) be the unique minimal period-two points of F in $\text{int}(Q_2(\bar{\mathbf{x}}))\cap\mathcal{R}$ and $\text{int}(Q_4(\bar{\mathbf{x}}))\cap\mathcal{R}$, respectively.

(i) Let $\mathbf{x}\in\mathcal{W}_+$. By Theorem 3.1, there is an $n_0\in\mathbb{N}$ such that for $n\geq n_0$, both $F^{2n}(\mathbf{x})\in\text{int}(Q_4(\bar{\mathbf{x}}))\cap\mathcal{R}$ and $F^{2n+1}(\mathbf{x})\in\text{int}(Q_2(\bar{\mathbf{x}}))\cap\mathcal{R}$. Let $n\geq n_0$ and let $\mathbf{x_0}=F^{2n}(\mathbf{x})$ and $\mathbf{x_1}=F^{2n+1}(\mathbf{x})$.

By Lemma 3.1, there exists $\mathbf{c}\in\text{int}(Q_4(\bar{\mathbf{x}}))\cap\mathcal{R}$ such that $\mathbf{c}\prec_{SE}\mathbf{x_0}$ and $F^2(\mathbf{c})\gg_{SE}\mathbf{c}$. Let $\mathbf{d}=(a_2,b_1)$. Since $\mathbf{d}$ is the south-east corner of $\mathcal{R}$, we have $\mathbf{d}\in\text{int}(Q_4(\bar{\mathbf{x}}))\cap\mathcal{R}$, $\mathbf{d}\succeq_{SE}\mathbf{x_0}$, and $F^2(\mathbf{d})\ll_{SE}\mathbf{d}$.

Since F^2 strongly preserves the south-east ordering in $\text{int}(\mathcal{R})$, the sequence $\{F^{2n}(\mathbf{c})\}$ is strictly increasing with respect to the south-east ordering and $\{F^{2n}(\mathbf{d})\}$ is strictly decreasing with respect to the south-east ordering.

Since $F^{2n}(\mathbf{c})\ll_{SE}F^{2n}(\mathbf{x_0})\preceq_{SE}F^{2n}(\mathbf{d})$ for all $n\geq n_0$ and (v_2,w_2) is the unique fixed point of F^2 in $\text{int}(Q_4(\bar{\mathbf{x}}))\cap\mathcal{R}$, $\{F^{2n}(\mathbf{x_0})\}\to(v_2,w_2)$, which means that $\{x_{2n}\}\to v_2$ and $\{y_{2n}\}\to w_2$.

By Lemma 3.1, there exists $\mathbf{p}\in\text{int}(Q_2(\bar{\mathbf{x}}))\cap\mathcal{R}$ such that $\mathbf{p}\succ_{SE}\mathbf{x_1}$ and $F^2(\mathbf{p})\ll_{SE}\mathbf{p}$. Let $\mathbf{q}=(a_1,b_2)$. Since $\mathbf{q}$ is the north-west corner of $\mathcal{R}$, we have $\mathbf{q}\in\text{int}(Q_2(\bar{\mathbf{x}}))\cap\mathcal{R}$, $\mathbf{q}\preceq_{SE}\mathbf{x_1}$, and $F^2(\mathbf{q})\gg_{SE}\mathbf{q}$.

Since F^2 strongly preserves the south-east ordering, the sequence $\{F^{2n}(\mathbf{p})\}$ is strictly decreasing with respect to the south-east ordering and $\{F^{2n}(\mathbf{q})\}$ is strictly increasing with respect to the south-east ordering. Since $F^{2n}(\mathbf{p})\gg_{SE}F^{2n}(\mathbf{x_1})\succeq_{SE}F^{2n}(\mathbf{q})$ for all $n\geq n_0$ and (v_1,w_1) is the unique fixed point of F^2 in $\text{int}(Q_2(\bar{\mathbf{x}}))\cap\mathcal{R}$, $\{F^{2n}(\mathbf{x_1})\}\to(v_1,w_1)$, which means that $\{x_{2n+1}\}\to v_1$ and $\{y_{2n+1}\}\to w_1$.

(ii) The proof for $\mathbf{x}\in\mathcal{W}_-$ is similar and will be omitted.

