

SMALL FUNCTIONS IN DISKS OF $\mathbb{C}_p$

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ABSTRACT. Small functions were defined in complex analysis and next in ultrametric analysis. Order of growth and type of growth were also defined in complex analysis and have a similar definition in ultrametric analysis. Here we compare these two notions in the same way, on a complete ultrametric algebraically closed field $\mathbb{K}$ of characteristic 0 such as $\mathbb{C}_p$. Small functions with respect to an entire function f were studied in several articles. Inside an "open" disk, small functions also exist. After a general study, here we examine how two analytic functions inside an open disk can share three small functions, ignoring multiplicity and we give sufficient conditions proving that these two functions are equal.

1. INTRODUCTION AND GENERAL PROPERTIES OF THE GROWTH

In complex analysis, a notion of small functions with respect to a "big" holomorphic function was introduced and particularly applied to the Nevanlinna theory [2], [5]. On the other hand, as far as entire functions are concerned, a notion of order of growth was also examined and led to another kind of small function [6], [8], [10]. These studies suggested symmetric studies in ultrametric analysis [1], [2], [3], [4], [6], [8], [10]. Then, inside an open disk of $\mathbb{C}_p$, it is also possible to define an order of growth and to make the same kind of work, defining small functions in a classical way in an open disk E and then defining the order of growth in E and comparing the order of growth of two analytic functions in E [2]. Here we will first recall result obtained in [2]. Thus, two notions of "small function" with respect to a "big function" appear and we can compare them, as it was done for entire functions [7]. Next, when 3 functions are shared (ignoring multiplicity) by two functions, we can obtain criteria proving that the shared functions are small and hence satisfy the hypothesis of known theorems.

Notations and definitions. We denote by $\mathbb{K}$ a complete algebraically closed field of characteristic 0 such as $\mathbb{C}_p$ and given $a \in \mathbb{K}$ and $r > 0$, we denote by $d(a, r)$ the disk $\{x \in \mathbb{K} \mid |x - a| \leq r\}$ and by $d(a, r^-)$ the disk $\{x \in \mathbb{K} \mid |x - a| < r\}$. More precisely throughout the paper, we put $E = d(0, R^-)$ where R is a fixed positive radius.

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Given $r_1, r_2 \in \mathbb{R}$ such that $0 < r_1 < r_2$, we denote by $\Gamma(0, r_1, r_2)$ the annulus $\{x \in \mathbb{K} \mid r_1 < |x| < r_2\}$. We denote by Log the Napierian logarithm.

Now, we denote by $\mathcal{A}(E)$ the $\mathbb{K}$ -algebra of analytic functions in E i.e. the set of power series $\sum_{n=0}^{\infty} a_n x^n$ converging in E and by $\mathcal{A}_{ub}(E)$ the set of $f \in \mathcal{A}(E)$ that are unbounded in E [4], [7]. According to classical notation [4], given $f \in \mathcal{A}(E)$ we put $|f|(r) = \sup\{|f(x)| \mid |x| \leq r\}$, ($0 < r < R$) [4]. Then $|\cdot|$ is an ultrametric multiplicative norm on $\mathcal{A}(E)$, and $|f|(r) = \lim_{|x| \rightarrow r, |x| \neq r} |f(x)|$ [4].

Now, given $f \in \mathcal{A}(E)$, we define

$$\rho(f) = \limsup_{r \rightarrow R^-} \frac{\text{Log}(\text{Log}(|f|(r)))}{(-\text{Log}(R-r))}, \quad \tilde{\rho}(f) = \liminf_{r \rightarrow R^-} \frac{\text{Log}(\text{Log}(|f|(r)))}{(-\text{Log}(R-r))}.$$

Next, if $0 < \rho(f) < +\infty$, we put $\sigma(f, r) = \text{Log}(|f|(r)(R-r)^{\rho(f)})$, $\sigma(f) = \limsup_{r \rightarrow R^-} \sigma(f, r)$ and $\tilde{\sigma}(f) = \liminf_{r \rightarrow R^-} \sigma(f, r)$. Then, $\rho(f)$ is called *the order of growth of f* , $\tilde{\rho}(f)$ is called *the lower order of growth of f* , $\sigma(f)$ is called *the type of growth of f* .

On the other hand, for every $r \in]0, R[$, we denote by $s(r, f)$ the number of zeros of f in $d(0, r)$, taking multiplicity into account. If the set of the $y > 0$ such that $\lim_{r \rightarrow R^-} s(r, f)(R-r)^y = 0$ is empty, we put $\theta(f) = +\infty$. We then denote by $\theta(f)$ the lower bound of the $y > 0$ such that $\lim_{r \rightarrow R^-} s(r, f)(R-r)^y = 0$. Similarly, if the set of the $y > 0$ such that $\lim_{r \rightarrow R^-} \text{Log}(|f|(r))(R-r)^y = 0$ is empty, we put $\lambda(f) = +\infty$. Also, we denote by $\lambda(f)$ the lower bound of the $y > 0$ such that $\lim_{r \rightarrow R^-} \text{Log}(|f|(r))(R-r)^y = 0$. And if $0 < \rho(f) < +\infty$, we put $\psi(f, r) = s(r, f)(R-r)^{\rho(f)}$ and $\psi(f) = \limsup_{r \rightarrow R^-} \psi(f, r)$.

