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CATEGORY OF DIRECTED COMPLETE L-SLICES

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ABSTRACT. A new notion of action of a locale L on P is developed in the context of point free topology, given a locale L and a directed complete join semilattice P with bottom element 0_P. This work establishes the existence of a contravariant functor from the category **TopDWMod** of topological directed weak L-modules and continuous weak L-module homomorphisms to the category **DL-slice** of directed complete L-slices and directed complete L-slice homomorphisms.

1. INTRODUCTION

From 1914 onwards, it was recognised that a set with a lattice of open subsets is a topological space. Marshall Stone did the initial research on the relationship between lattices and topology. After his work on topological representation of Boolean algebras and distributive lattices [13–15], the relation between topology and lattice theory began to be explored. Johnstone [4] defined pointless topology as the complete lattice that satisfies an infinite distributive law. Following them, the majority of topological concepts were examined inside a localic context. Theory of frames is the opposite of the idea of theory of locales. Studies in localic backgrounds are topological, whereas those utilising frame theory are more algebraic.

Given a complete semiring $(S,+,.,0_S,1_S)$ where finite products $\cdot$ distribute over infinite sums $+$ and a dcpo monoid $(M,*,0_M)$, a directed complete weak S-module is defined to be an action of S on $(M,*,0_M)$. We have defined a directed complete weak S-module homomorphism between two weak S-modules $(\delta,M),(\gamma,N)$ and it is proved that if $(N,*')$ is commutative, then the collection of all weak S-module homomorphisms from (δ,M) to (γ,N) is a weak S-module.

A contravariant functor is known to exist between the categories of idempotent topological monoids and continuous monoid homomorphisms and the category of join semilattices with bottom element and semilattice homomorphisms. It is a matter of interest to investigate to which algebraic structure, the category **DL-slice** of directed complete L-slices and directed complete L-slice homomorphisms can be associated. It has been demonstrated in this study that there exists a contravariant

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functor from the category **DL-slice** of a locale L to the category **TopDWMod** of directed complete weak topological L-modules

2. PRELIMINARIES

Definition 2.1. *[8] A frame (or a locale) is a complete lattice L satisfying the infinite distributivity law $a \wedge \bigvee B = \bigvee\{a \wedge b; b \in B\}$ for all $a \in L$ and $B \subseteq L$.*

Definition 2.2. *[8] A frame homomorphism is a map $h : L \to M$ that preserves all finite meets (including the top 1) and all joins (including the bottom 0).*

Frm indicates the category of frames. The category **Loc** of locales is the opposite of **Frm**.

Examples 2.1. *i. The lattice of open subsets of a topological space.*
ii. In logic, the collection of all propositions under the partial order $p_1 \leq p_2$ if and only if $p_1 \Rightarrow p_2$.

Definition 2.3. *[5] A subset I of a locale L is said to be an ideal if*
i) I is a sub-join-semilattice of of L; that is $0 \in I$ and $a \in I, b \in I$ implies $a \vee b \in I$;and
ii) I is a lower set; that is $a \in I$ and $b \leq a$ imply $b \in I$.

Definition 2.4. *[8] A subset F of a locale L is said to be a filter if*
i) F is a sub-meet-semi lattice of L; that is $1 \in F$ and $a \in F$, $b \in F$ imply $a \wedge b \in F$
ii) F is an upper set; that is $a \in F$ and $a \leq b$ imply $b \in F$.

Definition 2.5. *[8] A filter F of a locale L is proper if $F \neq L$, that is if $0 \notin F$. A proper filter F in a locale L is prime if $a_1 \vee a_2 \in F$ implies that $a_1 \in F$ or $a_2 \in F$.*

Definition 2.6. *[8] A proper filter F in a locale L is a completely prime filter if for any index set J, $\bigvee a_i \in F$ implies that there exists $i \in J$ such that $a_i \in F$.*

Definition 2.7. *A proper filter F in a locale L is a partially completely prime filter if for any index set J, $\bigvee_{i \in J} a_i \in F$ implies that there exists an arbitrary positive integer n and $a_1, a_2....a_n \in L$ such that $a_1 \vee a_2 \vee ... \vee a_n \in F$.*

Definition 2.8. *A partially ordered set is a directed-complete partial order (dcpo) if each of its directed subsets has a supremum.*

Definition 2.9. *A semiring is a triple $(S, +, \cdot)$, where S is a set and + and $\cdot$ are binary operations, such that + is commutative, both $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws hold for all $x, y, z \in S$.*
1. $x.(y+z) = x.y + x.z$.
2. $(x+y).z = (x.z) + (y.z)$.
If $(S, \cdot)$ is a monoid,then $(S, +, \cdot)$ is a semiring with 1.

Definition 2.10. *A complete semiring is a semiring for which the addition monoid is a complete monoid and the following infinitary distributive laws hold $\Sigma(a.a_i) = a.\Sigma a_i$ and $\Sigma(a_i.a) = (\Sigma a_i).a$.*

Definition 2.11. *A topological semiring is a semiring S together with a a topology under which the semiring operations are continuous.*

3. DIRECTED COMPLETE L-SLICES

Definition 3.1. *Let L be a locale and P be a directed complete partially ordered set (dcpo) with bottom element 0_P.We define the action of L on P as a function $\mu : L \times P \to P$ that satisfies the subsequent requirements.*

1. *If D is a directed subset of P, then $\{\mu(a,d) : d \in D\}$ is directed and $\mu(a, \bigvee D) = \bigvee_{d\in D} \mu(a,d)$ for all $a \in L$.*
2. *$\mu(a, 0_P) = 0_P$ for $a \in L$.*
3. *$\mu(a \sqcap b, x) = \mu(a, \mu(b,x)) = \mu(b, \mu(a,x))$ for $a, b \in L, x \in P$.*
4. *$\mu(1_L, x) = x$ and $\mu(0_L, x) = 0_P$ for $x \in P$.*
5. *If $\{\mu(a_\alpha, x); a_\alpha \in L\}$ is directed subset of P, then $\mu(\bigsqcup a_\alpha, x) = \bigvee_\alpha \mu(a_\alpha, x)$ for $x \in P$.*

(μ, P) is referred to as a directed complete L-slice.

