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EXTENDED GENERALIZED FIBONACCI AND TRIBONACCI POLYNOMIALS WITH SOME PROPERTIES

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ABSTRACT. In this paper, we introduced the extended generalized Fibonacci polynomial sequence $\{Y_{2,n}\}$ and extended generalized Tribonacci polynomial $\{Y_{3,n}\}$ with arbitrary initial values and established a recursive matrix and then presented some properties of these. Further, we investigated some well-known identities like Binet's formula, Catalan's identity, Cassini's identity, d'Ocagne's identity, generating function, explicit sum formula, sum of first n terms for the extended generalized Fibonacci polynomial sequence and extended generalized Tribonacci polynomial sequence.

1. INTRODUCTION

Many studies of numerical sequences and polynomials have been published recently and they have been applied extensively in a variety of fields of study, including physics, engineering, architecture, nature and the arts. The Fibonacci number sequence F_n is determined by the second-order linear recurrence sequence, as following

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \text{ where } F_0 = 0, F_1 = 1.$$

Fibonacci polynomials are a generalization of the sequence F_n defined by a second-order linear recurrence sequence

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad \text{with } F_0(x) = 0, F_1(x) = 1, n \geq 2.$$

The process of evaluating the polynomials $F_n(x)$ at $x = 1$ yields the Fibonacci numbers. Numerous intriguing characteristics and applications in practically every discipline may be found in the Fibonacci numbers, polynomials, and their generalisations. Regarding particular examples of second-order linear recurrence sequences of integers and polynomials, see, for example, [8, 9, 13] for books and [1–3, 5–7, 10, 14, 15] for papers.

In [4],G. Lee and M. Asci introduced a generalization of $h(x)$- Fibonacci polynomials called (p,q)-Fibonacci polynomials and defined by

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$F_{p,q,n+1}(x) = p(x)F_{p,q,n}(x) + q(x)F_{p,q,n-1}(x)$, $n \geq 1$ with $F_{p,q,0}(x) = 0, F_{p,q,1}(x) = 1$, where $p(x)$ and $q(x)$ be polynomials with real coefficients.
A matrix corresponding to the above polynomial sequence is given by

$$Q^n_{p,q}(x) = \begin{pmatrix} F_{p,q,n+1}(x) & q(x)F_{p,q,n}(x) \\ F_{p,q,n}(x) & q(x)F_{p,q,n-1}(x) \end{pmatrix}.$$

In [12], Y. Soykan generalized Fibonacci polynomials and provided sum formulas, generating functions, and Simson's formulas using a matrix technique for the generalized Fibonacci polynomials sequence. Additionally, he generalized Tribonacci polynomials in [11] and provided several identities for them.

In this paper, we have generalized the Fibonacci polynomial sequence and the Tribonacci polynomial sequence. In addition, some matrix identities, Binet's formulas, explicit sum formulas, etc. for the new generalized Fibonacci and Tribonacci polynomials were presented.

2. The $\{Y_{2,n}\}$ Extended Generalized Fibonacci Polynomial Sequence and some properties

Consider the second order linear difference equation given by

$$Y_{2,n}(x) = u(x)Y_{2,n-1}(x) + v(x)Y_{2,n-2}(x), \quad n \geq 2 \tag{2.1}$$

with $Y_{2,0}(x) = g(x) + h(x)$ and$Y_{2,1}(x) = 2g(x) + 1$
where, $g(x), h(x), u(x)$ and $v(x)$ are polynomials with real coefficients with $u(x) \neq 0, v(x) \neq 0$.
Similar to the Fibonacci Polynomial sequence, the sequence $Y_{2,n}(x)$ can also be extended in the negative direction by arranging equation (2.1) as

$$Y_{2,-n}(x) = \frac{1}{v(x)}Y_{2,-n+2}(x) - \frac{u(x)}{v(x)}Y_{2,-n+1}(x)$$

for $n = 1, 2, 3 \ldots$, with the same initial value and $v(x) \neq 0$.
Thus, the first few terms of the polynomial sequence are as follows:
$Y_{2,0}(x) = g(x) + h(x)$
$Y_{2,1}(x) = 2g(x) + 1$
$Y_{2,2}(x) = u(x)(2g(x) + 1) + v(x)(g(x) + h(x))$
$Y_{2,3}(x) = u^2(x)(2g(x) + 1) + u(x)v(x)(g(x) + h(x)) + v(x)(2g(x) + 1)$
$Y_{2,4}(x) = u^3(x)(2g(x) + 1) + u^2(x)v(x)(g(x) + h(x)) + 2u(x)v(x)(2g(x) + 1)$
$\quad + v^2(x)(g(x) + h(x))$.

Remark 2.1. For a polynomial sequence $Y_{2,n}(x)$ satisfying equation (2.1), we have,

$$Y_{2,n}(x) = (g(x) + h(x))v(x)f_{n-1}(x) + (2g(x) + 1)f_n(x)$$

where $f_n(x) = u(x)f_{n-1}(x) + v(x)f_{n-2}(x)$, $n \geq 2$ with $f_0(x) = 0, f_1(x) = 1$.

Note: For the sake of simplicity throughout the rest of the paper, we use $Y_{2,n}, u, v, g, h, Y_{2,0}, Y_{2,1}, f_n$ instead of $Y_{2,n}(x), u(x), v(x), g(x), h(x), Y_{2,0}(x), Y_{2,1}(x), f_n(x)$ respectively.

2.1. **Matrix Formation**

The matrix $\{M_2^n\}_{n\geq 0}$ of the polynomial sequence $Y_{2,n}$ is defined as,

$$M_2^n = \begin{pmatrix} Y_{2,n+1} & Y_{2,n} \\ Y_{2,n} & Y_{2,n-1} \end{pmatrix} = \begin{pmatrix} u & v \\ 1 & 0 \end{pmatrix}^n M_2^0, \tag{2.2}$$

where $M_2^0 = \begin{pmatrix} 2g+1 & g+h \\ g+h & \frac{1}{v}[2g+1-u(g+h)] \end{pmatrix}$.
In the next theorem and results, we present some interesting recursive and explicit formulas for the matrix M_2^n associated with the generalized Fibonacci polynomial matrix.