(iii) Let $\mathbf{x}\in\mathcal{C}$. By Corollary 3.1, $F(\mathbf{x})\in\mathcal{C}$. Since $\mathcal{C}$ is a subset of the basin of attraction of $\bar{\mathbf{x}}$ under F^2, we have $\{F^{2n}(\mathbf{x})\}\to\bar{\mathbf{x}}$ and $\{F^{2n+1}(\mathbf{x})\}\to\bar{\mathbf{x}}$, so $\{(x_n,y_n)\}\to\bar{\mathbf{x}}$.

(b) Suppose there are no minimal period-two points of F.

(i) Let $\mathbf{x}\in\mathcal{W}_+$. By Theorem 3.1, there is an $n_0\in\mathbb{N}$ such that for $n\geq n_0$, both $F^{2n}(\mathbf{x})\in\text{int}(Q_4(\bar{\mathbf{x}}))\cap\mathcal{R}$ and $F^{2n+1}(\mathbf{x})\in\text{int}(Q_2(\bar{\mathbf{x}}))\cap\mathcal{R}$. Let $n\geq n_0$ and let $\mathbf{x_0}=F^{2n}(\mathbf{x})$ and $\mathbf{x_1}=F^{2n+1}(\mathbf{x})$.

By Lemma 3.1, there exists $\mathbf{c}\in\text{int}(Q_4(\bar{\mathbf{x}}))\cap\mathcal{R}$ such that $\mathbf{c}\prec_{SE}\mathbf{x_0}$ and $F^2(\mathbf{c})\gg_{SE}\mathbf{c}$. Since F^2 strongly preserves the south-east ordering, the sequence $\{F^{2n}(\mathbf{c})\}$ is strictly increasing with respect to the south-east ordering.

The second component of $\{F^{2n}(\mathbf{c})\}$ is decreasing and bounded below by b_1, so it must converge. Since $F^{2n}(\mathbf{x_0})\gg_{SE}F^{2n}(\mathbf{c})$ for $n\geq n_0$, $\{y_{2n}\}$ must be bounded.

Combining the facts that F^2 has no fixed-points in $\text{int}(Q_4(\bar{\mathbf{x}}))\cap\mathcal{R}$, the set $\mathcal{R}$ contains its limit-points, and the second component of $\{F^{2n}(\mathbf{c})\}$

is convergent, we see that the first component of $\{F^{2n}(\mathbf{c})\}$ is strictly increasing and must be unbounded. Since $F^{2n}(\mathbf{x_0}) \gg_{SE} F^{2n}(\mathbf{c})$ for $n \geq n_0$, we have $\{x_{2n}\} \to \infty$.

By Lemma 3.1, there exists $\mathbf{d} \in \text{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$ such that $\mathbf{d} \succ_{SE} \mathbf{x_1}$ and $F^2(\mathbf{d}) \ll_{SE} \mathbf{d}$. Since F^2 strongly preserves the south-east ordering, the sequence $\{F^{2n}(\mathbf{d})\}$ is strictly decreasing with respect to the south-east ordering.

The first component of $\{F^{2n}(\mathbf{d})\}$ is decreasing and bounded from below by a_1, so it must converge. Since $F^{2n}(\mathbf{x_1}) \ll_{SE} F^{2n}(\mathbf{d})$ for $n \geq n_0$, the first component of $F^{2n}(\mathbf{x_1})$ must be bounded. Thus, $\{x_{2n+1}\}$ must be bounded.

Combining the facts that F^2 has no fixed-points in $\text{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$, the set $\mathcal{R}$ contains its limit-points, and the first component of $\{F^{2n}(\mathbf{d})\}$ converges, we see that the second component of $\{F^{2n}(\mathbf{d})\}$ is strictly increasing and must be unbounded. Since $F^{2n}(\mathbf{x_1}) \ll_{SE} F^{2n}(\mathbf{d})$ for $n \geq n_0$, we have $\{y_{2n+1}\} \to \infty$.

(ii) The proof for $\mathbf{x} \in \mathcal{W}_-$ is similar and will be omitted.

(iii) The proof for $\mathbf{x} \in \mathcal{C}$ is the same as in part (a).

□

4. Example: A Weakly Anti-Competitive Map

Before we begin our analysis, it is helpful to prove a lemma that makes it easier to show that hypothesis (b) in Theorem 3.1 is satisfied.