We call $\psi(f)$ *the cotype of growth of f* .

A function $f \in \mathcal{A}(E)$ is said to be *regular* if $\tilde{\rho}(f) = \rho(f)$ and it is said to be *clean* if $\tilde{\sigma}(f) = \sigma(f)$.

We can now state some results on the function ρ [2]. Theorem 1.1 is easily proved.

Theorem 1.1. *Let $f, g \in \mathcal{A}(E)$. Then,*

- i) $\rho(f+g) \leq \max(\rho(f), \rho(g))$ and if $\rho(g) < \rho(f)$, then $\rho(f+g) = \rho(f)$,
- ii) $\rho(fg) = \max(\rho(f), \rho(g))$.

Theorem 1.2. *Let $f \in \mathcal{A}_{ub}(E)$. Then, $\forall n \in \mathbb{N}^*$, $\rho(f^n) = \rho(f)$. Let $P \in \mathbb{K}[x]$. Then $\rho(P \circ f) = \rho(f)$.*

Proof. Let $n = \deg(P)$. For r close enough to R , we have

$$\begin{aligned} \text{Log}(\text{Log}(|f|(r))) &\leq \text{Log}(\text{Log}(|P \circ f|(r))) \leq \text{Log}((n+1)\text{Log}(|f|(r))) \\ &= \text{Log}(n+1) + \text{Log}(\text{Log}(|f|(r))). \end{aligned}$$

Consequently, the statement is immediate. $\square$

Definitions and notations: Given $t > 0$, we denote by $\widehat{\mathcal{A}}_t(E)$ the set of functions $h \in \mathcal{A}(E)$ such that $\rho(h) < t$. Given $f, h \in \mathcal{A}(E)$, h is called a *small function with respect to f* if $\lim_{r \rightarrow R^-} \frac{\text{Log}(|h|(r))}{\text{Log}(|f|(r))} = 0$ and we denote by $\mathcal{A}_f(E)$ the set of functions $h \in \mathcal{A}(E)$ that are small functions with respect to f .

On the other hand we will say that a function $h \in \mathcal{A}(E)$ is a *very small function with respect to a function $f \in \mathcal{A}(E)$* if $\rho(h) < \widetilde{\rho}(f)$. Given $t > 0$, we denote by $\widehat{\mathcal{A}}_t(E)$ the set of $h \in \mathcal{A}(E)$ such that $\rho(h) < t$.

By Theorem 1.1 we have the following corollary.

Corollary 1.1. *For any $t > 0$, $\widehat{\mathcal{A}}_t(E)$ is a $\mathbb{K}$ -subalgebra of $\mathcal{A}(E)$.*

Theorem 1.3. *Let $f(x) \in \mathcal{A}(E)$ be such that $\widetilde{\sigma}(f) > 0$. Then f is regular.*

Proof. Suppose first that $\rho(f) \neq 0$. Let $a = \widetilde{\sigma}(f)$, and $b = \frac{a}{2}$. Then,

$$a = \liminf_{r \rightarrow R^-} |f|(r)(R-r)^{\rho(f)},$$

hence there exists $R' \in]0, R[$ such that $\text{Log}(|f|(r))(R-r)^{\rho(f)} \geq b \forall r \in]R', R[$, hence $\text{Log}(|f|(r))(R-r)^{\rho(f)} > b \forall r >]R', R[$ and hence

$$\text{Log}(\text{Log}(|f|(r))) + \rho(f)\text{Log}(R-r) \geq \text{Log}(b) \forall r \in]R', R[.$$

Consequently,

$$\text{Log}(\text{Log}(|f|(r))) \geq (-\rho(f)\text{Log}(R-r)) + \text{Log}(b) \forall r \in]R', R[,$$

and hence

$$\frac{\text{Log}(\text{Log}(|f|(r)))}{-\text{Log}(R-r)} \geq \rho(f) + \frac{\text{Log}(b)}{-\text{Log}(R-r)}.$$

When r tends to R , we can see that $\widetilde{\rho}(f) \geq \rho(f)$, hence f is regular. $\square$

Corollary 1.2. *Let $f(x) \in \mathcal{A}(E)$ be clean and such that $\sigma(f) > 0$. Then f is regular.*

Theorem 1.4. *Let $f, g \in \mathcal{A}(E)$. Then $\sigma(fg) \leq \sigma(f) + \sigma(g)$. If $\rho(f) \geq \rho(g)$, then $\sigma(f) \leq \sigma(fg)$. If $\rho(f) = \rho(g)$, then $\max(\sigma(f), \sigma(g)) \leq \sigma(fg)$.*

If $\rho(f) = \rho(g)$ and $\sigma(f) > \sigma(g)$, then $\sigma(f+g) \geq \sigma(f)$. If $\rho(f+g) = \rho(f) \geq \rho(g)$, then $\sigma(f+g) \leq \max(\sigma(f), \sigma(g))$.