Theorem 2.1. *The determinant of the matrix M_2^n is given by,*

$$det(M_2^n) = (-v)^{n-1}[u(2g+1)(h+g) + v(g+h)^2 - (2g+1)^2].$$

Proof. To prove it, we use the following result on Generalized Fibonacci polynomials

$$f_n f_{m+1} - f_m f_{n+1} = (-v)^n f_{n-m}.$$

Therefore,

$$\begin{aligned}
Det(M_2^n) &= Y_{2,n+1}Y_{2,n-1} - Y_{2,n}^2 \\
&= [(g+h)vf_n + (2g+1)f_{n+1}][(g+h)vf_{n-2} + (2g+1)f_{n-1}] \\
&- [(g+h)vf_{n-1} + (2g+1)f_n]^2 \\
&= (g+h)^2 f_n f_{n-2} + (g+h)(2g+1)vf_nf_{n-1} + (g+h)(2g+1)f_{n+1}f_{n-2} \\
&+ (2g+1)^2 f_{n+1}f_{n-1} - (g+h)^2v^2f_{n-1}^2 - (2g+1)^2 f_n^2 \\
&- 2(g+h)(2g+1)vf_{n-1}f_n \\
&= (g+h)^2v^2(f_nf_{n-2} - f_{n-1}^2) + (g+h)(2g+1)v(f_{n+1}f_{n-2} - f_{n-1}f_n) \\
&+ (2g+1)^2(f_{n+1}f_{n-1} - f_n^2) \\
&= (g+h)^2v^2(-1)^{n-1}v^{n-1}v^(-1) + (g+h)(2g+1)v(-1)^{n+1}v^{n-2}u \\
&+ (2g+1)^2(-v)^n v^{-1} \\
&= (-v)^{n-1}[(g+h)^2v + (g+h)(2g+1)u - (2g+1)^2]
\end{aligned}$$

as required. □

Remark 2.2. (1) (Fibonacci polynomial matrix). For $g=0, h=0, u=x$ and $v=1$, we have $det(M_2^n) = (-1)^n$.

(2) (Generalized Fibonacci polynomial matrix). For $g=0$ and $h=0$, we have $det(M_2^n)=(-1)^n v$.
(3) (Lucas polynomial matrix). For $2g=x-1, 2h=5-x, u=x$ and $v=1$, we have $det(M_2^n)=(-1)^n(x^2+4)$.

Theorem 2.2. *Let M_2^n be the matrix as defined above and Q_2^n the generalized Fibonacci polynomial matrix, then*

$$M_2^n=Q_2^n M_2^0=M_2^0 Q_2^n,\ \forall n\in\mathbb{Z} \qquad \textit{where}\quad Q_2^n=\begin{pmatrix} f_{n+1} & vf_n \\ f_n & vf_{n-1}\end{pmatrix}.$$

Proof. We have,

$$\begin{aligned}
Q_2^n M_2^0 &= \begin{pmatrix} f_{n+1} & vf_n \\ f_n & vf_{n-1}\end{pmatrix}\begin{pmatrix} 2g+1 & g+h \\ g+h & \frac{1}{v}[2g+1-u(g+h)]\end{pmatrix}\\
&=\begin{pmatrix} (2g+1)f_{n+1}+(g+h)vf_n & (2g+1)f_n+(g+h)f_{n+1}-u(g+h)f_n \\ (2g+1)f_n+(g+h)vf_{n-1} & (2g+1)f_{n-1}+(g+h)f_n-u(g+h)f_{n-1}\end{pmatrix} \ using (2.1),\\
&=\begin{pmatrix} Y_{2,n+1} & Y_{2,n} \\ Y_{2,n} & Y_{2,n-1}\end{pmatrix}\\
&=M_2^n.
\end{aligned}$$

By a similar argument, we have $M_2^0 Q_2^n=M_2^n$. ☐

Theorem 2.3. *Let M_2^n be the matrix as defined in* (2.2)*, then*

$$M_2^n M_2^{-n}=(M_2^0)^2.$$

Proof. From the above theorem, we have

$$\begin{aligned}
M_2^n M_2^{-n} &= Q_2^n M_2^0 Q_2^{-n} M_2^0\\
&= M_2^0 Q_2^n Q_2^{-n} M_2^0\\
&= M_2^0 I M_2^0=(M_2^0)^2.
\end{aligned}$$

☐

2.2. **Binet's formula, Generating Function and Some identities**

The characteristic equation for the equation $Y_{2,n}(x)=u(x)Y_{2,n-1}(x)+v(x)Y_{2,n-2}(x)$, $n\geq 2$ with initial condition $Y_{2,0}(x)=g(x)+h(x)$ and $Y_{2,1}(x)=2g(x)+1$ is as follows

$$d^2-ud-v=0 \tag{2.3}$$

This equation has the roots $\mu_1(x)=\frac{u(x)+\sqrt{u^2(x)+4v(x)}}{2}$ and $\mu_2(x)=\frac{u(x)-\sqrt{u^2(x)+4v(x)}}{2}$. Also, we have that $\mu_1(x)+\mu_2(x)=u(x)$, $\mu_1(x)\mu_2(x)=-v(x)$ and $\mu_1(x)-\mu_2(x)=\sqrt{u^2(x)+4v(x)}$.