Proposition 3.1. Let L be a locale, and let S be a set of order preserving maps $L \to L$ such that:

(i) The constant map $\mathbf{0} \in S$($\mathbf{0}$ takes everything to 0).
(ii) If $f_\alpha \in S$, then $\bigvee f_\alpha \in S$.
(iii) For all $a \in L$ and for all $f \in S$, the meet of the constant map $\mathbf{a}$ and f is in S (i.e. $f \wedge \mathbf{a} \in S$).

Then the map $\mu : L \times S \to S$ defined by $\mu(a, f)(x) = f(x) \sqcap a$ is a directed complete L-slice.

Proof. By the hypothesis, S is a complete join semilattice with bottom element $\mathbf{0}$, and hence S is a dcpo with a bottom element. Also the map μ is well defined.

1. $\mu(a, \bigvee f_\alpha)(x) = (\bigvee f_\alpha)(x) \sqcap a = (\bigvee f_\alpha(x)) \sqcap a = \bigvee (f_\alpha(x) \sqcap a) = \bigvee \mu(a, f_\alpha)(x) = (\bigvee \mu(a, f_\alpha))(x)$.
2. $\mu(a, \mathbf{0})(x) = \mathbf{0}(x) \sqcap a = 0 \sqcap a = 0 = \mathbf{0}(x)$.
3. $\mu(a \sqcap b, f)(x) = f(x) \sqcap (a \sqcap b) = a \sqcap (f(x) \sqcap b) = a \sqcap \mu(b, f)(x) = \mu(a, \mu(b, f))(x) = \mu(b, \mu(a, f))(x)$.
4. $\mu(1_L, f)(x) = f(x) \sqcap 1_L = f(x)$ and $\mu(0_L, f)(x) = f(x) \sqcap 0_L = 0 = \mathbf{0}(x)$.
5. $\mu(\bigsqcup a_\alpha, f)(x) = f(x) \sqcap (\bigsqcup a_\alpha) = \bigsqcup (f(x) \sqcap a_\alpha) = \bigsqcup (f \sqcap a_\alpha)(x) = (\bigvee \mu(a_\alpha, f))(x)$.

□

Examples 3.1. *1. Let L be a locale and I be any ideal of L with the property that it is closed under arbitrary joins. Consider each $x \in I$ as the constant map $\boldsymbol{x} : L \to L$. Consequently by 3.1 proposition, (μ, I) is a directed complete L-slice. (μ, L) is specifically a directed complete L-slice.*
2. Let L be a chain with Top and Bottom elements and P be any dcpo with a bottom

element. Define $\mu : L \times P \to P$ by $\mu(a,p) = p$ for $a \neq 0$ and $\mu(0_L, p) = 0_P$. Consequently μ is an action of L on P and (μ,P) is a directed complete L-slice.

Proposition 3.2. Let (μ,P) be a directed complete L-slice of a locale L. For each $x \in P$, let $F_x = \{a \in L; \mu(a,x) = x\}$. Then F_x is a filter in L.

Proof. By 3.1 Definition(4), $1 \in F_x$. Hence F_x is non empty. Let $a,b \in F_x$. Then $\mu(a,x) = x, \mu(b,x) = x$. $\mu(a \sqcap b,x) = \mu(a,\mu(b,x)) = \mu(a,x) = x$. Hence $a \sqcap b \in F_x$. Let $a \in F_x$ and $c \in L$ such that $a \leq c$. $\mu(a,x) = \mu(a \sqcap c,x) = \mu(c \sqcap a,x) = \mu(c,\mu(a,x)) = \mu(c,x)$. Hence $c \in F_x$. Thus F_x is a filter in L. □

Definition 3.2. *An element $x \in P$ is said to be a compact element of the directed complete L-slice (μ,P) for a locale L, if for any collection $\{l_\alpha\}$ of L whenever $\mu(\bigsqcup l_\alpha, x) = x$, there exists a finite sub collection $\{l_1, l_2, ...l_n\}$ of $\{l_\alpha\}$ such that $\mu(l_1,x) \vee \mu(l_2,x) \vee ... \vee \mu(l_n,x) = x$. A directed complete L-slice (μ,P) is compact if each element $x \in P$ is compact.*

Example 3.2. *Let (μ,P) be any directed complete L-slice. Then 0_P is a compact element.*

Proposition 3.3. Let (μ,P) be any directed complete L-slice of a locale L and $x \in P$. Then x is a compact element of the directed complete L-slice (μ,P) if and only if F_x is a partially completely prime filter in L.

Proof. Suppose x is a compact element of the directed complete L-slice (μ,P). By proposition 3.2, F_x is a filter. Let $\bigsqcup l_\alpha \in F_x$. Then we have $\mu(\bigsqcup l_\alpha, x) = x$. x being a compact element, a finite collection $\{l_1, l_2, ...l_n\}$ of $\{l_\alpha\}$ exists with the property that $\mu(l_1,x) \vee \mu(l_2,x) \vee ... \vee \mu(l_n,x) = x$. That is $\mu(l_1 \sqcup l_2 \sqcup ... \sqcup l_n, x) = x$. So $l_1 \sqcup l_2 \sqcup ... \sqcup l_n \in F_x$ and hence F_x is a partially completely prime filter.

Conversely assume F_x is partially completely prime. Let $\{l_\alpha\} \in L$ be such that $\mu(\bigsqcup l_\alpha, x) = x$. Then we have $\bigsqcup l_\alpha \in F_x$. The filter F_x being partially completely prime, there is a finite collection $\{l_1, l_2, ...l_n\}$ of $\{l_\alpha\}$ such that $l_1 \sqcup l_2 \sqcup ... \sqcup l_n \in F_x$. Hence $\mu(l_1,x) \vee \vee \mu(l_2,x) \vee ... \vee \mu(l_n,x) = x$. So x is a compact element of the directed complete L-slice (μ,P). □

Definition 3.3. *Let (μ,P) be a directed complete L-slice. A directed complete subset P', containing 0_P, of P is said to be a directed complete L-subslice of (μ,P) if P' is closed under action by elements of L.*

Example 3.3. *Let L be a locale and the collection of all order preserving maps $L \to L$ is indicated by the notation $O(L)$. Then $(\mu, O(L))$ is a directed complete L-slice in which $\mu : L \times O(L) \to O(L)$ is defined by $\mu(a, f) = f_a$ where $f_a : L \to L$ is defined by $f_a(x) = f(x) \wedge a$. Let $K = \{f \in O(L) : f(x) \leq x, \forall x \in L\}$. Then (μ,K) is a directed complete L-subslice of the directed complete L-slice $(\mu, O(L))$.*

Definition 3.4. *A directed complete L-subslice (μ,I) of a directed complete L-slice (μ,P) is said to be ideal of (μ,P) if I is a lower set.*

Proposition 3.4. Let (μ,P) be a directed complete L-slice of a locale L and let $\{(\mu,I_\alpha) : \alpha \in J\}$ be a collection of ideals of (μ,P). Then, $(\mu,\bigcap\limits_{\alpha\in J} I_\alpha)$ is an ideal in (μ,P).