Lemma 4.1. *Let F be an anti-competitive map on a closed and bounded rectangle $\mathcal{R} \subset \mathbb{R}^2$, such that the north-west and south-east corners of $\mathcal{R}$ are minimal period-two points of F that are mapped onto each other. Let F^2 be a strongly competitive map on $\mathcal{R}$.*

Then, there is a fixed-point of F in $\text{int}(\mathcal{R})$.

Proof.

Let $\mathbf{p_1}$ and $\mathbf{p_2}$ be the north-west and south-east corners of $\mathcal{R}$, respectively. Let $\mathbf{x} \in \mathcal{R} \backslash (\mathbf{p_1} \cup \mathbf{p_2})$. Then, $\mathbf{p_1} \prec_{SE} \mathbf{x} \prec_{SE} \mathbf{p_2}$.

Since F is anti-competitive on $\mathcal{R}$, we have $F(\mathbf{p_1}) \succeq_{SE} F(\mathbf{x}) \succeq_{SE} F(\mathbf{p_2})$. So, $\mathbf{p_2} \succeq_{SE} F(\mathbf{x}) \succeq_{SE} \mathbf{p_1}$ and thus, $F(\mathcal{R}) \subset \mathcal{R}$.

Since F^2 is strongly competitive on $\mathcal{R}$, we have $F^2(\mathbf{p_1}) \ll_{SE} F^2(\mathbf{x}) \ll_{SE} F^2(\mathbf{p_2})$. So, $\mathbf{p_1} \ll_{SE} F^2(\mathbf{x}) \ll_{SE} \mathbf{p_2}$ and thus, $F^2(\mathbf{x}) \in \text{int}(\mathcal{R})$.

By the Brouwer Fixed-Point Theorem, there is a fixed-point $\bar{\mathbf{x}}$ of F in $\mathcal{R}$.

Since $F^2(\mathcal{R} \backslash (\mathbf{p_1} \cup \mathbf{p_2})) \in \text{int}(\mathcal{R})$ and $\bar{\mathbf{x}}$ is a fixed-point of F^2, $\bar{\mathbf{x}} \in \text{int}(\mathcal{R})$. □

Consider the system of difference equations

$$\left.\begin{aligned} x_{n+1} &= \tfrac{x_n+y_n}{x_n} \\ y_{n+1} &= \tfrac{\beta_2 x_n}{A_2+x_n} \end{aligned}\right\} \quad \text{for } n = 0,1,2,\ldots \tag{4.1}$$

where β_2, A_2, and x_0 are positive and y_0 is nonnegative.

Let F be the map corresponding to system (4.1). Then,

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x+y}{x} \\ \frac{\beta_2 x}{A_2+x} \end{pmatrix}.$$

For all $n \geq 0$, $x_n > 0$. So, for all $n \geq 1$, $y_n > 0$.

For $n \geq 0$,

$$x_{n+1} = \frac{x_n+y_n}{x_n} \geq 1 \text{ and } y_{n+1} = \frac{\beta_2 x_n}{A_2+x_n} < \beta_2.$$

So, for all $n \geq 1$, $x_{n+1} > 1$ and

$$x_{n+1} = \frac{x_n+y_n}{x_n} = 1 + \frac{y_n}{x_n} < 1 + \frac{\beta_2}{1} = 1+\beta_2.$$

Let $\mathcal{R} = [1, 1+\beta_2] \times [0, \beta_2]$.

Then, $\mathcal{R}$ is an attracting, invariant rectangle. Also, $F(\text{int}(\mathcal{R})) \subset \text{int}(\mathcal{R})$ and if $\mathbf{x_0} \in (0,\infty) \times [0,\infty)$, then for $n \geq 2$, $\mathbf{x_n} \in \text{int}(\mathcal{R})$.

The Jacobian matrix of F is

$$J_F = \begin{bmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ \frac{\beta_2 A_2}{(A_2+x)^2} & 0 \end{bmatrix}.$$

In the region $\mathcal{R}$, y is nonnegative, while x and all of the parameters are positive. From the sign configuration of the Jacobian matrix and Lemma 2.1, we see that F is anti-competitive, but not strongly anti-competitive, on $\mathcal{R}$.