Theorem 2.4. *(Binet's formula). For $n\geq 0$, we have,*

$$Y_{2,n}=-R\mu_1^n+S\mu_2^n=\frac{1}{\sqrt{u^2+4v}}[\{(g+h)\mu_2-(2g+1)\}\mu_1^n+\{(g+h)\mu_1-(2g+1)\}\mu_2^n] \tag{2.4}$$

Proof. The solution of equation (2.1) can be expressed in general form

$$Y_{2,n} = c_1\mu_1^n + c_2\mu_2^n, \tag{2.5}$$

where c_1 and c_2 are constant and $\mu_1 = \dfrac{u+\sqrt{u^2+4v}}{2}$ and $\mu_2 = \dfrac{u-\sqrt{u^2+4v}}{2}$.
For $n = 0$ and $n = 1$, we have $Y_{2,0} = c_1 + c_2$ and $Y_{2,1} = c_1\mu_1 + c_2\mu_2$, which gives $c_1 + c_2 = g + h$ and $c_1\mu_1 + c_2\mu_2 = 2g + 1$.
After solving above equation, we get

$$c_1 = \frac{-(g+h)\mu_2 + (2g+1)}{\mu_1-\mu_2} \text{ and } c_2 = \frac{(g+h)\mu_1 - (2g+1)}{\mu_1-\mu_2}.$$

Thus, from equation (2.5), we obtain

$$Y_{2,n} = -R\mu_1^n + S\mu_2^n,$$

where $R = \dfrac{(g+h)\mu_2 - (2g+1)}{\mu_1-\mu_2}$ and $S = \dfrac{(g+h)\mu_1 - (2g+1)}{\mu_1-\mu_2}$,
or
$$Y_{2,n} = -R\mu_1^n + S\mu_2^n = \frac{1}{\sqrt{u^2+4v}}[\{(g+h)\mu_2-(2g+1)\}\mu_1^n + \{(g+h)\mu_1-(2g+1)\}\mu_2^n].$$
□

Corollary 2.1. *For $n \in \mathbb{N}$, we have,* $Y_{2,(-n)} = (-1)^n(s)^{(-n)}\dfrac{-R\mu_1^n + S\mu_2^n}{\mu_1-\mu_2}$.

Proof. Replace n by $(-n)$ in equation (2.4) and after solving the equation, we get $Y_{2,(-n)} = (-1)^n(s)^{(-n)}\dfrac{-R\mu_1^n + S\mu_2^n}{\mu_1-\mu_2}$. □

Remark 2.3. $-R+S = (g+h), -R\mu_1 + S\mu_2 = 2g+1$, and $-R\mu_2 + S\mu_1 = u(g+h) - (2g+1)$.

Theorem 2.5. *(Catalan's identity). For the sequence* (2.1)*, we have*

$$Y_{2,n-m}Y_{2,n+m} - Y_{2,n}^2 = \frac{(-1)^n v^{n-m}}{2^m(u^2+4v)}[(2g+1)^2 - v(g+h)^2 - (2g+1)(g+h)u]L,$$

where,

$$L = [2^{m+1}v^m - (-u^2-2v+u\sqrt{u^2+4v})^m - (-u^2-2v-u\sqrt{u^2+4v})^m].$$

Proof. Using Binet's formula (2.4), we have

$$Y_{2,n-m}Y_{2,n+m} - Y_{2,n}^2 = \left(\frac{-R\mu_1^{n-m} + S\mu_2^{n-m}}{\mu_1-\mu_2}\right)\left(\frac{-R\mu_1^{n+m} + S\mu_2^{n+m}}{\mu_1-\mu_2}\right) - \left(\frac{-R\mu_1^n + S\mu_2^n}{\mu_1-\mu_2}\right)^2$$
$$= \frac{1}{(\mu_1-\mu_2)^2}\left[RS\left(2\mu_1^n\mu_2^n - \mu_1^{n-m}\mu_2^{n+m} - \mu_1^{n+m}\mu_2^{n-m}\right)\right]$$

$= \frac{1}{(\mu_1-\mu_2)^2} RS\mu_1^n\mu_2^n \left[2 - \left(\frac{\mu_1}{\mu_2}\right)^m - \left(\frac{\mu_2}{\mu_1}\right)^m\right]$

$= \frac{1}{(\mu_1-\mu_2)^2} RS\mu_1^n\mu_2^n \left[2 - \left(\frac{u^2+2v-u\sqrt{u^2+4v}}{-2v}\right)^m - \left(\frac{u^2+2v+u\sqrt{u^2+4v}}{-2v}\right)^m\right]$

$= \frac{(-v)^n RS}{(\mu_1-\mu_2)^2 (2v)^m} \left[2^{m+1}v^m - \left(-u^2-2v+u\sqrt{u^2+4v}\right)^m - \left(-u^2-2v-u\sqrt{u^2+4v}\right)^m\right]$

$= \frac{(-1)^n v^{n-m}}{2^m(u^2+4v)} \left[(2g+1)^2 - v(g+h)^2 - (2g+1)(g+h)u\right] \left[2^{m+1}v^m\right.$

$\left. -(-u^2-2v+u\sqrt{u^2+4v})^m - (-u^2-2v-u\sqrt{u^2+4v})^m\right]$

as required. □

Corollary 2.2. *(Cassini's identity). For $m = 1$ in the above theorem, we have,*

$$Y_{2,n-1}Y_{2,n+1} - Y_{2,n}^2 = (-1)^n v^{n-1}[(2g+1)^2 - v(g+h)^2 - (2g+1)(g+h)u].$$

Theorem 2.6. *(d'Ocagne's identity). For $m,n \in \mathbb{N}$, we have,*

$$Y_{2,n}Y_{2,m+1} - Y_{2,n+1}Y_{2,m} = \frac{[(2g+1)^2 - v(g+h)^2 - (2g+1)(g+h)u]}{\sqrt{u^2+4v}}(\mu_1^n\mu_2^m - \mu_1^m\mu_2^n).$$

Proof. Using Binet's formula (2.4), we have

$Y_{2,n}Y_{2,m+1} - Y_{2,n+1}Y_{2,m} = \left(\frac{-R\mu_1^n+S\mu_2^n}{\mu_1-\mu_2}\right)\left(\frac{-R\mu_1^{n+1}+S\mu_2^{n+1}}{\mu_1-\mu_2}\right) - \left(\frac{-R\mu_1^{n+1}+S\mu_2^{n+1}}{\mu_1-\mu_2}\right)\left(\frac{-R\mu_1^m+S\mu_2^m}{\mu_1-\mu_2}\right)$