Proof. Let $y = \rho(f)$, $t = \rho(g)$ and suppose $y \geq t$. When r is close enough to R , we have $\max(\text{Log}(|f|(r)), \text{Log}(|g|(r))) \leq \text{Log}(|f.g|(r)) = \text{Log}(|f|(r)) + \text{Log}(|g|(r))$ and by Theorem 1.1, we have $\rho(fg) = y$. Therefore,

$$\begin{aligned} \sigma(fg) &= \limsup_{r \rightarrow R^-} (\text{Log}(|f.g|(r))(R-r)^u) \\ &\leq \limsup_{r \rightarrow R^-} (\text{Log}(|f|(r))(R-r)^u) + \limsup_{r \rightarrow R^-} (\text{Log}(|g|(r))(R-r)^t) = \sigma(f) + \sigma(g). \end{aligned}$$

On the other hand,

$$\sigma(f) = \limsup_{r \rightarrow R^-} \text{Log}(|f|(r))(R-r)^u \leq \limsup_{r \rightarrow +R^-} (\text{Log}(|fg|(r))(R-r)^u).$$

But $\rho(fg) = y$, hence $\sigma(f) \leq \sigma(fg)$. Particularly, if $\rho(f) = \rho(g)$, then

$$\max(\sigma(f), \sigma(g)) \leq \sigma(fg).$$

Now, suppose again that $\rho(f) = \rho(g) = y$ and suppose $\sigma(f) > \sigma(g)$. Let $y = \rho(f)$, $b = \sigma(f)$. Then, $b > 0$. Let $(r_n)_{n \in \mathbb{N}}$ be a sequence such that $\lim_{n \rightarrow +\infty} r_n = R$ and $\lim_{n \rightarrow +\infty} (\text{Log}(|f|(r_n))(R-r_n)^y) = b$. Since $\sigma(g) < \sigma(f)$, we notice that when n is big enough we have $|g|(r_n) < |f|(r_n)$. Consequently, when n is big enough, we have $|f+g|(r_n) = |f|(r_n)$ and hence

$$\lim_{n \rightarrow +\infty} (\text{Log}(|f+g|(r_n))(R-r_n)^y) = b. \quad (1.1)$$

By definition of σ we have $\sigma(f+g) \geq \lim_{n \rightarrow +\infty} (\text{Log}(|f+g|(r_n))(R-r_n)^{\rho(f+g)})$, and by Theorem 1.1, we have $\rho(f+g) \leq y$, hence

$$\begin{aligned} \sigma(f+g) &\geq \lim_{n \rightarrow +\infty} (\text{Log}(|f+g|(r_n))(R-r_n)^{\rho(f+g)}) \geq \lim_{n \rightarrow +\infty} (\text{Log}(|f+g|(r_n))(R-r_n)^y) \\ &= \lim_{n \rightarrow +\infty} \text{Log}(|f|(r_n))(R-r_n)^y = \sigma(f) \end{aligned}$$

therefore, by (1.1), $\sigma(f+g) \geq \sigma(f)$. Finally, suppose now that $\rho(f+g) = \rho(f) \geq \rho(g)$. Let $y = \rho(f)$ and $t = \rho(g)$. Then,

$$\begin{aligned} \sigma(f+g) &= \limsup_{r \rightarrow R^-} (\text{Log}(|f+g|(r))(R-r)^y) \\ &\leq \max \left(\limsup_{r \rightarrow R^-} (\text{Log}(|f|(r))(R-r)^y), \limsup_{r \rightarrow R^-} (\text{Log}(|g|(r))(R-r)^y) \right) \\ &\leq \max \left(\limsup_{r \rightarrow R^-} (\text{Log}(|f|(r))(R-r)^y), \limsup_{r \rightarrow R^-} (\text{Log}(|g|(r))(R-r)^t) \right) \\ &= \max(\sigma(f), \sigma(g)), \end{aligned}$$

which ends the proof. $\square$

Corollary 1.3. *Let $f, g \in \mathcal{A}(E)$ be such that $\rho(f) \neq \rho(g)$. Then,*

$$\sigma(f+g) \leq \max(\sigma(f), \sigma(g)).$$

Theorem 1.5. *Let $f(x) \in \mathcal{A}(E)$. Then, $\mathcal{A}_f(E)$ is a $\mathbb{K}$ -subalgebra of $\mathcal{A}(E)$.*

