SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.21.01.03
Vol.21 (34), No.1 (2025), 21–28

# COMPLEMENTARY DISTANCE AND RECIPROCAL COMPLEMENTARY DISTANCE SPECTRUM FOR INDU-BALA PRODUCT OF GRAPHS

HARISHCHANDRA S. RAMANE AND DANESHWARI D. PATIL

ABSTRACT. For two graphs $G_1$ and $G_2$, graph obtained with two disjoint copies of join structure $G_1 \vee G_2$ by joining the corresponding vertices in $G_2$'s, is the Indu–Bala product $G_1 \blacktriangledown G_2$. Present work focuses on the study of complementary distance ($\mathcal{CD}$) and reciprocal complementary distance ($\mathcal{RCD}$) spectrum for Indu–Bala product of regular graphs via the concept of equitable partition. Hence note $\mathcal{CD}$ and $\mathcal{RCD}$ spectrum of dumbbell graph as a particular case of the Indu–Bala product.

## 1. INTRODUCTION

In 2016 [4], Indulal and Balakrishnan defined a new graph operation, namely, Indu–Bala product of graphs and studied its distance spectrum using eigen vertor technique. Recently in 2019 [8], S. Patil and M. Mathapati studied adjacency, Laplacian and signless Laplacian spectra of Indu–Bala product of graphs using the concept of coronal. Present work focuses on the study of $\mathcal{CD}$ and $\mathcal{RCD}$ spectrum for Indu–Bala product of regular graphs via the concept of equitable partition. As a consequence, we provide $\mathcal{CD}$ and $\mathcal{RCD}$ spectrum of dumbbell graph as a particular case of the Indu–Bala product. Product structure considered here involves join structure $G_1 \vee G_2$ as well as Cartesian product structure $K_2 \Box G_2$. Literature related to the study of $\mathcal{CD}$ and $\mathcal{RCD}$ spectrum involving join structure can be seen in [9–12].

## 2. PRELIMINARIES

All the graphs that are taken into consideration are finite, simple and undirected. An ordered pair $(V(G), E(G))$ comprising of vertex set $V(G)$ and edge set $E(G)$ accompanied by a relation of incidence between them, is defined as a graph $G$. Order of $G$ is equal to its vertex count. Those vertices which are connected by an edge are adjacent or neighbors. Number of edges at the vertex $v$ amounts to its degree $d_i$, if $d_i = r$ (a constant) for every vertex $v_i$ then $G$ is a graph with regularity $r$ ($r$-regular graph). An alternating sequence of vertices and edges of $G$ with distinct vertices

2020 *Mathematics Subject Classification.* 05C12, 05C50.
*Key words and phrases.* Join of two graphs; Indu-Bala product; $\mathcal{CD}$-spectrum; $\mathcal{RCD}$-spectrum.

and edges is defined as a path, distance $d_{ij}$ between two vertices $v_i$ and $v_j$ is the minimal length of the path connecting them, maximal distance between the vertices is diameter ($D$) of $G$. Graph obtained from $G$, with same vertex count as $G$ and adjacency relation is such that two vertices are connected by an edge if and only if they are not connected in $G$ is the complement $\overline{G}$. For a graph $G$ with vertex count $n$, adjacency matrix is $\mathcal{A}(G) = [a_{ij}]_{n\times n}$ with $a_{ij} = 1$ if there is adjacency among vertices $v_i$ and $v_j$ and $a_{ij} = 0$, in other case. Eigenvalues associated to $\mathcal{A}(G)$ are adjacency eigenvalues ($\lambda_i$'s) and their collection is the $\mathcal{A}$-spectrum ($\sigma_{\mathcal{A}}$). $J$ denote all 1's matrix, $I$ is the unit matrix, $C_n$ is cycle graph on $n$ vertices, $\overline{K_m}$ is the empty graph on $m$ vertices. We adhere to the book [3] for undefined graph theoretical terms and notations.