4.1. Analysis for the Case $A_2 \leq 1$ or $\beta_2 \leq \frac{A_2(A_2+1)^2}{(A_2-1)^2}$

In this section, we use Theorem 2.5 to show that in this region of the parameter space, every solution of System (4.1) converges to the unique equilibrium point.

Solving the equilibrium equations shows that in all regions of the parameter space, there is a unique interior equilibrium point $\bar{\mathbf{x}} = (\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1-A_2+\sqrt{(1-A_2)^2+4(A_2+\beta_2)}}{2} \quad \text{and} \quad \bar{y} = \frac{\beta_2 \bar{x}}{A_2+\bar{x}}. \tag{4.2}$$

Suppose (v_1, w_1) and (v_2, w_2) are distinct minimal period-two points of System (4.1). Then, we have the following equations:

$$(a)\ v_1 = \tfrac{v_2+w_2}{v_2}, \qquad (b)\ v_2 = \tfrac{v_1+w_1}{v_1},$$

$$(c)\ w_1 = \tfrac{\beta_2 v_2}{A_2+v_2}, \qquad (d)\ w_2 = \tfrac{\beta_2 v_1}{A_2+v_1}.$$

Subtracting Equation (a) from Equation (b), we obtain

$$v_1 - v_2 = w_2 - w_1. \tag{4.3}$$

This shows that $v_1 = v_2$, if and only if, $w_1 = w_2$. So, to have a minimal period-two solution, we must have $v_1 \neq v_2$ and $w_1 \neq w_2$.

Subtracting Equation (c) from Equation (d), we obtain

$$A_2(w_2 - w_1) + v_1 w_2 - v_2 w_1 = \beta_2(v_1 - v_2).$$

Using the additive identity, we have

$$A_2(w_2 - w_1) + v_1 w_2 - v_1 w_1 + v_1 w_1 - v_2 w_1 = \beta_2(v_1 - v_2).$$

So,

$$A_2(w_2 - w_1) + v_1(w_2 - w_1) + w_1(v_1 - v_2) = \beta_2(v_1 - v_2).$$

Using Equation (4.3) and the fact that $w_1 \neq w_2$, we obtain

$$w_1 = -v_1 + \beta_2 - A_2.$$

So, a minimal period-two point (v, w) must be on the line $w = -v + \beta_2 - A_2$.

This fact, combined with Equation (c) gives us

$$w_1 = -v_1 + \beta_2 - A_2 = \frac{\beta_2 v_2}{A_2 + v_2}.$$

Solving for v_2 in terms of v_1 yields

$$v_2 = \frac{-A_2 v_1 + \beta_2 A_2 - A_2^2}{v_1 + A_2}. \tag{4.4}$$

Using Equation (c), we have

$$v_1 = \frac{v_2 + w_2}{v_2} = \frac{v_2 - v_2 + \beta_2 - A_2}{v_2} = \frac{\beta_2 - A_2}{v_2}.$$

Substituting for v_2 using Equation (4.4), we see that in order to have minimal period-two solutions, we must have

$$v_1 = \frac{(1-A_2)(A_2-\beta_2) - \sqrt{(1-A_2)^2(A_2-\beta_2)^2 + 4A_2^2(A_2-\beta_2)}}{2A_2};$$

$$v_2 = \frac{(1-A_2)(A_2-\beta_2) + \sqrt{(1-A_2)^2(A_2-\beta_2)^2 + 4A_2^2(A_2-\beta_2)}}{2A_2};$$

$$w_1 = -v_1 + \beta_2 - A_2;$$

$$w_2 = -v_2 + \beta_2 - A_2.$$

If $\beta_2 \leq A_2$, then $w_1, w_2 \leq 0$ and no minimal period-two solution exists. Also, if $\beta_2 > A_2$ and $A_2 \leq 1$, then $v_1 < 0$, and no minimal period-two solution exists.

Suppose $\beta_2 > A_2 > 1$.

If

$$\beta_2 < \frac{A_2 (A_2+1)^2}{(A_2-1)^2},$$

then the discriminant in the formula for v_1 is negative and no minimal period-two solution exists.