Proof. Let D be a directed subset of $(\mu,\bigcap\limits_{\alpha\in J} I_\alpha)$. Since $\bigcap\limits_{\alpha\in J} I_\alpha \subseteq I_\alpha$, D is a directed subset of $I_\alpha, \forall\alpha \in J$. Since each (μ,I_α) are ideals in (μ,P), $\bigvee D \in (\mu,I_\alpha)$, for all $\alpha \in J$. Hence $\bigvee D \in (\mu,\bigcap\limits_{\alpha\in J} I_\alpha)$ and $(\mu,\bigcap\limits_{\alpha\in J} I_\alpha)$ is a directed complete subset of (μ,P).
Let $x \in (\mu,\bigcap\limits_{\alpha\in J} I_\alpha), a \in L$. Then $x \in (\mu,I_\alpha), \forall\alpha \in J$. Since all (μ,I_α) are ideals in (μ,P), $\mu(a,x) \in (\mu,I_\alpha), \forall\alpha \in J$. Hence $\mu(a,x) \in (\mu,\bigcap\limits_{\alpha\in J} I_\alpha)$.
$(\mu,\bigcap\limits_{\alpha\in J} I_\alpha)$ is a lower set since each (μ,I_α) is a lower set. Hence $(\mu,\bigcap\limits_{\alpha\in J} I_\alpha)$ is an ideal in (μ,P). □

Definition 3.5. *A proper ideal I of a directed complete L-slice (μ,P) is a prime ideal if $\mu(a \sqcap b,x) \in I$ for $a,b \in L, x \in P$ implies that $\mu(a,x) \in I$ or $\mu(b,x) \in I$.*

Definition 3.6. *Let (μ_1,P) and (μ_2,Q) be directed complete L-slices of a locale L. A map $f : (\mu_1,P) \to (\mu_2,Q)$ is said to be directed complete L-slice homomorphism if :*

1. *The collection $\{f(d) : d \in D\}$ is directed whenever D is directed and $f(\bigvee D) = \bigvee\limits_{d\in D} f(d)$.*
2. *$f(\mu_1(a,x)) = \mu_2(a, f(x))$, $a \in L$, $x \in P$.*

Examples 3.4. *i. Let (μ,P) be a directed complete L-slice and (μ,P') be a directed complete L-subslice of (μ,P). Then the inclusion map $i : (\mu,P') \to (\mu,P)$ is a directed complete L-slice homomorphism.*
ii.Given the locale L, let $I =\downarrow(a), J =\downarrow(b)$ be its principal ideals. Hence the directed complete L-slices $(\sqcap,I)$, $(\sqcap,J)$ are formed. In that case the directed complete L-slice homomorphism $f : (\sqcap,I) \to (\sqcap,J)$ is defined by $f(x) = x \sqcap b$ for every $x \in I$.

Proposition 3.5. If $f : (\mu_1,P) \to (\mu_2,Q)$ is a directed complete L-slice homomorphism, then $f(0_P) = 0_Q$.

Proof. Let $(\mu_1,P),(\mu_2,Q)$ be directed complete L-slices of a locale L and $f : (\mu_1,P) \to (\mu_2,Q)$ be a directed complete L-slice homomorphism. Then, we have that $f(0_P) = f(\mu_1(0,x)) = \mu_2(0, f(x)) = 0_Q$. □

Proposition 3.6. The composition of two directed complete L-slice homomorphisms of a locale L is a directed complete L-slice homomorphism.

Proof. Let $(\mu_1,P_1),(\mu_2,P_2),(\mu_3,P_3)$ be directed complete L-slices of a locale L. Let $f:(\mu_1,P_1)\to(\mu_2,P_2)$ and $g:(\mu_2,P_2)\to(\mu_3,P_3)$ be directed complete L-slice homomorphisms.
$(g\circ f)(\bigvee D)=g(f(\bigvee D))=g(\bigvee_{d\in D}f(d))=\bigvee_{d\in D}g(f(d))=\bigvee_{d\in D}(g\circ f)(d)$.
$(g\circ f)(\mu_1(a,x))=g(f(\mu_1(a,x)))=g(\mu_2(a,f(x)))=\mu_3(a,g(f(x)))=\mu_3(a,(g\circ f)(x))$ for every $a\in L$ and $x\in P_1$. $\square$

Proposition 3.7. Let (μ_1,P) and (μ_2,Q) be directed complete L-slices of a locale L and let $f:(\mu_1,P)\to(\mu_2,Q)$ be a directed complete L-slice homomorphism. Let (μ_2,I) be an ideal of (μ_2,Q), then $(\mu_1,f^{-1}(I))$ is an ideal of (μ_1,P). Specifically, $(\mu_1,f^{-1}(I))$ is a prime ideal of (μ_1,P) if (μ_2,I) is a prime ideal.