Due to the significant interest in obtaining supplementary structural descriptors for QSPR and QSAR models, O. Ivanciuc et al. [5] defined $\mathcal{CD}$-matrix and $\mathcal{RCD}$-matrix, indicating new molecular graph metric derived from graph distances. $\mathcal{CD}$ and $\mathcal{RCD}$ matrices are notable in graph theoretical matrices in the field of chemistry [6]. For a graph $G$ with vertex count $n$ and diameter $D$, complementary distance matrix $\mathcal{CD}(G) = [c_{ij}]$ with $c_{ij} = 1 + D - d_{ij}$ if $i \neq j$ and 0 otherwise. Eigenvalues associated to $\mathcal{CD}(G)$ are $\mathcal{CD}$-eigenvalues and their collection is the $\mathcal{CD}$-spectrum. For a graph $G$ with vertex count $n$ and diameter $D$, reciprocal complementary distance matrix $\mathcal{RCD}(G) = [rc_{ij}]$ with $rc_{ij} = \dfrac{1}{1+D-d_{ij}}$ if $i \neq j$ and 0 otherwise. Eigenvalues associated to $\mathcal{RCD}(G)$ are $\mathcal{RCD}$-eigenvalues and their collection is the $\mathcal{RCD}$-spectrum.

**Definition 2.1.** *[3] Graph having vertex set as $V(G_1)\bigcup V(G_2)$ and $E(G_1)\bigcup E(G_2)\bigcup\{uv : u \in V(G_1), v \in V(G_2)\}$ as set of edges is the join structure $G_1 \vee G_2$ of graphs $G_1$ and $G_2$.*

**Definition 2.2.** *[3] Cartesian product $K_2\Box G_2$ of graphs $K_2$ and $G_2$ is a graph obtained from two disjoint copies of $G_2$ by joining corresponding vertices in two copies of $G_2$.*

**Definition 2.3.** *[4] For two graphs $G_1$ and $G_2$, graph obtained from two disjoint copies of join structure $G_1 \vee G_2$ by joining the corresponding vertices in $G_2$'s, is the Indu–Bala product $G_1 \blacktriangledown G_2$.*

**Definition 2.4.** *[13] Partition $\Pi : V_1 \bigcup V_2 \bigcup \ldots \bigcup V_k$ of set of vertices in graph $G$ is equitable if for two partite sets $V_i$ and $V_j$ of partition $\Pi$, there is a constant $q_{ij}$ such that a vertex $v_i \in V_i$ has exactly $q_{ij}$ neighbors in $V_j$, regardless of the choice of $v_i$.*

The matrix associated with the equitable partition is called quotient matrix ($Q$). In the structure of join of two regular graphs, whole vertex set of that regular graph will be taken as a partite set for the equitable partition (which results in two partite sets).

**Definition 2.5.** *[1] Generalized wheel graph $W_{m,n}$ is defined as the join $C_n \vee \overline{K_m}$, for $m \geq 2, n \geq 3$.*

**Definition 2.6.** *[7] Dumbbell graph $DB(W_{m,n})$, is obtained by connecting m vertices at the centres of two generalized wheel graphs $W_{m,n}$ through m edges.*

It is noted that, $DB(W_{m,n}) = C_n \blacktriangledown \overline{K_m}$.

**Lemma 2.1.** *[2] For a 2 × 2 block symmetric matrix $D = \begin{bmatrix} D_0 & D_1 \\ D_1 & D_0 \end{bmatrix}$, eigenvalues of D are those of $D_0 + D_1$ and $D_0 - D_1$.*

**Lemma 2.2.** *[3] $\mathcal{A}$-spectrum of cycle graph $C_n$ and empty graph $\overline{K_m}$ are:*

$$\sigma_{\mathcal{A}}(C_n) = \left\{ 2cos\left(\frac{2\pi k}{n}\right) : k = 0, 1, \ldots, n-1 \right\}$$

*and*

$$\sigma_{\mathcal{A}}(\overline{K_m}) = \left\{ 0 : \textit{ appearing m times} \right\}.$$

**Theorem 2.1.** [13] *For equitable partition $\Pi : V_1 \bigcup V_2 \bigcup \ldots \bigcup V_k$ of the vertex set $V(G)$ of a graph G, spectrum due to the quotient matrix Q will be a part of that due to the adjacency matrix $\mathcal{A}(G)$.*