Proof. Suppose first $f \in \mathcal{A}(E)$ and let $g, h \in \mathcal{A}_f(E)$. Consider

$$\text{Log}(|g+h|(r)) \leq \text{Log}(|g|(r) + |h|(r)) \leq \text{Log}(|g|(r)) + \text{Log}(|h|(r)).$$

But by hypotheses,

$$\lim_{r \rightarrow R^-} \frac{\text{Log}(|g|(r))}{\text{Log}(|f|(r))} = \lim_{r \rightarrow R^-} \frac{\text{Log}(|h|(r))}{\text{Log}(|f|(r))} = 0,$$

hence $g+h \in \mathcal{A}_f(E)$.

Next, $\text{Log}(|gh|(r)) \leq \text{Log}(|g|(r)|h|(r)) = \text{Log}(|g|(r)) + \text{Log}(|h|(r))$ hence

$$\lim_{r \rightarrow R^-} \frac{\text{Log}(|gh|(r))}{\text{Log}(|f|(r))} = 0,$$

which shows that $gh \in \mathcal{A}_f(E)$. Consequently, $\mathcal{A}_f(E)$ is a subalgebra of $\mathcal{A}(E)$. $\square$

Theorem 1.6. *Let $f \in \mathcal{A}(E)$ be such that $\tilde{\rho}(f) > 0$. Then the $\mathbb{K}$ -algebra $\widehat{\mathcal{A}_{\tilde{\rho}(f)}}(E)$ is included in $\mathcal{A}_f(E)$.*

Proof. Let $g \in \mathcal{A}(E)$ be such that $\rho(g) = \tilde{\rho}(f) - b$, with $b > 0$. There exists $R' < R$ such that $\frac{\text{Log}(\text{Log}(|f|(r)))}{-\text{Log}(R-r)} \geq \tilde{\rho}(f) - b \forall r \in [R', R[$. Consequently when r tends to R , we have

$$\frac{\text{Log}(\text{Log}(|g|(r)))}{-\text{Log}(R-r)} \leq \frac{\text{Log}(\text{Log}(|f|(r)))}{-\text{Log}(R-r)} - b$$

hence $\text{Log}(\text{Log}(|g|(r))) \leq \text{Log}(\text{Log}(|f|(r))) - b(-\text{Log}(R-r))$. Therefore,

$$\text{Log}(|g|(r)) \leq (\text{Log}(|f|(r))(R-r)^b,$$

which leads to

$$\frac{\text{Log}(|g|(r))}{\text{Log}(|f|(r))} \leq (R-r)^b.$$

This proves that $g \in \mathcal{A}_f(E)$. $\square$

Remark 1.1. In other words, very small functions with respect to f are small functions with respect to f . As is shown in Remark 1.2 below, the converse does not hold in the general case.

Corollary 1.4. *Let $f(x) \in \mathcal{A}(E)$ be regular. Then, the $\mathbb{K}$ -algebra $\widehat{\mathcal{A}_{\rho(f)}}(E)$ is included in $\mathcal{A}_f(E)$.*

Corollary 1.5. *Let $f(x) \in \mathcal{A}(E)$ be such that $\tilde{\sigma}(f) > 0$. Then, $\widehat{\mathcal{A}_{\tilde{\rho}(f)}}(E)$ is a $\mathbb{K}$ -subalgebra of $\mathcal{A}_f(E)$.*

Corollary 1.6. *Let $f(x) \in \mathcal{A}(E)$ be clean and such that $\sigma(f) > 0$. Then, $\widehat{\mathcal{A}_{\rho(f)}}(E)$ is a $\mathbb{K}$ -subalgebra of $\mathcal{A}_f(E)$.*

Theorem 1.7. *Let $f \in \mathcal{A}(E)$ be clean, such that $\sigma(f) > 0$ and let $h \in \mathcal{A}(E)$ be such that $\rho(f) = \rho(h)$. Then, $h \in \mathcal{A}_f(E)$ if and only if $\sigma(h) = 0$.*

Proof. Let $t = \rho(f)$. Since f is clean, we have

$$\frac{\sigma(h)}{\sigma(f)} = \frac{\limsup_{r \rightarrow R^-} \text{Log}(|h|(r))(R-r)^t}{\lim_{r \rightarrow R^-} \text{Log}(|f|(r))(R-r)^t} = \limsup_{r \rightarrow R^-} \frac{\text{Log}(|h|(r))}{\log(|f|(r))}.$$

Now, h lies in $\mathcal{A}_f(E)$ if and only if $\limsup_{r \rightarrow R^-} \frac{\text{Log}(|h|(r))}{\log(|f|(r))} = 0$, i.e., $\sigma(h) = 0$. $\square$

Thanks to the classical inequality $|f'|(r) \leq \frac{|f|(r)}{r}$ [4], the following theorem is then immediate.