Proof. Let D be a directed subset of $f^{-1}(I)$. For $d\in D\subseteq f^{-1}(I), f(d)\in I$. Thus $\{f(d):d\in D\}$ is a directed subset of I and $f(\bigvee D)=\bigvee_{d\in D}f(d)\in I$. Hence $\bigvee D\in f^{-1}(I)$. Since $0_Q\in I, 0_P\in f^{-1}(I)$. Hence $f^{-1}(I)$ is a directed complete subset of (μ_1,P).
Also for each $x\in f^{-1}(I)$ and $a\in L$, $f(\mu_1(a,x))=\mu_2(a,f(x))\in I$. Hence $(\mu,f^{-1}(I))$ is a directed complete L-subslice of (μ_1,P).
Let $x\in f^{-1}(I)$ and $y\in P$ be such that $y\leq x$. Since f preserves a directed join, f preserves order. Hence $f(y)\leq f(x)$. Since $f(x)\in I$ and I is an ideal of (μ_1,P), $f(y)\in I$. Hence $y\in f^{-1}(I)$. Thus $f^{-1}(I)$ is an ideal of (μ_1,P).
Now let I be prime ideal of (μ_2,Q). Suppose $\mu_1(a\sqcap b,x)\in f^{-1}(I)$, then $f(\mu_1(a\sqcap b,x))=\mu_2(a\sqcap b,f(x))\in I$. Since I is prime, either $f(\mu_1(a,x))=\mu_2(a,f(x))\in I$ or $f(\mu_1(b,x))=\mu_2(b,f(x))\in I$. So either $\mu_1(a,x)\in f^{-1}(I)$ or $\mu_2(b,x)\in f^{-1}(I)$. Hence $(\mu_1,f^{-1}(I))$ is a prime ideal of (μ_1,P). $\square$

Proposition 3.8. Let (μ_1,P) and (μ_2,Q) be directed complete L-slices of a locale L and let $f:(\mu_1,P)\to(\mu_2,Q)$ be a directed complete L-slice homomorphism. Then for any $z\in Q, f^{-1}(\downarrow z)=\downarrow f^{-1}(z)$.

Proof. Let $y\in f^{-1}(\downarrow z)$. Then $f(y)\in\downarrow z$. So $f(y)\leq z$ or $y\leq f^{-1}(z)$. Thus $y\in\downarrow f^{-1}(z)$. Hence $f^{-1}(\downarrow z)\subseteq\downarrow f^{-1}(z)$.
Now let $y\in\downarrow f^{-1}(z)$. Then $y\leq f^{-1}(z)$ or $f(y)\leq z$. Hence $f(y)\in\downarrow z$ and so $y\in f^{-1}(\downarrow z)$. Hence $f^{-1}(\downarrow z)\subseteq f^{-1}(\downarrow z)$. $\square$

Definition 3.7. *Let (μ_1,P) and (μ_2,Q) be directed complete L-slices of a locale L. A directed complete L-slice homomorphism $f:(\mu_1,P)\to(\mu_2,Q)$ is said to be a directed complete L-slice isomorphism if f is a bijection.*

Let (μ_1,P), (μ_2,Q) be directed complete L-slices of a locale L, where Q is a complete sup lattice. Let $DL-Hom(P,Q)$ denote the collection of all directed complete L-slice homomorphisms from (μ_1,P) to (μ_2,Q). The collection $DL-Hom(P,Q)$ is a poset under the partial order relation $f\leq g$ if and only if $f(x)\leq g(x)$ for all $x\in P$.

Lemma 3.5. *Let (μ_1, P), (μ_2, Q) be directed complete L-slices of a locale L, where Q is a complete sup lattice.*

i. The map $\boldsymbol{0} : (\mu_1, P) \to (\mu_2, Q)$ defined by $\boldsymbol{0}(x) = 0_Q$ for $x \in P$ is a directed complete L-slice homomorphism.
ii. If $\{f_\alpha; \alpha \in J\}$ for any indexing set J is a directed subset of $DL - Hom(P, Q)$, the map $\bigvee_{\alpha\in J} f_\alpha : (\mu_1, P) \to (\mu_2, Q)$ defined by $(\bigvee_{\alpha\in J} f_\alpha)(x) = \bigvee_{\alpha\in J}(f_\alpha(x))$ is a directed complete L-slice homomorphism.

Proof. Let D be any directed subset of (μ_1, P) and $a \in L, x \in P$.
i. The collection $\{\mathbf{0}(d); d \in D\}$ is directed and $\mathbf{0}(\bigvee D) = 0_Q = \bigvee_{d\in D} \mathbf{0}(d)$.
$\mathbf{0}(\mu_1(a,x)) = 0_Q = \mu_2(a, \mathbf{0}(x))$. Hence $\mathbf{0}$ is a directed complete L-slice homomorphism.
ii. Since Q is complete sup lattice, the mapping $\bigvee_{\alpha\in J} f_\alpha$ is well defined and $\{\bigvee_{\alpha\in J} f_\alpha(d); d \in D\}$ is directed.
$(\bigvee_{\alpha\in J} f_\alpha)(\bigvee D) = \bigvee_{\alpha\in J}(f_\alpha(\bigvee D)) = \bigvee_{\alpha\in J}\bigvee_{d\in D}(f_\alpha(d)) = \bigvee_{d\in D}\bigvee_{\alpha\in J}(f_\alpha(d)) = \bigvee_{d\in D}(\bigvee_{\alpha\in J} f_\alpha)(d)$.

$(\bigvee_{\alpha\in J} f_\alpha)(\mu_1(a,x)) = \bigvee_{\alpha\in J}(f_\alpha(\mu_1(a,x)) = \bigvee_{\alpha\in J}\mu_2(a, f_\alpha(x)) = \mu_2(a, \bigvee_{\alpha\in J}(f_\alpha(x))) =$
$\mu_2(a, (\bigvee_{\alpha\in J} f_\alpha)(x))$.
Hence the map $\bigvee_{\alpha\in J} f_\alpha : (\mu_1, P) \to (\mu_2, Q)$ defined by $(\bigvee_{\alpha\in J} f_\alpha)(x) = \bigvee_{\alpha\in J}(f_\alpha(x))$ is a directed complete L-slice homomorphism. □

Proposition 3.9. Let (μ_1, P) and (μ_2, Q) be directed complete L-slices of a locale L, where Q is a complete sup lattice. Then the collection $DL - Hom(P, Q)$ of all directed complete L-slice homomorphisms from (μ_1, P) to (μ_2, Q) is a directed complete L-slice.