### 2.1. Complementary distance spectrum for Indu-Bala product of graphs

**Theorem 2.2.** *$\mathcal{CD}$-spectrum of Indu-Bala product $G_1 \blacktriangledown G_2$ of two regular graphs $G_1$ and $G_2$ with orders $n_1, n_2$ and regularity $r_1, r_2$ respectively, consists of the following:*

i. $\lambda_i(G_1) - 2$ *appearing twice for* $i = 2, 3, \ldots, n_1$

ii. $2\lambda_i(G_2)$ *for* $i = 2, 3, \ldots, n_2$

iii. $-4$ *appearing* $(n_2 - 1)$ *times*

iv. $\frac{n_1}{2} + \frac{n_2}{2} + \frac{r_1}{2} - 3 \pm \frac{1}{2}\sqrt{n_1^2 + n_2^2 + r_1^2 + 2(n_1 - 2)n_2 + 2(n_1 - n_2 + 2)r_1 + 4n_1 + 4}$

*and*

$\frac{3n_1}{2} + \frac{3n_2}{2} + \frac{r_1}{2} + r_2 - 1 \pm$

$\frac{1}{2}\sqrt{9n_1^2 + 9n_2^2 + r_1^2 + 4r_2^2 + 2(41n_1 + 6)n_2 + 2(3n_1 - 3n_2 - 2)r_1 - 4(3n_1 - 3n_2 + r_1 - 2)r_2 - 12n_1 + 4}.$

*Proof.* With a proper labeling of the vertices forming the product structure, general $\mathcal{CD}$ matrix is:

$$\mathcal{CD}(G_1 \blacktriangledown G_2) = \begin{bmatrix} 3A(G_1) + 2A(\overline{G_1}) & J_{n_1 \times n_1} & 3J_{n_1 \times n_2} & 2J_{n_1 \times n_2} \\ J_{n_1 \times n_1} & 3A(G_1) + 2A(\overline{G_1}) & 2J_{n_1 \times n_2} & 3_{n_1 \times n_2} \\ 3J_{n_2 \times n_1} & 2J_{n_2 \times n_1} & 3A(G_2) + 2A(\overline{G_2}) & 3I_{n_2} + 2A(G_2) + A(\overline{G_2}) \\ 2J_{n_2 \times n_1} & 3J_{n_2 \times n_1} & 3I_{n_2} + 2A(G_2) + A(\overline{G_2}) & 3A(G_2) + 2A(\overline{G_2}) \end{bmatrix}.$$

As $G_1, G_2$ are regular graphs,

(a) vertex $v_i$ in $V(G_1)$ has exactly $r_1+n_2$ neighbors in $V(G_2)$, regardless of the choice of $v_i$ (same follows for other copy of $G_1$)
(b) vertex $v_i$ in $V(G_2)$ has exactly $r_2+n_1$ neighbors in $V(G_1)$, regardless of the choice of $v_i$
(c) vertex $v_i$ in $V(G_2)$ has exactly $r_2+1$ neighbors in $V(G_2)$ (with the other copy of $G_2$), regardless of the choice of $v_i$ (same follows for other copy of $G_2$).

Thus, we can have the equitable partition $\Pi: V(G_1)\bigcup V(G_1)\bigcup V(G_2)\bigcup V(G_2)$. Due to this equitable partition, $\mathcal{CD}(G_1\blacktriangledown G_2)$ is reduced to a smaller matrix (order $4\times 4$), as each entry of the $4\times 4$ block matrix has constant row sums.
Therefore,

$$Q_{\mathcal{CD}}(G_1\blacktriangledown G_2)=\begin{bmatrix} 2n_1+r_1-2 & n_1 & 3n_2 & 2n_2 \\ n_1 & 2n_1+r_1-2 & 2n_2 & 3n_2 \\ 3n_1 & 2n_1 & 2n_2+r_2-2 & n_2+r_2+2 \\ 2n_1 & 3n_1 & n_2+r_2+2 & 2n_2+r_2-2 \end{bmatrix}.$$