Theorem 1.8. *Suppose $\mathbb{K}$ has characteristic 0. Let $f \in \mathcal{A}(E)$. Then $\rho(f') \leq \rho(f)$.*

Remark 1.2. In a field of characteristic $p \neq 0$, certain analytic functions have a null derivative. This is why we must suppose that $\mathbb{K}$ has characteristic 0 in all statements involving derivatives.

In complex analysis, many estimates were given concerning the growth order of solutions of linear differential equations. Here, from Theorems 1.1 and 1.8 we can immediately obtain the following corollary.

Corollary 1.7. *Suppose $\mathbb{K}$ has characteristic 0. Consider the differential equation*

$$(\mathcal{E}) \quad f^{(n)} + a_{n-1}(x)f^{(n-1)}(x) + \dots + a_0(x)f(x) = 0$$

with $a_j \in \mathcal{A}(E)$, $j = 0, \dots, n-1$ and $\rho(a_j) < \rho(a_0) \quad \forall j = 1, \dots, n-1$. Then every non-trivial solution f of $(\mathcal{E})$ satisfies $\rho(f) \geq \rho(a_0)$.

In [4], the following theorem is easily shown.

Theorem 1.9. *Suppose $\mathbb{K}$ has residue characteristic 0. Then for every $f \in \mathcal{A}(E)$ we have $\rho(f') = \rho(f)$, $\theta(f') = \theta(f)$, $\sigma(f') = \sigma(f)$ and $\psi(f') = \psi(f)$.*

Remark 1.3. Theorem 1.9 does not hold for residue characteristic $p > 0$ because there exist functions $f \in \mathcal{A}(E)$ such that $\rho(f) > 0$ and that f' is bounded, as shows the following example with $R = 1$: $g(x) = \sum_{m=0}^{\infty} \frac{x^{p^m}}{p^m}$. We can see that $g'(x) = \sum_{n=0}^{\infty} x^{p^m-1}$ hence g' is bounded and therefore $\rho(g') = 0$. However, consider the sequence $(r_m)_{m \in \mathbb{N}}$ defined as $r_m = 1 - \frac{1}{p^m}$. We can check that $|g|(r_m) \geq p^m(r_m)^{p^m}$, hence

$$\text{Log}(|g|(r_m)) \geq m + p^m \text{Log}(r_m) = m + p^m \text{Log}\left(1 - \frac{1}{p^m}\right).$$

When m is big enough, we have $\text{Log}\left(1 - \frac{1}{p^m}\right) \geq \frac{-2}{p^m}$, hence

$$\text{Log}(|g|(r_m)) \geq m - p^m \left(\frac{2}{p^m}\right) = m - 2.$$

Therefore, when m is big enough, we have

$$\frac{\text{Log}(\text{Log}(|g|(r_m)))}{-\text{Log}(r_m)} \geq \frac{\text{Log}(m-2)}{-\text{Log}\left(1 - \frac{1}{p^m}\right)} > \frac{\text{Log}(m-2)}{\frac{2}{p^m}} = \frac{p^m}{2} \text{Log}(m-2).$$

Thus, we have $\rho(g) = +\infty$.

Theorem 1.10. *Let $f \in \mathcal{A}(E)$. Then $\lambda(f) = \rho(f)$.*

Proof. First we will prove that $\rho(f) \leq \lambda(f)$. Obviously, we can assume that $\lambda(f) < +\infty$. Let y be such that $\lim_{r \rightarrow R^-} \text{Log}(|f|(r))(R-r)^y = 0$. Let us fix $\varepsilon > 0$. For r close enough to R , we have $\text{Log}(|f|(r))(R-r)^y \leq \varepsilon$, hence $\text{Log}(|f|(r)) \leq \frac{\varepsilon}{(R-r)^y}$, therefore $\text{Log}(\text{Log}(|f|(r))) \leq \text{Log} \varepsilon - y \text{Log}(R-r)$ hence

$$\frac{\text{Log}(\text{Log}(|f|(r)))}{(-\text{Log}(R-r))} \leq \frac{\text{Log}(\varepsilon)}{(-\text{Log}(R-r))} + y,$$

and hence

$$\limsup_{r \rightarrow R^-} \frac{\text{Log}(\text{Log}(|f|(r)))}{(-\text{Log}(R-r))} \leq y,$$

i.e., $\rho(f) \leq y$. This is true for every y such that $\lim_{r \rightarrow R^-} \text{Log}(|f|(r))(R-r)^y = 0$ and hence $\rho(f) \leq \lambda(f)$.