$= \frac{1}{(\mu_1-\mu_2)^2}(RS\mu_1^{n+1}\mu_2^m + RS\mu_1^m\mu_2^{n+1} - RS\mu_1^n\mu_2^{m+1} - RS\mu_1^{m+1}\mu_2^n)$

$= \frac{RS[\mu_1^n\mu_2^m(\mu_1-\mu_2) - \mu_1^m\mu_2^n(\mu_1-\mu_2)]}{(\mu_1-\mu_2)^2}$

$= \frac{[(2g+1)^2 - v(g+h)^2 - (2g+1)(g+h)u]}{\sqrt{u^2+4v}}(\mu_1^n\mu_2^m - \mu_1^m\mu_2^n).$ □

Theorem 2.7. *(Generating Function). The Generating function for the sequence $Y_{2,n}$ is given by*

$$\sum_{n=0}^{\infty} Y_{2,n}t^n = \frac{(g+h) + ((2g+1) - u(g+h))t}{1-ut-vt^2}.$$

Proof. Let $g(t) = \sum_{n=0}^{\infty} Y_{2,n}t^n$, then from equation (2.1) by multiplying t^{n+2} with summation from 0 to ∞, we get:

$\sum_{n=0}^{\infty} t^{n+2}Y_{2,n+2} - \sum_{n=0}^{\infty} t^{n+2}uY_{2,n+1} - \sum_{n=0}^{\infty} t^{n+2}vY_{2,n} = 0$

$\Rightarrow (g(t) - Y_{2,0} - Y_{2,1}t) - u(g(t) - Y_{2,0})t - vg(t)t^2 = 0$

$\Rightarrow g(t)(1-ut-vt^2) - (Y_{2,0} + Y_{2,1}t - uY_{2,0}t) = 0$

$\Rightarrow g(t) = \frac{Y_{2,0} + (Y_{2,1} - uY_{2,0}t)}{1-ut-vt^2}$ so

$g(t) = \sum_{n=0}^{\infty} Y_{2,n}t^n = \frac{(g+h) + ((2g+1) - u(g+h))t)}{1-ut-vt^2}.$ □

Proposition 2.1. *(Explicit sum Formula) Let $Y_{2,n}$ be the n^{th} Extended Generalized Fibonacci polynomial, then*

$$Y_{2,n} = (g+h)\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} u^{n-2k} v^k + [(2g+1)-u(g+h)] \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} u^{n-2k-1} v^k,$$

where $\lfloor n \rfloor$ is the greatest integer less than or equal to n.

Proof. Using the generating function, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} Y_{2,n} t^n &= \frac{(g+h)+((2g+1)-u(g+h))t}{1-ut-vt^2} \\
&= [(g+h)+((2g+1)-u(g+h))t][1-(u-vt)t]^{-1} \\
&= [(g+h)+((2g+1)-u(g+h))t] \sum_{n=0}^{\infty} (u+vt)^n t^n \\
&= [(g+h)+((2g+1)-u(g+h))t] \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k} u^{n-k} (vt)^k \\
&= [(g+h)+((2g+1)-u(g+h))t] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} u^{n-k} v^k t^{n+k}.
\end{aligned}$$

Taking $n+k$ instead of n, we have

$$\begin{aligned}
&= [(g+h)+((2g+1)-u(g+h))t] \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!n!} u^n v^k t^{n+2k} \\
&= [(g+h)+((2g+1)-u(g+h))t] \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k)!}{k!(n-2k)!} u^{n-2k} v^k t^n
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \left[(g+h) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k)!}{k!(n-2k)!} u^{n-2k} v^k \right] t^n + \sum_{n=0}^{\infty} \left[[(2g+1)-u(g+h)] \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-k)!}{k!(n-2k)!} u^{n-2k} v^k \right] t^{n+1}.$$

Taking the coefficient of t^n, we get

$$Y_{2,n} = (g+h)\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} u^{n-2k} v^k + [(2g+1)-u(g+h)] \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k} u^{n-2k-1} v^k.$$

□

Proposition 2.2. *(Sum of first n^{th} terms). The sum of the first n^{th} terms of the extended generalized Fibonacci polynomials is given as follows*

$$\sum_{k=0}^{n-1} Y_{2,k}(x) = \frac{Y_{2,n}(x) + vY_{2,n-1}(x) - [(2g+1)-u(g+h)] - (g+h)}{u+v-1}.$$

Proof. Using Binet's formula, we have

$$\sum_{k=0}^{n-1} Y_{2,k}(x) = \sum_{k=0}^{n-1} (-R\mu_1^k(x) + S\mu_2^k(x)),$$

where $R = \dfrac{(g+h)\mu_2 - (2g+1)}{\mu_1 - \mu_2}$ and $S = \dfrac{(g+h)\mu_1 - (2g+1)}{\mu_1 - \mu_2}$.
Thus,

$$\begin{aligned}\sum_{k=0}^{n-1} Y_{2,k}(x) &= -R\sum_{k=0}^{n-1}\mu_1^k(x) + S\sum_{k=0}^{n-1}\mu_2^k(x)\\ &= -R\frac{(\mu_1^n(x)-1)}{\mu_1 - 1} + S\frac{(\mu_2^n(x)-1)}{\mu_2 - 1}\\ &= \frac{-R+S-(-R\mu_2(x)+S\mu_1(x))-(-R\mu_1^n(x)+S\mu_2^n(x))}{\mu_1(x)\mu_2(x)-\mu_1(x)-\mu_2(x)+1} + \frac{\mu_1(x)\mu_2(x)(-R\mu_1^{n-1}(x)+S\mu_1^{n-1}(x))}{\mu_1(x)\mu_2(x)-\mu_1(x)-\mu_2(x)+1}.\end{aligned}$$