Since, $Q_{\mathcal{CD}}$ is a reduced matrix, eigenvalues of $Q_{\mathcal{CD}}$ are those of $\mathcal{CD}(G_1\blacktriangledown G_2)$. Expanding determinant of $Q_{\mathcal{CD}}$, we have:

$$\begin{aligned}\sigma_{Q_{\mathcal{CD}}} &= \Bigg\{\frac{n_1}{2}+\frac{n_2}{2}+\frac{r_1}{2}-3\pm\frac{1}{2}\sqrt{n_1^2+n_2^2+r_1^2+2(n_1-2)n_2+2(n_1-n_2+2)r_1+4n_1+4}\,, \\ &\quad \frac{3n_1}{2}+\frac{3n_2}{2}+\frac{r_1}{2}+r_2-1\pm \\ &\frac{1}{2}\sqrt{9n_1^2+9n_2^2+r_1^2+4r_2^2+2(41n_1+6)n_2+2(3n_1-3n_2-2)r_1-4(3n_1-3n_2+r_1-2)r_2-12n_1+4}\Bigg\}.\end{aligned} \tag{2.1}$$

For the remaining eigenvalues, we have to examine $G_1$ and $G_2$ structures.
Due to $G_1$, we have

$$\lambda_i(G_1)-2 \text{ appearing twice (as } G_1 \text{ is appearing twice in } G_1\blacktriangledown G_2) \text{ for } i=2,3,\ldots,n_1, \tag{2.2}$$

which is obtained from $3\mathcal{A}(G_1)+2\mathcal{A}(\overline{G_1})$, as graphs $G_1$ and $G_2$ connected through the join structure. Due to $G_2$, we have the matrix

$$\begin{bmatrix} 3\mathcal{A}(G_2)+2\mathcal{A}(\overline{G_2}) & 3I_{n_2}+2\mathcal{A}(G_2)+\mathcal{A}(\overline{G_2}) \\ 3I_{n_2}+2\mathcal{A}(G_2)+\mathcal{A}(\overline{G_2}) & 3\mathcal{A}(G_2)+2\mathcal{A}(\overline{G_2}) \end{bmatrix}.$$

From Lemma 2.1, $3\mathcal{A}(G_2)+2\mathcal{A}(\overline{G_2})+3I_{n_2}+2\mathcal{A}(G_2)+\mathcal{A}(\overline{G_2})$ and $3\mathcal{A}(G_2)+2\mathcal{A}(\overline{G_2})-3I_{n_2}-2\mathcal{A}(G_2)-\mathcal{A}(\overline{G_2})$ have part of the spectrum as $V(G_2)$'s make partite sets, which give

$$2\lambda_i(G_2) \ \text{ for } i=2,3,\ldots,n_2, \tag{2.3}$$

obtained from $3\mathcal{A}(G_2)+2\mathcal{A}(\overline{G_2})+3I_{n_2}+2\mathcal{A}(G_2)+\mathcal{A}(\overline{G_2})$.

$$-4 \ \text{ for } i=2,3,\ldots,n_2, \tag{2.4}$$

obtained from $3\mathcal{A}(G_2)+2\mathcal{A}(\overline{G_2})-3I_{n_2}-2\mathcal{A}(G_2)-\mathcal{A}(\overline{G_2})$.
From equations (2.1), (2.2), (2.3) and (2.4) result follows. □

*Remark* 2.1. Concept of dumbbell graph defined by S. Kaliyaperumal et al. [1] in 2022, is the particular case of Indu-Bala product [4]. Thus, $\mathcal{CD}$-spectrum of dumbbell graph consists of the following:

i. $2\,cos\left(\frac{2\pi k}{n}\right)-2$ appearing twice for $k=1,2,3,\dots,n-1$

ii. 0 appearing $(m-1)$ times

iii. $-4$ appearing $(m-1)$ times

iv. $\frac{n}{2}+\frac{m}{2}-2\pm\frac{1}{2}\sqrt{n^2+m^2+2mn-8m+8n+16}$ and $\frac{3n}{2}+\frac{3m}{2}\pm\frac{1}{2}\sqrt{9n^2+9m^2+82mn}.$

*Proof.* Proof follows from Theorem 2.2 and Lemma 2.2, where $G_1=C_n$ and $G_2=\overline{K_m}$. □