On the other hand, we notice that, by definition of $\lambda(f)$, either $\lambda(f) = 0$ and then $\lambda(f) \leq \rho(f)$, or

$$\lambda(f) = \sup\{y \in]0, +\infty[\mid \limsup_{r \rightarrow R^-} \text{Log}(|f|(r))(R-r)^y > 0\}.$$

Thus, suppose that $\lambda(f) > 0$. Let us take $y \in]0, \lambda(f)[$. We have a number $b > 0$ such that

$$\limsup_{r \rightarrow R^-} (\text{Log}(|f|(r))(R-r)^y) \geq b > 0.$$

Let us fix $\varepsilon \in]0, b[$. There exists a sequence $(r_n)_{n \in \mathbb{N}}$ in $]0, R[$ such that $\lim_{n \rightarrow +\infty} r_n = R$ and such that, when n is big enough, we have $b - \varepsilon \leq \text{Log}(|f|(r_n))(R-r_n)^y$, hence $-y \text{Log}(R-r_n) + \text{Log}(b - \varepsilon) < \text{Log}(\text{Log}(|f|(r_n)))$ therefore

$$y + \frac{\text{Log}(b - \varepsilon)}{(-\text{Log}(R-r_n))} \leq \frac{\text{Log}(\text{Log}(|f|(r_n)))}{(-\text{Log}(R-r_n))}.$$

Consequently, $\limsup_{n \rightarrow +\infty} \frac{\text{Log}(\text{Log}(|f|(r_n)))}{(-\text{Log}(R-r_n))} \geq y$, therefore $\rho(f) \geq y$. But this holds for every $y < \lambda(f)$. Thus, $\rho(f) \geq \lambda(f)$ and finally, $\rho(f) = \lambda(f)$. $\square$

Remark 1.4. Let $f \in \mathcal{A}(E)$ and $g \in \mathcal{A}_f(E)$. Can we tell that $g \in \mathcal{A}_{\rho(f)}(E)$? Suppose that $\rho(f) = 1$ and consider a function $h \in \mathcal{A}(E)$ such that $\text{Log}(|h|(r))$ is of the form $\omega(r) \text{Log}(|f|(r))$ with $\omega(r) \in [\frac{a}{\text{Log}(\text{Log}(|f|(r)))}, \frac{b}{\text{Log}(\text{Log}(|f|(r)))}]$ and $0 < a < b$. Then,

$$\frac{\text{Log}(\text{Log}(|h|(r)))}{\text{Log}(\text{Log}(|f|(r)))} = \frac{\text{Log}(\omega(r)) + \text{Log}(\text{Log}(|f|(r)))}{\text{Log}(\text{Log}(|f|(r)))} = \frac{\text{Log}(\omega(r))}{\text{Log}(\text{Log}(|f|(r)))} + 1$$

and that belongs to

$$\left[\frac{\text{Log}(a) - \text{Log}(\text{Log}(\text{Log}(|f|(r))))}{\text{Log}(\text{Log}(|f|(r)))} + 1, \frac{\text{Log}(b) - \text{Log}(\text{Log}(\text{Log}(|f|(r))))}{\text{Log}(\text{Log}(|f|(r)))} + 1 \right],$$

therefore

$$\lim_{r \rightarrow R^-} \frac{\text{Log}(\text{Log}(|h|(r)))}{\text{Log}(\text{Log}(|f|(r)))} = 1,$$

and hence $\rho(h) = \rho(f)$.

On the other hand, by definition,

$$\frac{\text{Log}(|h|(r))}{\text{Log}(|f|(r))} = \omega(r) \in \left[\frac{a}{\text{Log}(\text{Log}(|f|(r)))}, \frac{b}{\text{Log}(\text{Log}(|f|(r)))} \right],$$

hence $\lim_{r \rightarrow R^-} \frac{\text{Log}(|h|(r))}{\text{Log}(|f|(r))} = 0$ and hence h belongs to $\mathcal{A}_f(E)$.

Remark 1.5. When f is not regular, a function $h \in \widehat{\mathcal{A}_{\rho(f)}}(E)$ does not always lie in $\mathcal{A}_f(E)$. Indeed, suppose that $\rho(f) - \tilde{\rho}(f) = 2b$, with $b > 0$. We can construct a function $h \in \mathcal{A}(E)$ such that $\rho(h) \leq \rho(f) - b$ and such that there exists a sequence $(r_n)_{n \in \mathbb{N}}$ where $\lim_{n \rightarrow +\infty} r_n = R$, satisfying

$$\frac{\text{Log}(\text{Log}(|f|(r_n)))}{-\text{Log}(R - r_n)} \leq \frac{\text{Log}(\text{Log}(|h|(r_n)))}{-\text{Log}(R - r_n)} \leq \frac{\text{Log}(\text{Log}(|f|(r_n)))}{-\text{Log}(R - r_n)} - b$$

So, we have

$$\text{Log}(\text{Log}(|f|(r_n))) \leq \text{Log}(\text{Log}(|h|(r_n))) \leq \text{Log}(\text{Log}(|f|(r_n))) - b \text{Log}(r_n)$$

hence $\text{Log}(|f|(r_n)) \leq \text{Log}(|h|(r_n))$ and hence $1 \leq \frac{\text{Log}(|h|(r_n))}{\text{Log}(|f|(r_n))}$ which proves that $h \notin \mathcal{A}_f(E)$, although $h \in \widehat{\mathcal{A}_{\rho(f)}}(E)$.