Let $(\bar{x}, \bar{y})$ be the unique equilibrium point of System (4.1). If

$$\beta_2 = \frac{A_2 (A_2+1)^2}{(A_2-1)^2},$$

then the discriminant in the formula for v_1 and v_2 is zero and

$$v_1 = v_2 = \frac{(1-A_2)(A_2-\beta_2)}{2A_2} = \bar{x}.$$

One can also show that $w_1 = w_2 = \bar{y}$ and no minimal period-two solution exists.

By Theorem 2.5, when $A_2 \leq 1$ or $\beta_2 \leq \frac{A_2(A_2+1)^2}{(A_2-1)^2}$, every solution of System (4.1) converges to the unique equilibrium point.

4.2. **Analysis for the Case** $1 < A_2 < \frac{A_2(A_2+1)^2}{(A_2-1)^2} < \beta_2$

In this section, we use Theorem 3.1 and Theorem 3.2 to show that in this region of the parameter space, almost every solution of System (4.1) converges to the unique minimal period-two solution. This shows that System (4.1) possesses a period-doubling bifurcation.

To determine the local stability of the unique fixed-point $\bar{\mathbf{x}} = (\bar{x}, \bar{y})$, we examine the simplified Jacobian matrix of F evaluated at the fixed-point,

$$J_F(\bar{x}, \bar{y}) = \begin{bmatrix} \frac{-\bar{y}}{\bar{x}^2} & \frac{1}{\bar{x}} \\ \\ \frac{A_2\bar{y}^2}{\beta_2\bar{x}^2} & 0 \end{bmatrix}. \tag{4.5}$$

Both eigenvalues of Matrix (4.5) are inside the unit circle, if and only if,

$$\frac{\bar{y}}{\bar{x}^2} < 1 - \frac{A_2\bar{y}^2}{\beta_2\bar{x}^3} < 2.$$

The second inequality is true in all regions of the parameter space. Thus, one eigenvalue is always inside the unit circle.

The inequality

$$\frac{\bar{y}}{\bar{x}^2} < 1 - \frac{A_2\bar{y}^2}{\beta_2\bar{x}^3}$$

is true, if and only if,

$$\beta_2\bar{x}\bar{y} < \beta_2\bar{x}^3 - A_2\bar{y}^2. \tag{4.6}$$

Using the first equation in System 4.1, we see that $\bar{y} = \bar{x}(\bar{x}-1)$ with $\bar{x} > 1$. Substituting this into (4.6) and simplifying, we obtain

$$A_2(\bar{x}-1)^2 < \beta_2. \tag{4.7}$$

Using the second equation in System 4.1, we see that

$$\beta_2 = \frac{\bar{y}(A_2+\bar{x})}{\bar{x}}.$$

Substituting $\bar{x}(\bar{x}-1)$ for $\bar{y}$, we get

$$\beta_2 = (\bar{x}-1)(A_2+\bar{x}).$$

Substituting for β_2 in (4.7), we get

$$A_2(\bar{x}-1) < A_2+\bar{x}.$$

So,

$$\bar{x}(A_2-1) < 2A_2,$$

which is always true if $A_2 \leq 1$.

Let $A_2 > 1$. Then, the equilibrium point is locally asymptotically stable, if and only if,

$$\bar{x} < \frac{2A_2}{A_2-1}.$$

Using the formula for $\bar{x}$ in (4.2) and a bit of algebra, we arrive at the following analysis.

(a) The equilibrium point is locally asymptotically stable when $A_2 \leq 1$ or $\beta_2 < \frac{A_2(A_2+1)^2}{(A_2-1)^2}$.

(b) The equilibrium point is non-hyperbolic with one eigenvalue inside the unit circle when $1 < A_2 < \beta_2 = \frac{A_2(A_2+1)^2}{(A_2-1)^2}$.

(c) The equilibrium point is a saddle point when $1 < A_2 < \frac{A_2(A_2+1)^2}{(A_2-1)^2} < \beta_2$.

Thus, the equilibrium point is a saddle-point when the minimal period-two points exist.

Let F be the map corresponding to system (4.1). From Section 4, for $\mathcal{R} = [1, 1+\beta_2]\times[0,\beta_2]$, $F(\mathcal{R}) \subset \mathcal{R}$, and F is anti-competitive, but not strongly anti-competitive, on $\mathcal{R}$.