## 3. Reciprocal complementary distance spectrum for Indu-Bala product of graphs

**Theorem 3.1.** *$\mathcal{RCD}$-spectrum of Indu-Bala product $G_1\blacktriangledown G_2$ of two regular graphs $G_1$ and $G_2$ with orders $n_1,n_2$ and regularity $r_1,r_2$ respectively, consists of the following:*

i. $-\frac{\lambda_i(G_1)}{6}-\frac{1}{2}$ *appearing twice for* $i=2,3,\dots,n_1$

ii. $-\frac{2\lambda_i(G_2)}{3}-\frac{7}{6}$ *for* $i=2,3,\dots,n_2$

iii. $\frac{\lambda_i(G_2)}{3}+\frac{1}{6}$ *for* $i=2,3,\dots,n_2$

iv. $\frac{3n_1}{4}+\frac{3n_2}{4}-\frac{r_1}{12}-\frac{r_2}{3}-\frac{5}{6}\pm$
$\frac{1}{12}\sqrt{81n_1^2-2(31n_1+36)n_2+81n_2^2-2(9n_1-9n_2+4)r_1+r_1^2+8(9n_1-9n_2-r_1+4)r_2+16r_2^2+72n_1+16}$
*and*
$-\frac{n_1}{4}-\frac{n_2}{4}-\frac{r_1}{12}+\frac{r_2}{6}-\frac{1}{6}\pm$
$\frac{1}{12}\sqrt{9n_1^2-2(7n_1+12)n_2+9n_2^2+2(3n_1-3n_2+4)r_1+r_1^2+4(3n_1-3n_2+r_1+4)r_2+4r_2^2+24n_1+16}.$

*Proof.* With a proper labeling of the vertices forming the product structure, general $\mathcal{RCD}$ matrix is:

$$\mathcal{RCD}(G_1\blacktriangledown G_2)=\begin{bmatrix} \frac{1}{3}A(G_1)+\frac{1}{2}A(\overline{G_1}) & J_{n_1\times n_1} & \frac{1}{3}J_{n_1\times n_2} & \frac{1}{2}J_{n_1\times n_2} \\ J_{n_1\times n_1} & \frac{1}{3}A(G_1)+\frac{1}{2}A(\overline{G_1}) & \frac{1}{2}J_{n_1\times n_2} & \frac{1}{3}J_{n_1\times n_2} \\ \frac{1}{3}J_{n_2\times n_1} & \frac{1}{2}J_{n_2\times n_1} & \frac{1}{3}A(G_2)+\frac{1}{2}A(\overline{G_2}) & \frac{1}{3}I_{n_2}+\frac{1}{2}A(G_2)+A(\overline{G_2}) \\ \frac{1}{2}J_{n_2\times n_1} & \frac{1}{3}J_{n_2\times n_1} & \frac{1}{3}I_{n_2}+\frac{1}{2}A(G_2)+A(\overline{G_2}) & \frac{1}{3}A(G_2)+\frac{1}{2}A(\overline{G_2}) \end{bmatrix}.$$

As $G_1,G_2$ are regular graphs,

(a) vertex $v_i$ in $V(G_1)$ has exactly $r_1+n_2$ neighbors in $V(G_2)$, regardless of the choice of $v_i$ (same follows for other copy of $G_1$)
(b) vertex $v_i$ in $V(G_2)$ has exactly $r_2+n_1$ neighbors in $V(G_1)$, regardless of the choice of $v_i$
(c) vertex $v_i$ in $V(G_2)$ has exactly $r_2+1$ neighbors in $V(G_2)$ (with the other copy of $G_2$), regardless of the choice of $v_i$ (same follows for other copy of $G_2$).

Thus, we can have the equitable partition $\Pi : V(G_1)\bigcup V(G_1)\bigcup V(G_2)\bigcup V(G_2)$. Due to this equitable partition, $\mathcal{RCD}(G_1 \blacktriangledown G_2)$ is reduced to a smaller matrix (order $4\times 4$), as each entry of the $4\times 4$ block matrix has constant row sums.
Therefore,

$$Q_{\mathcal{RCD}}(G_1 \blacktriangledown G_2) = \begin{bmatrix} \frac{n_1}{2}-\frac{r_1}{6}-\frac{1}{2} & n_1 & \frac{n_2}{3} & \frac{n_2}{2} \\ n_1 & \frac{n_1}{2}-\frac{r_1}{6}-\frac{1}{2} & \frac{n_2}{2} & \frac{n_2}{3} \\ \frac{n_1}{3} & \frac{n_1}{2} & \frac{n_2}{2}-\frac{r_2}{6}-\frac{1}{2} & n_2-\frac{r_2}{2}-\frac{2}{3} \\ \frac{n_1}{2} & \frac{n_1}{3} & n_2-\frac{r_2}{2}-\frac{2}{3} & \frac{n_2}{2}-\frac{r_2}{6}-\frac{1}{2} \end{bmatrix}.$$