Theorem 1.11. *Let $f \in \mathcal{A}(E)$. Then $\psi(fg) \leq \psi(f) + \psi(g)$. Moreover, if $\rho(f) = \rho(g)$ then $\max(\psi(f), \psi(g)) \leq \psi(fg)$.*

Proof. Set $\rho(f) = y$, $\rho(g) = t$. Without loss of generality we can assume $y \geq t$. By Theorem 1.1 we have $\rho(f.g) = \rho(f) = y$. Now, for each $r > 0$, we have $s(r, f.g) = s(r, f) + s(r, g)$ hence

$$\psi(fg) = \limsup_{r \rightarrow R^-} (s(r, f) + s(r, g))(R - r)^y \leq \limsup_{r \rightarrow R^-} s(r, f)(R - r)^y + \limsup_{r \rightarrow R^-} s(r, g)(R - r)^t$$

hence $\psi(fg) \leq \psi(f) + \psi(g)$. Now, suppose $y = t$. Then,

$$\begin{aligned} \psi(fg) &= \limsup_{r \rightarrow R^-} (s(r, f) + s(r, g))(R - r)^y \geq \limsup_{r \rightarrow R^-} \max(s(r, f), s(r, g))(R - r)^y \\ &= \max(\psi(f), \psi(g)). \end{aligned}$$

□

Remark 1.6. Let $f \in \mathcal{A}(E)$. If $y > \theta(f)$, then by definition, $\lim_{r \rightarrow R^-} s(r, f)(R - r)^y = 0$. But if $y < \theta(f)$, then $\limsup_{r \rightarrow R^-} s(r, f)(R - r)^y = +\infty$ because if $\limsup_{r \rightarrow R^-} s(r, f)(R - r)^y < +\infty$

$+\infty$, we can find $y' \in]y, \theta(f)[$ and then we can check that $\lim_{r \rightarrow R^-} s(r, f)(R - r)^{y'} = 0$, a contradiction.

2. COUNTING FUNCTIONS AND APPLICATIONS

We can now examine the deep role of the order of growth. For convenience, we need to define the counting function of zeros of an analytic function in the disk E : let $f \in \mathcal{A}(E)$. We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ defined in the following way.

Let (a_n) , $(1 \leq n \leq q(r))$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, of respective order s_n . We set $Z(r, f) = \max(\omega_0(f), 0)\text{Log}r + \sum_{n=1}^{q(r)} s_n(\text{Log}r - \text{Log}|a_n|)$ and so, $Z(r, f)$ is called *the counting function of zeros of f in $d(0, r)$, counting multiplicity*.

In order to define the counting function of zeros of f ignoring multiplicity, we put $\overline{\omega}_0(f) = 0$ if $\omega_0(f) = 0$ and $\overline{\omega}_0(f) = 1$ if $\omega_0(f) \geq 1$.

Now, we denote by $\overline{Z}(f, r)$ the counting function of zeros of f ignoring multiplicity: $\overline{Z}(f, r) = \overline{\omega}_0(f)\text{Log}r + \sum_{n=1}^{q(r)} (\text{Log}r - \text{Log}|a_n|)$ and so, $\overline{Z}(f, r)$ is called *the counting function of zeros of f in $d(0, r)$ ignoring multiplicity*.

And we denote by $Z^0(f', r)$ the counting function of the zeros of f' that are zeros of $f - a_n$ for any $n \leq q(r)$. Now, by Theorem B 13.24 in [4], we have the following theorem.

Theorem 2.1. *Let $f \in \mathcal{A}(E)$ and let $r_1, r_2 \in]0, R[$ satisfy $r_1 < r_2$. If f admits exactly q zeros in $d(0, r_1)$ (taking multiplicity into account) and t different zeros α_j , of respective multiplicity order m_j ($1 \leq j \leq t$) in $\Gamma(0, r_1, r_2)$, then f satisfies*

$$Z(r_2, f) - Z(r_1, f) = \sum_{j=1}^t m_j(\text{Log}(r_2) - \text{Log}(|\alpha_j|)) + q(\text{Log}(r_2) - \text{Log}(r_1)).$$

Corollary 2.1. *Let $f(x) \in \mathcal{A}(d(0, R^-))$ be such that $f(0) \neq 0$, let $r \in]0, R[$ and let a_j , $1 \leq j \leq q$ be the zeros of f in $d(0, r)$, of respective multiplicity m_j . Then,*

$$Z(r, f) = \text{Log}(|f(0)|) + \sum_{j=1}^q m_j(\text{Log}(r) - \text{Log}(|a_j|)).$$

Corollary 2.2. (Schwarz Lemma) *Let $D = d(0, s)$, let $f \in \mathcal{A}(E)$ have at least (resp. at most) q zeros in $d(0, r)$ with $q > 0$ and $0 < r < s$. Then, we have $\frac{|f|(s)}{|f|(r)} \geq (\frac{s}{r})^q$,*

(resp. $\frac{|f|(s)}{|f|(r)} \leq (\frac{s}{r})^q$).