Since $\mu_1(x)+\mu_2(x) = u, \mu_1(x)\mu_2(x) = -v$ and by Remark 2.3, we have

$$\sum_{k=0}^{n-1} Y_{2,k}(x) = \frac{Y_{2,n}(x) + vY_{2,n-1}(x) - [(2g+1) - u(g+h)] - (g+h)}{u+v-1}. \qquad \square$$

3. The $\{Y_{3,n}\}$ Extended Generalized Tribonacci Polynomial Sequence and some properties

Consider the third order linear difference equation given by

$$Y_{3,n}(x) = u^2(x)Y_{2,n-1}(x) + u(x)v(x)Y_{2,n-2}(x) + w^2(x)Y_{3,n-3}, \quad n \geq 3, \tag{3.1}$$

with $Y_{3,0}(x) = j(x), Y_{3,1}(x) = g(x)+h(x)$ and $Y_{3,2}(x) = 2g(x)+1$, where, $j(x)$, $g(x)$, $h(x)$, $u(x)$ and $v(x)$ are polynomials with real coefficients with $u(x) \neq 0, v(x) \neq 0, w(x) \neq 0$.
The relation can be extended in negative terms as follows

$$Y_{3,-n} = \frac{1}{w^2(x)}\left(Y_{3,-n+3} - u^2(x)Y_{3,-n+2} - u(x)v(x)Y_{3,-n+1}\right),$$

for $n = 1,2,3\ldots$, with the same initial value and $w(x) \neq 0$.
In particular $j(x) = 0, g(x) = 0, h(x) = 0$ and $u(x) = 1, v(x) = 1, w(x) = 1$, so equation (3.1) becomes the Tribonacci sequence and for $j(x) = 3, g(x) = 1, h(x) = 0$ and $u(x) = 1, v(x) = 1, w(x) = 1$, so equation (3.1) becomes the Tribonacci Lucas sequence.

Thus, the first few terms of the polynomial sequence $Y_{3,n}$ are as follows

$Y_{3,0}(x) = j(x)$

$Y_{3,1}(x) = g(x) + h(x)$

$Y_{3,2}(x) = 2g(x) + 1$

$Y_{3,3}(x) = u^2(x)(2g+1) + u(x)v(x)(g+h) + j(x)w^2(x)$

$Y_{3,4}(x) = (u^4(x) + u(x)v(x))(2g+1) + (u^3(x)v(x) + w^2(x))(g(x)+h(x)) + j(x)u^2(x)w^2(x).$

For the sake of simplicity, we use $Y_{3,n}, u, v, w, j, g, h, Y_{3,0}, Y_{3,1}, f_n$ instead of $Y_{3,n}(x)$, $u(x)$, $v(x)$, $w(x)$, $j(x)$, $g(x)$, $h(x)$, $Y_{3,0}(x)$, $Y_{3,1}(x)$, $f_n(x)$, respectively.

3.1. **Matrix Representation**

The matrix $\{M_3^n\}_{n\geq 0}$ of the polynomial sequence $Y_{3,n}$ is defined as follows

$$M_3^n = \begin{pmatrix} Y_{3,n+2} & uvY_{3,n+1} + w^2Y_{3,n} & w^2Y_{3,n+1} \\ Y_{3,n+1} & uvY_{3,n} + w^2Y_{3,n-1} & w^2Y_{3,n} \\ Y_{3,n} & uvY_{3,n-1} + w^2Y_{3,n-2} & w^2Y_{3,n-1} \end{pmatrix} = \begin{pmatrix} u^2 & uv & w^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n M_3^0, \quad (3.2)$$

where $$M_3^0 = \begin{pmatrix} 2g+1 & uv(g+h)+w^2j & w^2(g+h) \\ g+h & (2g+1)-u^2(g+h) & w^2j \\ j & (g+h)-u^2j & (2g+1)-u^2(g+h)-uvj \end{pmatrix}.$$

Theorem 3.1. *Let $\{f_{3,n}\}_{n\geq 0}$ be the Tribonacci polynomial sequence defined as*

$$f_{3,n+3} = u^2 f_{3,n+2} + uvf_{3,n+1} + w^2 f_{3,n} \quad with f_{3,0} = 0, f_{3,1} = 0, f_{3,2} = 1, \qquad (3.3)$$

then

$$Y_{3,n} = (g+h)(f_{3,n+1} - u^2 f_{3,n}) + jw^2 f_{3,n-1} + (2g+1) f_{3,n}, \forall n \in \mathbb{Z}.$$

Proof. We prove it using mathematical induction on n. For $n = 0$, the result obviously holds. For $n = 1$, we have

$$Y_{3,1} = (g+h)(f_{3,2} - u^2 f_{3,1}) + jw^2 f_{3,0} + (2g+1) f_{3,1} = g+h.$$

$$Y_{3,2} = (g+h)(f_{3,3} - u^2 f_{3,2}) + jw^2 f_{3,1} + (2g+1) f_{3,2} = 2g+1.$$

Now assume the result is true for $n = k$. For $n = k+1$, we write

$Y_{3,k+1} = u^2 Y_{2,k} + uvY_{2,k-1}(x) + w^2 Y_{3,k-2}$
$= u^2[(g+h)(f_{3,k+1} - u^2 f_{3,k}) + jw^2 f_{3,k-1} + (2g+1) f_{3,k}] + uv[(g+h)(f_{3,k} - u^2 f_{3,k-1}) + jw^2 f_{3,k-2} + (2g+1) f_{3,k-1}] + w^2[(g+h)(f_{3,k-1} - u^2 f_{3,k-2}) + jw^2 f_{3,k-3} + (2g+1) f_{3,k-2}]$
$= (g+h)[u^2 f_{3,k+1} - u^4 f_{3,k} + uvf_{3,k} - u^3 v f_{3,k-1} + w^2 f_{3,k-1} - u^2 w^2 f_{3,k-2}] + jw^2[u^2 f_{3,k-1} + uvf_{3,k-2} + w^2 f_{3,k-3}] + (2g+1)[u^2 f_{3,k} + uvf_{3,k-1} + w^2 f_{3,k-2}]$.