Since, $Q_{\mathcal{RCD}}$ is a reduced matrix, eigenvalues of $Q_{\mathcal{RCD}}$ are those of $\mathcal{RCD}(G_1 \blacktriangledown G_2)$.
Expanding determinant of $Q_{\mathcal{RCD}}$, we have:

$$\begin{aligned} \sigma_{Q_{\mathcal{RCD}}} = \Bigg\{ & \frac{3n_1}{4}+\frac{3n_2}{4}-\frac{r_1}{12}-\frac{r_2}{3}-\frac{5}{6}\pm \\ & \frac{1}{12}\sqrt{81n_1^2+81n_2^2+r_1^2+16r_2^2-2(31n_1+36)n_2-2(9n_1-9n_2+4)r_1+8(9n_1-9n_2-r_1+4)r_2+72n_1+16}, \\ & -\frac{n_1}{4}-\frac{n_2}{4}-\frac{r_1}{12}+\frac{r_2}{6}-\frac{1}{6}\pm \\ & \frac{1}{12}\sqrt{9n_1^2+9n_2^2+r_1^2+4r_2^2-2(7n_1+12)n_2+2(3n_1-3n_2+4)r_1+4(3n_1-3n_2+r_1+4)r_2+24n_1+16}\Bigg\}. \end{aligned} \tag{3.1}$$

For the remaining eigenvalues, we have to examine $G_1$ and $G_2$ structures.
Due to $G_1$, we have
$-\frac{\lambda_i(G_1)}{6}-\frac{1}{2}$ appearing twice (as $G_1$ is appearing twice in $G_1 \blacktriangledown G_2$) for $i=2,3,\ldots,n_1$, which is obtained from $\frac{1}{3}\mathcal{A}(G_1)+\frac{1}{2}\mathcal{A}(\overline{G_1})$, as graphs $G_1$ and $G_2$ are connected through the join structure.
Due to $G_2$, we have the matrix

$$\begin{bmatrix} \frac{1}{3}\mathcal{A}(G_2)+\frac{1}{2}\mathcal{A}(\overline{G_2}) & \frac{1}{3}I_{n_2}+\frac{1}{2}\mathcal{A}(G_2)+\mathcal{A}(\overline{G_2}) \\ \frac{1}{3}I_{n_2}+\frac{1}{2}\mathcal{A}(G_2)+\mathcal{A}(\overline{G_2}) & \frac{1}{3}\mathcal{A}(G_2)+\frac{1}{2}\mathcal{A}(\overline{G_2}) \end{bmatrix}.$$

From Lemma 2.1,

$$\frac{1}{3}\mathcal{A}(G_2)+\frac{1}{2}\mathcal{A}(\overline{G_2})+\frac{1}{3}I_{n_2}+\frac{1}{2}\mathcal{A}(G_2)+\mathcal{A}(\overline{G_2})$$

and

$$\frac{1}{3}\mathcal{A}(G_2)+\frac{1}{2}\mathcal{A}(\overline{G_2})-\frac{1}{3}I_{n_2}-\frac{1}{2}\mathcal{A}(G_2)-\mathcal{A}(\overline{G_2})$$

have part of the spectrum as $V(G_2)$'s make partite sets.
Which give

$$\frac{-2\lambda_i(G_2)}{3}-\frac{7}{6}\ \text{ for } i=2,3,\ldots,n_2, \tag{3.2}$$

obtained from $\frac{1}{3}\mathcal{A}(G_2)+\frac{1}{2}\mathcal{A}(\overline{G_2})+\frac{1}{3}I_{n_2}+\frac{1}{2}\mathcal{A}(G_2)+\mathcal{A}(\overline{G_2})$

$$\frac{\lambda_i(G_2)}{3}+\frac{1}{6}\ \text{ for } i=2,3,\ldots,n_2, \tag{3.3}$$

obtained from and $\frac{1}{3}\mathcal{A}(G_2)+\frac{1}{2}\mathcal{A}(\overline{G_2})-\frac{1}{3}I_{n_2}-\frac{1}{2}\mathcal{A}(G_2)-\mathcal{A}(\overline{G_2})$.
From equations (3.1), (3.2) and (3.3) result follows. □