Proof. Define $\delta : L \times DL - Hom(P, Q) \to DL - Hom(P, Q)$ as follows. For each $a \in L$, $f \in DL - Hom(P, Q)$, let $\delta(a, f) : (\mu_1, P) \to (\mu_2, Q)$ be defined by $\delta(a, f)(x) = \mu_2(a, f(x)), x \in P$.
Let D be a directed subset of (μ_1, P). Then $\{\delta(a, f)(d) : d \in D\}$ is directed and $\delta(a, f)(\bigvee D) = \mu_2(a, f(\bigvee D)) = \mu_2(a, \bigvee_{d\in D} f(d)) = \bigvee_{d\in D}\mu_2(a, f(d)) = \bigvee_{d\in D}\delta(a, f)(d)$.
$\delta(a, f)(\mu_1(b,x)) = \mu_2(a, f(\mu_1(b,x)) = \mu_2(a, \mu_2(b, f(x))) = \mu_2(b, \mu_2(a, f(x))) =$
$\mu_2(b, \delta(a, f)(x))$.
Hence the map $\delta(a, f) \in DL - Hom(P, Q)$. Also δ satisfies the following properties.

i. Let $\{f_\alpha : \alpha \in J\}$ be a directed subset of $DL-Hom(P,Q)$ for any indexing set J. $\delta(a, \bigvee_{\alpha\in J} f_\alpha)(x) = \mu_2(a,(\bigvee_{\alpha\in J} f_\alpha)(x)) = \mu_2(a, \bigvee_{\alpha\in J}(f_\alpha(x))) = \bigvee_{\alpha\in J}\mu_2(a, f_\alpha(x)) = \bigvee_{\alpha\in J}\delta(a, f_\alpha)(x)$.

ii. $\delta(a,\mathbf{0})(x) = \mu_2(a,\mathbf{0}(x)) = \mu_2(a,0_Q) = 0_Q = \mathbf{0}(x)$.

iii. $\delta(a\sqcap b, f)(x) = \mu_2(a\sqcap b, f(x)) = \mu_2(a,\mu_2(b, f(x))) = \mu_2(a,\delta(b, f)(x) = \delta(a,\delta(b, f))(x)$.
Similarly, $\delta(a\sqcap b, f)(x) = \delta(b,\delta(a, f))(x)$.

iv. $\delta(1_L, f)(x) = \mu_2(1_L, f(x)) = f(x)$ and $\delta(0_L, f)(x) = \mu_2(0_L, f(x)) = 0_Q$.

v. Let $\{\delta(a_\beta, f); \beta \in J\}$ be directed subset of DL-Hom(P, Q).
$\delta(\bigsqcup a_\beta, f)(x) = \mu_2(\bigsqcup a_\beta, f(x)) = \bigvee_{\beta\in J}\mu_2(a_\beta, f(x)) = \bigvee_{\beta\in J}\delta(a_\beta, f)(x)$.

Thus δ is an action of the locale L on $DL-Hom(P,Q)$ and $(\delta, DL-Hom(P,Q))$ is a directed complete L-slice. ☐

4. IDEALS $I_a, a \in L$ OF DIRECTED COMPLETE L-SLICE

Definition 4.1. *If (μ,I) is an ideal of a directed complete L-slice (μ,P) and $a \in L$, define $I_a = \{x \in (\mu,P) : \mu(a,x) \in I\}$, .*

Proposition 4.1. For any indexing set J, let (μ,P) be a directed complete L-slice for a locale L and (μ,I) be any ideal of (μ,P). For each $a \in L, (\mu,I_a)$ is an ideal in (μ,P).

Proof. Let $\{x_\alpha : \alpha \in J\}$ be a directed subset of (μ,I_a). Then $\{\mu(a,x_\alpha) : \alpha \in J\}$ is a directed subset of (μ,I). Since (μ,I) is an ideal in (μ,P), then $\bigvee_{\alpha\in J}\mu(a,x_\alpha) = \mu(a, \bigvee_{\alpha\in J} x_\alpha) \in I$. Hence $\bigvee_{\alpha\in J} x_\alpha \in I_a$. So, (μ,I_a) is a directed complete subset of (μ,P).
Let $x \in I_a$ and $b \in L$. Since $x \in I_a$, then $\mu(a,x) \in I$. Since (μ,I) is an ideal in (μ,P), then $\mu(b,\mu(a,x)) = \mu(a,\mu(b,x)) \in I$. Hence $\mu(b,x) \in I_a$. Thus (μ,I_a) is a directed complete L-subslice of (μ,P).
Let $x \in I_a$ and $y \leq x \in (\mu,P)$. Then $\mu(a,y) \leq \mu(a,x)$ and hence $\mu(a,y) \in I$. So, $y \in (\mu,I_a)$ and hence (μ,I_a) is an ideal in (μ,P). ☐

Proposition 4.2. Let (μ,I) be a prime ideal of a directed complete L-slice (μ,P). Then for each $a \in L, (\mu,I_a)$ is a prime ideal in (μ,P).

Proof. By 4.1, for each $a \in L, (\mu,I_a)$ is an ideal in (μ,P). Let $\mu(b\sqcap c,x) \in I_a$. Then $\mu(a,\mu(b\sqcap c,x)) = \mu(a\sqcap b\sqcap c,x) \in I$. Since the ideal (μ,I) is prime, either $\mu(a\sqcap b,x) = \mu(a,\mu(b,x)) \in I$ or $\mu(c,x) \in I$. Hence either $\mu(b,x) \in I_a$ or $\mu(c,x) \in I_a$. Hence for each $a \in L, (\mu,I_a)$ is a prime ideal in (μ,P). ☐

Proposition 4.3. Let (μ,P) be a directed complete L-slice for a locale L and (μ,I) be any ideal of (μ,P).

i. If $a,b\in L$ with $a\sqsubseteq b$, then $(\mu,I_b)\subseteq(\mu,I_a)$.
i. $(\mu,I)\subseteq(\mu,I_a)$ for every $a\in L$
ii. $(\mu,I_1)=(\mu,I)$.
iv. $(\mu,I_0)=(\mu,P)$.

Proof. Let (μ,P) be a directed complete L-slice for a locale L and (μ,I) be any ideal of (μ,P).

i. Let $a\sqsubseteq b$. $x\in(\mu,I_b)\Rightarrow\mu(b,x)\in(\mu,I)$. Since (μ,I) is a lower set and $a\sqsubseteq b$, $\mu(a,x)\in(\mu,I)$. So $x\in(\mu,I_a)$ and hence $(\mu,I_b)\subseteq(\mu,I_a)$.
ii. If $x\in(\mu,I),\mu(a,x)\in(\mu,I)$ for all $a\in L$. Hence $(\mu,I)\subseteq(\mu,I_a)$ for every $a\in L$.
iii. $I_1=\{x\in(\mu,P):\mu(1,x)=x\in(\mu,I)\}=(\mu,I)$.
iv. $(\mu,I_0)=\{x\in(\mu,P):\mu(0,x)\in(\mu,I)\}=(\mu,P)$.