The Jacobian matrix for F^2, the second iterate of F, is

$$J_{F^2} = \begin{bmatrix} \frac{\beta_2 x(xy+A_2x+2A_2y)}{(A_2+x)^2(x+y)^2} & -\frac{\beta_2 x^2}{(A_2+x)(x+y)^2} \\ \\ -\frac{A_2\beta_2 y}{(x+A_2x+y)^2} & \frac{A_2\beta_2 x}{(x+A_2x+y)^2} \end{bmatrix}.$$

In the interior of $\mathcal{R}$, $x > 1$ and $y > 0$. From the sign configuration of the Jacobian matrix and Lemma 2.1, we see that F^2 is strongly competitive on $\text{int}(\mathcal{R})$ and competitive on $\partial\mathcal{R}$.

In Section 4.1, we showed that v_1 and v_2 were the roots of a quadratic equation and $w_i = -v_i + \beta_2 - A_2$. Choose $v_1 < v_2$. Then, $(v_1, w_1) \ll_{SE} (v_2, w_2)$. Since F^2 is strongly competitive on $\text{int}(\mathcal{R})$, we can apply Lemma 4.1 on the rectangle formed by (v_1, w_1) and (v_2, w_2). Since $\bar{\mathbf{x}}$ is the unique fixed-point of F, we have $(v_1, w_1) \ll_{SE} \bar{\mathbf{x}} \ll_{SE} (v_2, w_2)$, and so $(v_1, w_1) \in \text{int}(Q_2(\bar{\mathbf{x}})) \cap \mathcal{R}$ and $(v_2, w_2) \in \text{int}(Q_4(\bar{\mathbf{x}})) \cap \mathcal{R}$. Since these are the only minimal period-two points of F, there are no minimal period-two points of F in $\text{int}(Q_1(\bar{\mathbf{x}}) \cup Q_3(\bar{\mathbf{x}})) \cap \mathcal{R}$.

Using the Jacobian matrix in (4.5), one can see that $\det(J_F(\bar{x}, \bar{y})) < 0$ in all regions of the parameter space. Also, the map is one-to-one.

This satisfies the hypotheses in Theorem 3.1 using the region $\mathcal{R}$.

The map F^2 moves the north-west corner of $\mathcal{R}$ to the south-east since

$$F^2(1, \beta_2) = \left(1 + \frac{\beta_2}{(1+A_2)(1+\beta_2)}, \frac{\beta_2(1+\beta_2)}{A_2+1+\beta_2}\right) \gg_{SE} (1, \beta_2).$$

The map F^2 moves the south-east corner of $\mathcal{R}$ to the north-west since

$$F^2(1+\beta_2, 0) = \left(1 + \frac{\beta_2(1+\beta_2)}{A_2+1+\beta_2}, \frac{\beta_2}{A_2+1}\right) \ll_{SE} (1+\beta_2, 0).$$

We can apply Theorem 3.2, using the region $\mathcal{R}$, to obtain the following result.

(i) If $\mathbf{x} \in \mathcal{W}_+$, $\{x_{2n}\} \to v_2$, $\{y_{2n}\} \to w_2$ and $\{x_{2n+1}\} \to v_1$, $\{y_{2n+1}\} \to w_1$.
(ii) If $\mathbf{x} \in \mathcal{W}_-$, $\{x_{2n}\} \to v_1$, $\{y_{2n}\} \to w_1$ and $\{x_{2n+1}\} \to v_2$, $\{y_{2n+1}\} \to w_2$.
(iii) If $\mathbf{x} \in \mathcal{C}$, $\{(x_n, y_n)\} \to \bar{\mathbf{x}}$.

5. Future Research

We plan to investigate the global dynamics of all anti-competitive systems of two rational-linear difference equations, so it was necessary to extend theorems to include more types of systems, such as weakly anti-competitive or homogeneous systems. While it may seem as though hypothesis (d) part (*ii*) was tacked on to the theorem just to include systems with homogeneous maps, there are many systems for which these hypotheses act as a toggle switch. That is, in one region of the parameter space, hypothesis (d) part (*i*) is true, and in the compliment of that region, part (*ii*) is true. In the future, we plan to investigate the many systems with this property.

We also plan to develop other theorems to determine the basin of attraction of fixed-points and periodic points for anti-competitive maps.

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