*Remark* 3.1. As dumbbell graph is a particular case of Indu-Bala product [4]. $\mathcal{RCD}$-spectrum of dumbbell graph consists of the following:

i. $-\frac{1}{3}\,cos\left(\frac{2\pi k}{n}\right)-\frac{1}{2}$ appearing twice for $k=1,2,3,\ldots,n-1$

ii. $-\frac{7}{6}$ appearing $(m-1)$ times

iii. $\frac{1}{6}$ appearing $(m-1)$ times

iv. $\frac{3n}{4}+\frac{3m}{4}-1\pm\frac{1}{12}\sqrt{81n^2+81m^2-62mn-36m+36n+4}$ and
$-\frac{n}{4}-\frac{m}{4}-\frac{1}{3}\pm\frac{1}{12}\sqrt{9n^2+9m^2-14mn-36m+36n+36}$.

*Proof.* Proof follows from Theorem 3.1 and Lemma 2.2, where $G_1=C_n$ and $G_2=\overline{K_m}$. □

## REFERENCES


[1] F. Buckley and F. Harary, *On the Euclidean dimension of a wheel*, Graphs Combin., 4 (1988), 23–30.
[2] P. J. Davis, *Circulant Matrices*, Wiley, New York, 1979.
[3] F. Harary, *Graph Theory*, Narosa Publishing House, New Delhi, 1998.
[4] G. Indulal and R. Balakrishnan, *Distance spectrum of Indu–Bala product of graphs*, AKCE Int. J. Graphs Comb., 13 (3) (2016) 230–234.

[5] O. Ivanciuc, T. Ivanciuc and A. T. Balban, *The complementary distance matrix, a new molecular graph metric*, ACH-Models Chem., 137 (2000), 57–82.
[6] D. Janežič, S. Nikolić, A. Miličević and N. Trinajstić, *Graph theoretical matrices in chemistry*, University of Kragujevac, Serbia, 2007.
[7] S. Kaliyaperumal and K. Desikan, *D–Spectrum of Hyper–Dumbbell graph*, preprint(2022). https://doi.org/10.21203/rs.3.rs-1763398/v1.
[8] S. Patil and M. Mathapati, *Spectra of Indu-Bala product of graphs and some new pairs of cospectral graphs*, Discrete Math. Algorithms Appl., 11 (5) (2019), # 1950056.
[9] H. S. Ramane and G. A. Gudodagi, *Reciprocal complementary distance equienergetic graphs*, Asian European J. Math., 9 (2016), 1650084.
[10] H. S. Ramane and M. M. Gundloor, *On the complementary distance energy of join of certain graphs*, Discrete Math. Lett., 2 (2019), 57–64.
[11] H. S. Ramane and D. D. Patil, *Complementary distance polynomial and energy of join of two graphs*, Ann. Math. Comput. Sci., 12 (2023), 46–51.
[12] H. S. Ramane and D. D. Patil, *Reciprocal complementary distance polynomial and energy of join of two graphs*, J. Adv. Math. Stud., 16 (3) (2023), 335–341.
[13] A. J. Schwenk, *Computing the characteristic polynomial of a graph*, Lecture Notes in Math., 406 (1974), 153–172.

(Received: November 06, 2024)
(Revised: July 18, 2025)

Harishchandra S. Ramane
Department of Mathematics
Karnatak University
Dharwad-580003
India
e-mail: *hsramane@yahoo.com*
*and*
Daneshwari D. Patil
Department of Mathematics
Sha. Sogamal Peeraju Oswal Government First Grade College
Muddebihal-586212
India
e-mail: *daneshwarip@gmail.com*