Using the Tribonacci polynomial sequence,we have

$$Y_{3,k+1} = (g+h)(f_{3,k+2} - u^2 f_{3,k+1}) + jw^2 f_{3,k} + (2g+1) f_{3,k+1}.$$

□

Theorem 3.2. *Let M_3^0 be the initial matrix defined in* (3.2) *and Q_3^n be the Tribonacci polynomial matrix of equation* (3.3) *defined as*

$$Q_3^n = \begin{pmatrix} f_{3,n+2} & uvf_{3,n+1}+w^2f_{3,n} & w^2f_{3,n+1} \\ f_{3,n+1} & uvf_{3,n}+w^2f_{3,n-1} & w^2f_{3,n} \\ f_{3,n} & uvf_{3,n-1}+w^2f_{3,n-2} & w^2f_{3,n-1} \end{pmatrix},$$

then, for all integers, we have $M_3^n = Q_3^nM_3^0 = M_3^0Q_3^n$.

Proof. Using mathematical induction this can be proved easily. $\square$

Theorem 3.3. *Let M_3^n be the matrix as defined in* (3.2) *, then for all integers*

$$M_3^nM_3^{-n} = (M_3^0)^2.$$

Proof. From the above theorem, we have

$$\begin{aligned} M_3^nM_3^{-n} &= Q_3^nM_3^0Q_3^{-n}M_3^0 \\ &= M_3^0Q_3^nQ_3^{-n}M_3^0 \\ &= M_3^0IM_3^0 = (M_3^0)^2. \end{aligned}$$

$\square$

3.2. **Binet's formula, Generating function and Identities of** $\{Y_{3,n}\}$

To establish any identity involving the n^{th} term of the sequence, the Binet formula plays an important role. It can be calculated using it's characteristic equation which is as follows

$$z^3 - u^2z^2 - uvz - w^2 = 0. \tag{3.4}$$

The roots of the characteristic equation (3.4) will be denoted by $\mu_1(x)$, $\mu_2(x)$, $\mu_3(x)$ and defined by

$$\mu_1(x) = \frac{u^2}{3} + a_1 + a_2,\ \mu_2(x) = \frac{u^2}{3} + \omega a_1 + \omega^2 a_2,\ \mu_3(x) = \frac{u^2}{3} + \omega^2 a_1 + \omega a_2,$$

where

$$\omega = \tfrac{-1+\sqrt{3}i}{2},\ a_1 = \left(\tfrac{u^6}{27} + \tfrac{u^3v}{6} + \tfrac{w^2}{2} + \sqrt{\triangle}\right)^{\frac{1}{3}},\ a_2 = \left(\tfrac{u^6}{27} + \tfrac{u^3v}{6} + \tfrac{w^2}{2} - \sqrt{\triangle}\right)^{\frac{1}{3}},$$

with $\triangle = \frac{u^6w^2}{27} - \frac{u^6v^2}{108} + \frac{u^3vw^2}{6} - \frac{u^3v^3}{27} + \frac{w^4}{4}$, and also satisfy

$$\mu_1(x)+\mu_2(x)+\mu_3(x) = u^2,\ \mu_1(x)\mu_2(x)+\mu_2(x)\mu_3(x)+\mu_3(x)\mu_1(x) = -uv,$$
$$\mu_1(x)\mu_2(x)\mu_3(x) = w^2.$$

Theorem 3.4. *(Binet's formula). For $n \geq 0$, we have*

$$Y_{3,n} = \frac{R\mu_1^n}{(\mu_1-\mu_2)(\mu_1-\mu_3)} + \frac{S\mu_2^n}{(\mu_2-\mu_1)(\mu_2-\mu_3)} + \frac{T\mu_3^n}{(\mu_3-\mu_1)(\mu_3-\mu_2)},$$

where

$$R = (2g+1)-(\mu_2+\mu_3)(g+h)+j\mu_2\mu_3,\ S = (2g+1)-(\mu_1+\mu_3)(g+h)+j\mu_1\mu_3,$$
$$T = (2g+1)-(\mu_1+\mu_2)(g+h)+j\mu_1\mu_2.$$

Proof. If μ_1, μ_2, μ_3 are distinct roots of the equation (3.4), then the general formula for $Y_{3,n}$ is in the following form

$$Y_{3,n} = c_1\mu_1^n + c_2\mu_2^n + c_3\mu_3^n \tag{3.5}$$

To determine the value of c_1, c_2 and c_3, we have

$$Y_{3,0} = c_1 + c_2 + c_3 \tag{3.6}$$

$$Y_{3,1} = c_1\mu_1 + c_2\mu_2 + c_3\mu_3 \tag{3.7}$$

$$Y_{3,2} = c_1\mu_1^2 + c_2\mu_2^2 + c_3\mu_3^2. \tag{3.8}$$

Now, we multiply equation (3.6) with μ_1 then subtract equation (3.7) and we multiply equation (3.7) with μ_1 then subtract equation (3.8), we get

$$(\mu_1 Y_{3,0} - Y_{3,1}) = c_2(\mu_1 - \mu_2) + c_3(\mu_1 - \mu_3) \tag{3.9}$$

$$(\mu_1 Y_{3,1} - Y_{3,2}) = c_2\mu_2(\mu_1 - \mu_2) + c_3\mu_3(\mu_1 - \mu_3) \tag{3.10}$$

If we multiply equation (3.9) with μ_2 then subtract equation (3.10), we get

$c_3 = \dfrac{\mu_1\mu_2 Y_{3,0} - (\mu_1 + \mu_2)Y_{3,1} + Y_{3,2}}{(\mu_3 - \mu_1)(\mu_3 - \mu_2)}$.