□

Proposition 4.4. Let (μ,P) be a directed complete L-slice for a locale L and (μ,I) be any ideal of (μ,P).

i. For any $a,b\in L,(\mu,I_a)\cap(\mu,I_b)=(\mu,I_{a\sqcup b})$.
ii. For any $a,b\in L,(\mu,I_a)\cup(\mu,I_b)\subseteq(\mu,I_{a\sqcap b})$.
iii. For any $a,b\in L,(\mu,I_{a\sqcap b})=(\mu,(I_a)_b)=(\mu,(I_b)_a)$.

Proof. Let (μ,I) be any ideal of a directed complete L-slice (μ,P) and $a,b\in L$.

i. $$\begin{aligned}x\in(\mu,I_a)\cap(\mu,I_b)&\Leftrightarrow\mu(a,x)\in I\text{ and }\mu(b,x)\in I\\&\Leftrightarrow\mu(a\sqcup b,x)=\mu(a,x)\vee\mu(b,x)\in(\mu,I)\\&\Leftrightarrow x\in(\mu,I_{a\sqcup b}).\end{aligned}$$

Hence $(\mu,I_a)\cap(\mu,I_b)=(\mu,I_{a\sqcup b})$.

ii. $x\in(\mu,I_a)\cup(\mu,I_b)$ implies $\mu(a,x)\in(\mu,I)$ or $\mu(b,x)\in(\mu,I)$. So $\mu(a\sqcap b,x)\in(\mu,I)$. Hence $(\mu,I_a)\cup(\mu,I_b)\subseteq(\mu,I_{a\sqcap b})$.

iii. $$\begin{aligned}x\in(\mu,I_{a\sqcap b})&\Leftrightarrow\mu(a\sqcap b,x)\in I\\&\Leftrightarrow\mu(a,\mu(b,x))\in I\\&\Leftrightarrow\mu(b,x)\in I_a\\&\Leftrightarrow x\in(I_a)_b.\end{aligned}$$

Hence $(\mu,I_{a\sqcap b})=(\mu,(I_a)_b)=(\mu,(I_b)_a)$. □

Proposition 4.5. Let $(\mu,I),(\mu,J)$ be two ideals of a directed complete L-slice (μ,P) and let $a\in L$. Then $(\mu,(I\cap J)_a)=(\mu,I_a)\cap(\mu,J_a)$.

Proof. Let $(\mu,I),(\mu,J)$ be two ideals of (μ,P) and $a\in L$.

$$\begin{aligned}x\in(\mu,(I\cap J)_a)&\Leftrightarrow\mu(a,x)\in(\mu,I\cap J)\\&\Leftrightarrow\mu(a,x)\in(\mu,I)\text{ and }\mu(a,x)\in(\mu,J)\\&\Leftrightarrow x\in(\mu,I_a)\text{ and }x\in(\mu,J_a)\\&\Leftrightarrow x\in(\mu,I_a)\cap(\mu,J_a)\ .\end{aligned}$$

Hence $(\mu,(I\cap J)_a) = (\mu,I_a)\cap(\mu,J_a)$. □

5. Directed complete weak S-module

Definition 5.1. *Let $(S,+,.,0_S,1_S)$ be a complete semiring where finite . distribute over infinite $+$ and let $(M,*,0_M)$ be a dcpo-monoid. By an action of S on M, we mean a function $\sigma : S\times M \to M$ that satisfies the following requirements.*

1. $\sigma(r+s,x) = \sigma(r,x)*\sigma(s,x)$ *for all $r,s\in S, x\in M$,*
2. $\sigma(r,x*y) = \sigma(r,x)*\sigma(r,y)$,
3. $\sigma(r,0_M) = 0_M$,
4. $\sigma(r.s,x) = \sigma(r,\sigma(s,x))$,
5. $\sigma(0_R,x) = 0_M$ *and* $\sigma(1_S,x) = x$,
6. *If D is a directed subset of M, then $\{\sigma(r,d); d\in D\}$ is directed and* $\sigma(r,\bigvee D) = \bigvee_{d\in D}\sigma(r,d)$.

If σ is an action of S on M, we call (σ,M) a directed complete weak S-module.

If $(S,\cdot)$ is commutative, then $\sigma(r.s,x) = \sigma(r,\sigma(s,x)) = \sigma(s,\sigma(r,x))$.

Example 5.1. *Every directed complete L-slice is an example of a directed complete weak L-module.*

Definition 5.2. *Let (δ,M) be a directed complete weak S-module, a directed complete submonoid M' of M is said to be a directed complete weak S-submodule of (δ,M) if M' is closed under action by elements of S.*

Definition 5.3. *A directed complete weak S-module homomorphism between weak S-modules $(\delta_1,M),(\delta_2,N)$ is a map $g : (\delta_1,M)\to(\delta_2,N)$ such that*

1. $g(x*y) = g(x)*'g(y)$
2. $g(\delta_1(r,x)) = \delta_2(r,g(x))$ *for all $x,y\in M, r\in S$.*
3. *If D is a directed subset of (δ_1,M), then $\{g(d) : d\in D\}$ is a directed subset of (δ_2,N) and* $g(\bigvee D) = \bigvee_{d\in D} g(d)$.

Proposition 5.1. Composition of two directed complete weak S-module homomorphisms is a directed complete weak S-module homomorphism.

Proof. Let $g : (\delta,M)\to(\delta',M')$, $h : (\delta',M')\to(\delta'',M'')$ be two directed complete weak S-module homomorphisms.
$(h\circ g)(x*y) = h(g(x*y)) = h(g(x)*'g(y)) = h(g(x))*''h(g(y))$.
$(h\circ g)(\delta(r,x)) = h(g(\delta(r,x))) = h(\delta'(r,g(x))) = \delta''(r,h(g(x))) = \delta''(r,(h\circ g)(x))$.
Let D be a directed subset of (δ_1,M). Since $g : (\delta,M)\to(\delta',M'), h : (\delta',M')\to(\delta'',M'')$ are directed complete weak S-module homomorphisms, $\{g(d); d\in D\}$ and $\{h(g(d)) : d\in D\}$ are directed.