Now, from (3.9), we have

$c_2 = \dfrac{\mu_1\mu_3 Y_{3,0} - (\mu_1 + \mu_3)Y_{3,1} + Y_{3,2}}{(\mu_2 - \mu_1)(\mu_2 - \mu_3)}$,

and from (3.6), we have

$c_1 = \dfrac{\mu_2\mu_3 Y_{3,0} - (\mu_2 + \mu_3)Y_{3,1} + Y_{3,2}}{(\mu_1 - \mu_2)(\mu_1 - \mu_3)}$.

Put the values of $c_1, c_2, c_3, Y_{3,0}, Y_{3,1}$ and $Y_{3,2}$ in the equation (3.5) and we get required result. □

Theorem 3.5. *(Generating Function). The Generating function for the sequence $Y_{3,n}$ is given by*

$$\sum_{n=0}^{\infty} Y_{3,n}t^n = \frac{j + [(g+h) - u^2 j]t + [(2g+1) - uvj - u^2(g+h)]t^2}{1 - u^2t - uvt^2 - w^2t^3}.$$

Proof. Let $g(t) = \sum_{n=0}^{\infty} Y_{3,n}t^n$, then from equation (3.1) by multiplying t^{n+3} with the summation from 0 to ∞, we get

$[g(t) - Y_{3,0} - Y_{3,1}t - Y_{3,2}t^2] - u^2[g(t) - Y_{3,0} - Y_{3,1}t]t - uv(g(t) - Y_{3,0})t^2 - w^2 g(t)t^3 = 0$

$g(t)[1 - u^2t - uvt^2 - w^2t^3] - Y_{3,0}(1 - u^2t - uvt^2) - Y_{3,1}(t - u^2t^2) - Y_{3,2}t^2 = 0$

$g(t) = \dfrac{Y_{3,2}t^2 + Y_{3,0}(1 - u^2t - uvt^2) + Y_{3,1}(t - u^2t^2)}{1 - u^2t - uvt^2 - w^2t^3}$

$g(t) = \dfrac{Y_{3,0} + (Y_{3,1} - u^2Y_{3,0})t + (Y_{3,2} - uvY_{3,0} - u^2Y_{3,1}t^2)}{1 - u^2t - uvt^2 - w^2t^3}$.

Thus,

$$\sum_{n=0}^{\infty} Y_{3,n}t^n = \frac{j + [(g+h) - u^2 j]t + [(2g+1) - uvj - u^2(g+h)]t^2}{1 - u^2t - uvt^2 - w^2t^3}. \qquad \square$$

Proposition 3.1. *The explicit sum formula for the sequence $Y_{3,n}$ is given below*
$$Y_{3,n} = jA + [(g+h) - u^2 j]B + [(2g+1) - u^2(g+h) - uvj]C,$$
where,
$$A = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{q=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-k-2q}{k+q} \binom{k+q}{q} (u^2)^{n-2k-3q} (uv)^k (w^2)^q,$$
$$B = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{q=0}^{\lfloor \frac{n-1}{3} \rfloor} \binom{n-k-2q-1}{k+q} \binom{k+q}{q} (u^2)^{n-2k-3q-1} (uv)^k (w^2)^q,$$
$$C = \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \sum_{q=0}^{\lfloor \frac{n-2}{3} \rfloor} \binom{n-k-2q-2}{k+q} \binom{k+q}{q} (u^2)^{n-2k-3q-2} (uv)^k (w^2)^q.$$

Proof. By the Generating function, we have

$$\sum_{n=0}^{\infty} Y_{3,n} t^n = [Y_{3,0} + (Y_{3,1} - u^2 Y_{3,0}])t + (Y_{3,2} - uvY_{3,0} - u^2 Y_{3,1})t^2][1 - (u^2 + uvt + w^2t^2)t]^{-1}$$
$$= [Y_{3,0} + (Y_{3,1} - u^2 Y_{3,0})t + (Y_{3,2} - uvY_{3,0} - u^2 Y_{3,1})t^2] \sum_{n=0}^{\infty} (u^2 + uvt + w^2t^2)^n t^n$$
$$= [Y_{3,0} + (Y_{3,1} - u^2 Y_{3,0})t + (Y_{3,2} - uvY_{3,0} - u^2 Y_{3,1})t^2] \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} \binom{n}{k} (u^2)^{n-k} (uvt + w^2t^2)^k$$
$$= [Y_{3,0} + (Y_{3,1} - u^2 Y_{3,0}t + (Y_{3,2} - uvY_{3,0} - u^2 Y_{3,1})t^2] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (u^2)^{n-k} (uv + w^2t)^k t^{n+k}$$
$$= [Y_{3,0} + (Y_{3,1} - u^2 Y_{3,0})t + (Y_{3,2} - uvY_{3,0} - u^2 Y_{3,1})t^2] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{q=0}^{k} \binom{n}{k} \binom{k}{q} (u^2)^{n-k} (uv)^{k-q} (w^2t)^q t^{n+k}$$
$$= [Y_{3,0} + (Y_{3,1} - u^2 Y_{3,0})t + (Y_{3,2} - uvY_{3,0} - u^2 Y_{3,1})t^2] \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{q=0}^{k} \binom{n}{k} \binom{k}{q} (u^2)^{n-k} (uv)^{k-q} (w^2)^q t^{n+k+q}.$$

Now replacing n with $n+k$ and k with $k+q$, we get

$$= [Y_{3,0} + (Y_{3,1} - u^2 Y_{3,0})t + (Y_{3,2} - uvY_{3,0} - u^2 Y_{3,1})t^2] \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} \binom{n+k}{k+q} \binom{k+q}{q} (u^2)^{n-q} (uv)^k (w^2)^q t^{n+2k+2q}$$
$$= [Y_{3,0} + (Y_{3,1} - u^2 Y_{3,0})t + (Y_{3,2} - uvY_{3,0} - u^2 Y_{3,1})t^2] \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{q=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-k-2q}{k+q} \binom{k+q}{q} (u^2)^{n-2k-3q} (uv)^k (w^2)^q t^n.$$