$(h\circ g)(\bigvee D) = h(g(\bigvee D)) = h(\bigvee_{d\in D} g(d)) = \bigvee_{d\in D} h(g(d))$. Hence $h\circ g$ is a weak S-module homomorphism. ☐

Proposition 5.2. Let $f : (\delta_1, M) \to (\delta_2, N)$ be a weak S-module homomorphism. Then, $kerf = \{x \in M : f(x) = 0_N\}$ is a weak S-submodule of (δ_1, M).

Proof. Let $f : (\delta_1, M) \to (\delta_2, N)$ be a weak S-module homomorphism. Let D be a directed subset of $kerf$. Then $\{f(d); d \in D\}$ is directed in (δ_2, N) and $f(\bigvee D) = \bigvee_{d\in D} f(d) = 0_N$. Hence $\bigvee D \in kerf$.
Let $x, y \in kerf, r \in S$. Then, $f(x*y) = f(x)*' f(y) = 0_N *' 0_N = 0_N$ and $f(\delta_1(r,x)) = \delta_2(r, f(x)) = \delta_2(r, 0_N) = 0_N$. So, $kerf$ is a directed complete weak S-submodule of (δ_1, M). ☐

Proposition 5.3. Let $f : (\delta, M) \to (\delta, M)$ be a weak S-module homomorphism. Then, $F = \{x \in (\delta, M) : f(x) = x\}$ is a directed complete weak S-submodule.

Proof. Let D be a directed subset of F. Then $\{f(d); d \in D\}$ is directed and $f(\bigvee D) = \bigvee_{d\in D} f(d) = \bigvee_{d\in D} d$. Hence $\bigvee D \in F$. Thus $(F, *)$ is a directed complete submonoid of $(M, *)$.
Let $r \in S, x, y \in F$. Then $f(x*y) = f(x)*f(y) = x*y$ and $f(\delta(r,x)) = \delta(r, f(x)) = \delta(r,x)$. Hence (δ, F) is a directed complete weak S-submodule of (δ, M). ☐

Definition 5.4. *A topological directed complete weak S-module is a* $(\delta, M, *, \leq, \tau)$*, where* τ *is a topology on directed complete weak S-module* $(\delta, M, *, \leq)$ *such that*
1. $* : M \times M \to M$ *is continuous*
2. $\delta_a : M \to M$ *defined by* $\delta_a(x) = \delta(a,x)$ *is continuous for every* $a \in S$.

Definition 5.5. *A morphism between topological directed complete weak S-modules* $(\delta_1, M, *, \leq, \tau_1)$, $(\delta_2, N, *', \leq, \tau_2)$ *is a directed complete weak S-module homomorphism* $h : (\delta_1, M, *, \leq) \to (\delta_2, N, *', \leq)$ *such that h is* $\tau_1 - \tau_2$ *continuous.*

6. TOPOLOGICAL DIRECTED WEAK L-MODULE ASSOCIATED WITH A DIRECTED COMPLETE L-SLICE

Let (μ, P) be a directed complete L-slice with bottom element $\mathbf{0}$. Let $Pt(P)$ denote the collection of all ideals in the directed complete L-slice (μ, P). Order $Pt(P)$ by $I \leq J$ if and only if $I \supseteq J$ and define the binary operation $*$ on $Pt(P)$ by $I * J = I \cap J$. Then $(Pt(P), *, P, \leq)$ is a commutative dcpo monoid with bottom element P. Define $\delta : L \times Pt(P) \to Pt(P)$ by $\delta(a, I) = I_a$ for every $a \in L, I \in Pt(P)$. In the next proposition, we will show that δ is an action of the complete semiring L on the dcpo monoid $(Pt(P), *, P, \leq)$.

Proposition 6.1. $(\delta, Pt(P), *, P, \leq)$ is a directed complete weak L-module.

Proof. $\delta : L \times Pt(P) \to Pt(P)$ is defined by $\delta(a,I) = I_a$. Let $a,b \in L$ and $I,J \in Pt(P)$.
1. $\delta(a+b,I) = \delta(a \sqcup b,I) = I_{a\sqcup b} = I_a \cap I_b = \delta(a,I) * \delta(b,I)$.
2. $\delta(a,I*J) = \delta(a,I\cap J) = (I\cap J)_a = I_a \cap J_a = \delta(a,I)*\delta(a,J)$.
3. $\delta(a,P) = P_a = P$.
4. $\delta(a.b,I) = \delta(a\sqcap b,I) = I_{a\sqcap b} = (I_a)_b = \delta(b,I_a) = \delta(b,\delta(a,I)) = \delta(a,\delta(b,I))$.
5. $\delta(0_L,I) = I_0 = P$, $\delta(1_L,I) = I_1 = I$.
6. Let $\{I_\alpha; \alpha \in \Lambda\}$ be a directed subset of $Pt(P)$ for some indexing set Λ. Since $Pt(P)$ is complete,$\{(I_\alpha)_a; \alpha \in \Lambda\}$ is directed and $\delta(a,\bigvee I_\alpha) = \delta(a,\bigcap I_\alpha) = (\bigcap I_\alpha)_a = \bigcap (I_\alpha)_a$.
Hence $(\delta, Pt(P), *, P, \leq)$ is a directed complete weak L-module. □

Proposition 6.2. For $x \in (\mu,P)$ define $\lambda_x = \{I \in Pt(P) : x \in I\}$.Then,

1. $\lambda_{\mathbf{0}} = Pt(P)$
2. $\lambda_x \cap \lambda_y = \lambda_{x\vee y}$ for every $x,y \in P$.

Proof. 1. $\lambda_{\mathbf{0}} = \{I \in Pt(P) : \mathbf{0} \in I\}$. Since the ideal of a directed complete L-slice is closed under taking lower elements, $\mathbf{0} \in I$, for every $I \in Pt(P)$. Hence $\lambda_{\mathbf{0}} = Pt(P)$.

$$\begin{aligned} I \in \lambda_x \cap \lambda_y &\Leftrightarrow x \in I \text{ and } y \in I \\ &\Leftrightarrow x \vee y \in I \\ &\Leftrightarrow I \in \lambda_{x\vee y}. \end{aligned}$$

Hence $\lambda_x \cap \lambda_y = \lambda_{x\vee y}$. □

By the above proposition $B = \{\lambda_x : x \in (\mu,P)\}$ is closed under finite intersection and hence B is a base for a unique topology τ on Pt(P).