Thus, the sum equals to

$$\sum_{n=0}^{\infty} [Y_{3,0} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{q=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-k-2q}{k+q} \binom{k+q}{q} (u^2)^{n-2k-3q} (uv)^k (w^2)^q t^n]$$
$$+ \sum_{n=0}^{\infty} [(Y_{3,1} - u^2 Y_{3,0}) \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{q=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n-k-2q}{k+q} \binom{k+q}{q} (u^2)^{n-2k-3q} (uv)^k (w^2)^q t^{n+1}]$$

$$+\sum_{n=0}^{\infty}[(Y_{3,2}-u^2Y_{3,1}-uvY_{3,0})\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{q=0}^{\lfloor\frac{n}{3}\rfloor}\binom{n-k-2q}{k+q}\binom{k+q}{q}(u^2)^{n-2k-3q}(uv)^k(w^2)^q t^{n+2}].$$

Equating the coefficient of t^n on both sides, we get

$$\begin{aligned}Y_{3,n} &= Y_{3,0}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{q=0}^{\lfloor\frac{n}{3}\rfloor}\binom{n-k-2q}{k+q}\binom{k+q}{q}(u^2)^{n-2k-3q}(uv)^k(w^2)^q\\ &+(Y_{3,1}-u^2Y_{3,0})\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{q=0}^{\lfloor\frac{n-1}{3}\rfloor}\binom{n-k-2q-1}{k+q}\binom{k+q}{q}(u^2)^{n-2k-3q-1}(uv)^k(w^2)^q\\ &+(Y_{3,2}-u^2Y_{3,1}-uvY_{3,0})\sum_{k=0}^{\lfloor\frac{n-2}{2}\rfloor}\sum_{q=0}^{\lfloor\frac{n-2}{3}\rfloor}\binom{n-k-2q-2}{k+q}\binom{k+q}{q}(u^2)^{n-2k-3q-2}(uv)^k(w^2)^q.\end{aligned}$$

□

Proposition 3.2. *(Sum of the first n terms). The sum of the first n^{th} terms of the extended generalized Tribonacci polynomials given as follows*

$$\sum_{k=0}^{n-1}Y_{3,k}(x)=\frac{Y_{3,n+1}+(1-u^2)Y_{3,n}+w^2Y_{3,n-1}-[Y_{3,2}-(u^2-1)Y_{3,1}-uvY_{3,0}]+(u^2-1)Y_{3,0}}{u^2+uv+w^2-1}.$$

Proof. Using Binet's formula, we get

$$\sum_{k=0}^{n-1}Y_{3,k}=\sum_{k=0}^{n-1}(A\mu_1^k+B\mu_2^k+C\mu_3^k)$$

where
$A=\frac{\mu_2\mu_3Y_{3,0}-(\mu_2+\mu_3)Y_{3,1}+Y_{3,2}}{(\mu_1-\mu_2)(\mu_1-\mu_3)}$, $B=\frac{\mu_1\mu_3Y_{3,0}-(\mu_1+\mu_3)Y_{3,1}+Y_{3,2}}{(\mu_2-\mu_1)(\mu_2-\mu_3)}$, $C=\frac{\mu_1\mu_2Y_{3,0}-(\mu_1+\mu_2)Y_{3,1}+Y_{3,2}}{(\mu_3-\mu_1)(\mu_3-\mu_2)}$.
It follows that

$$\begin{aligned}\sum_{k=0}^{n-1}Y_{3,k} &= A\sum_{k=0}^{n-1}\mu_1^k(x)+B\sum_{k=0}^{n-1}\mu_2^k(x)+C\sum_{k=0}^{n-1}\mu_3^k(x)\\ &= A\frac{\mu_1^n-1}{\mu_1-1}+B\frac{\mu_2^n-1}{\mu_2-1}+C\frac{\mu_3^n-1}{\mu_3-1}\\ &= \frac{A(\mu_1^n-1)(\mu_2-1)(\mu_3-1)+B(\mu_2^n-1)(\mu_1-1)(\mu_3-1)+C(\mu_3^n-1)(\mu_1-1)(\mu_2-1)}{(\mu_1-1)(\mu_2-1)(\mu_3-1)}\\ &= \frac{(u^2-1)(A+B+C)-(A\mu_1+B\mu_2+\mu_3)-(A\mu_2\mu_3+B\mu_1\mu_3+C\mu_1\mu_2)}{\mu_1\mu_2\mu_3+(\mu_1+\mu_2+\mu_3)-(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)-1}\\ &+\frac{(1-u^2)(A\mu_1^n+B\mu_2^n+C\mu_3^n)+(A\mu_1^{n+1}+B\mu_2^{n+1}+C\mu_3^{n+1})+\mu_1\mu_2\mu_3(A\mu_1^{n-1}+B\mu_2^{n-1}+C\mu_3^{n-1})}{\mu_1\mu_2\mu_3+(\mu_1+\mu_2+\mu_3)-(\mu_1\mu_2+\mu_2\mu_3+\mu_3\mu_1)-1}.\end{aligned}$$

After solving, we get

$\sum_{k=0}^{n-1}Y_{3,k}(x)=\frac{Y_{3,n+1}+(1-u^2)Y_{3,n}+w^2Y_{3,n-1}-[Y_{3,2}-(u^2-1)Y_{3,1}-uvY_{3,0}]+(u^2-1)Y_{3,0}}{u^2+uv+w^2-1}$. □

4. Conclusion

In this study, we generalized the Fibonacci polynomial sequence to obtain an extended generalized Fibonacci sequence, then we proved some identities like Binet's formula, Generating function, Catalan's identity, explicit sum formula and matrix properties. Further, we introduced a generalization of the Tribonacci polynomial sequence to obtain an extended generalized Tribonacci polynomial sequence, then investigated some identities like Binet's formula, generating function, explicit sum formula and matrix properties.

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