Proposition 6.3. $(\delta, Pt(P), *, P, \leq, \tau)$ is a topological directed weak L-module.

Proof. We have $(\delta, Pt(P), *, P, \leq)$ is a directed weak L-module. Let $f : Pt(P) \times Pt(P) \to Pt(P)$ be defined by $f(I,J) = I*J = I\cap J$ for every $I,J \in Pt(P)$. We will show that f is continuous with respect to the topology τ. Let U be any open set containing $f(I,J) = I\cap J$. Then there exist a basic open set λ_z such that $I\cap J \in \lambda_z$ and $\lambda_z \subseteq U$. $I\cap J \in \lambda_z$ implies that $z \in I$ and $z \in J$. So $I,J \in \lambda_z$ and $\lambda_z \times \lambda_z$ is an open set in $Pt(P)\times Pt(P)$ containing (I,J). Now we will show that $f(\lambda_z \times \lambda_z) \subseteq U$. Let $(I',J') \in \lambda_z \times \lambda_z$. Then $z \in I', z \in J'$ and so $z \in I' \cap J' = f(I',J')$. Hence $f(I',J') \in \lambda_z \subseteq U$. Thus $f(\lambda_z \times \lambda_z) \subseteq \lambda_z \subseteq U$. Hence $f : Pt(P)\times Pt(P) \to Pt(P)$ is continuous with respect to the topology τ.
Now we will show that for every $a \in L$ the map $\delta_a : Pt(P) \to Pt(P)$ defined by $\delta_a(I) = \delta(a,I) = I_a$ is continuous with respect to the topology τ. Let V be any open set containing $\delta(a,I) = I_a$. Then there exists a basic open set λ_x such that $I_a \in \lambda_x \subseteq V$. $I_a \in \lambda_x$ implies that $x \in I_a$. So $\mu(a,x) \in I$ or $I \in \lambda_{\mu(a,x)}$. Thus $\lambda_{\mu(a,x)}$ is an open set containing I. Let $J \in \lambda_{\mu(a,x)}$, then $\mu(a,x) \in J$ or $x \in J_a$. Hence $\delta_a(J) \in \lambda_x$. Thus δ_a is continuous with respect to the topology τ. Hence $(\delta, Pt(J), \tau)$ is a directed topological weak L-module. □

Proposition 6.4. If $f:(\mu_1,P_1)\to(\mu_2,P_2)$ is a directed complete L-slice homomorphism, then there is a morphism $\phi:(\delta_2,Pt(P_2),\tau_2)\to(\delta_1,Pt(P_1),\tau_1)$ in the category **TopDWMod** of topological directed weak L-modules.

Proof. Let $f:(\mu_1,P_1)\to(\mu_2,P_2)$ be a directed complete L-slice homomorphism. Define $\phi:(\delta_2,Pt(P_2),\tau_2)\to(\delta_1,Pt(P_1),\tau_1)$ by $\phi(I)=f^{-1}(I)$. Then $\phi(I*J)=\phi(I\cap J)=f^{-1}(I\cap J)=f^{-1}(I)\cap f^{-1}(J)=\phi(I)\cap\phi(J)=\phi(I)*\phi(J)$.Also, $\phi(\delta_2(a,I))=\phi(I_a)=f^{-1}(I_a)=(f^{-1}(I))_a=\delta_1(a,f^{-1}(I))=\delta_1(a,\phi(I))$.
Let $\{I_\alpha\}$ be a directed subset of $(\delta_2,Pt(P_2))$. Since $Pt(P_1)$ is complete, $\{\phi(I_\alpha)\}$ is directed and $\phi(\bigcap I_\alpha)=f^{-1}(\bigcap I_\alpha)=\bigcap f^{-1}(I_\alpha)=\bigcap\phi(I_\alpha)$. Thus ϕ is a directed complete L-slice homomorphism. In order to check the continuity of ϕ, let U be any open set containing $\phi(I)$. Then there exists a basic open set λ_x such that $f^{-1}(I)=\phi(I)\in\lambda_x\subseteq U$. Since $x\in f^{-1}(I)=\phi(I)$, $f(x)\in I$. Thus $\lambda_{f(x)}$ is an open subset of $(\delta_2,Pt(P_2))$ containing I. Now we will show that $\phi(\lambda_{f(x)})\subseteq U$. Let $J\in\lambda_{f(x)}$.Then $f(x)\in J$. Equivalently $f^{-1}(J)\in\lambda_x$. Hence Now we will show that $\phi(\lambda_{f(x)})\subseteq U$. Thus ϕ is continuous. Hence ϕ is a morphism in the category **TopDWMod**. $\square$

Proposition 6.5. There is contravarient functor from the category **DL-slice** to the category **TopDWMon**.

Proof. Define $\Psi:Ob(\textbf{DL-slice})\to Ob(\textbf{TopDWMod})$ by $\Psi(P)=Pt(P)$.
Also define $\Psi:Mor(\textbf{DL-slice})\to Mor(\textbf{TopDWMod})$ as follows.
Let $f:(\mu_1,P_1)\to(\mu_2,P_2)$ be a directed L-slice homomorphism. Define $\Psi(f):(\delta_2,Pt(P_2),\tau_2)\to(\delta_1,Pt(P_1),\tau_1)$ by $\Psi(f)(I)=f^{-1}(I)$. Then by the above proposition $\Psi(f)\in Mor(\textbf{TopDWMon})$.
Let $f:(\mu_1,P_1)\to(\mu_2,P_2)$ and $g:(\mu_2,P_2)\to(\mu_3,P_3)$ be directed L-slice homomorphisms. Then,

$$\begin{aligned}\Psi(g\circ f)(I)&=(g\circ f)^{-1}(I)=f^{-1}(g^{-1}(I))=\Psi(f)(g^{-1}(I))=\Psi(f)(\Psi(g)(I))\\&=\Psi(f)\circ\Psi(g)(I).\end{aligned}$$

Let $id:(\mu,P)\to(\mu,P)$ be an identity morphism in **DL-slice**. Then $\Psi(id)(I)=id^{-1}(I)=I$. Hence $\Psi(id)$ is an identity morphism in **TopDWMon**. This shows that Ψ is a contravarient functor from the category **DL-slice** to the category **TopDW-Mon**. $\square$

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