Now, we can prove the following two theorems.

Theorem 2.2. *Let $f \in \mathcal{A}(E)$ be such that, $\psi(f) < +\infty$. Then $\sigma(f) = 0$.*

Proof. Without generality, we can assume $f(0) \neq 0$. Let us fix $\varepsilon > 0$ and let R' be such that $\text{Log}(R) - \text{Log}(R') = \varepsilon$. Let $(a_n)_{n \in \mathbb{N}}$ be the sequence of zeros of f , for each $n \in \mathbb{N}$, let w_n be the order of a_n and let $r_n = |a_n|$. Now, let u be the biggest integer n such that $r_n < R'$ and for each $r > 0$, let $m(r)$ be the biggest integer n such that $r_n \leq r$.

Let $A_u = \sum_{n=0}^u w_n$ and let $B_u = \text{Log}(|f(0)|) + \sum_{n=0}^u w_n (\text{Log}(R') - \text{Log}(r_n))$. Let us take $r \in]R', R[$. Thanks to Theorem 2.1, we can write

$$\frac{\sigma(f, r)}{\psi(f, r)} = \frac{B_u + \sum_{n=u+1}^{m(r)} w_n (\text{Log}(r) - \text{Log}(r_n))}{A_u + \sum_{n=u+1}^{m(r)} w_n}.$$

But by hypothesis, $\text{Log}(r) - \text{Log}(r_n) \leq \varepsilon \forall n \geq u$, hence

$$\frac{\sigma(f, r)}{\psi(f, r)} \leq \frac{B_u + \varepsilon \sum_{n=u+1}^{m(r)} w_n}{A_u + \sum_{n=u+1}^{m(r)} w_n}.$$

Let us put $\phi(r) = \sum_{n=u+1}^{m(r)} w_n$. Thus,

$$\frac{\sigma(f, r)}{\psi(f, r)} \leq \frac{B_u + \varepsilon \phi(r)}{A_u + \phi(r)}.$$

But since f belongs to $\mathcal{A}(E)$, it has infinitely many zeros in E , hence $\phi(r)$ is an increasing unbounded function tending to $+\infty$ when r tends to R . Consequently, it is obvious that

$$\lim_{r \rightarrow R^-} \frac{\sigma(f, r)}{\psi(f, r)} = 0.$$

Therefore, if $\limsup_{r \rightarrow R^-} \psi(f, r) < +\infty$, then $\sigma(f) = 0$. □

Theorem 2.3. *Let $f \in \mathcal{A}(E)$ be such that $\rho(f) < +\infty$. Then $\theta(f) - 1 \leq \rho(f) \leq \theta(f)$. Moreover, if $\psi(f) < +\infty$, then $\theta(f) = \rho(f)$.*

Proof. We will denote $|\cdot|_\infty$ the Archimedean absolute value of $\mathbb{R}$. Let us first show that $\rho(f) \leq \theta(f)$. We choose $u > \theta(f)$. Then, $\lim_{r \rightarrow R^-} s(r, f)(R - r)^u = 0$. Now, since $\lim_{r \rightarrow R^-} |f|(r) = +\infty$, we can take $\ell \in]0, R[$ such that $|f|(\ell) > 1$. Then, we can take $b > 0$ such that

$$s(r, f) \leq b(R - r)^{-u} \forall r \in [\ell, R[.$$

Now, taking $r \in [\ell, R[$, by Theorem 2.2 we have

$$\text{Log}(|f|(r)) \leq \text{Log}(|f|(\ell)) + s(r, f) \left(\text{Log}\left(\frac{r}{\ell}\right) \right)$$

which leads to

$$\text{Log}(|f|(r)) \leq \text{Log}(|f|(\ell)) + b(R - r)^{-u} \left(\text{Log}\left(\frac{r}{\ell}\right) \right)$$

hence

$$\text{Log}(\text{Log}(|f|(r))) \leq \text{Log}\left(\text{Log}(|f|(\ell)) + b(R-r)^{-u}(\text{Log}(\frac{r}{\ell}))\right)$$

therefore, we can derive

$$\text{Log}(\text{Log}(|f|(r))) \leq \text{Log}(\text{Log}(|f|(\ell)) + \text{Log}\left(b(R-r)^{-u}(\text{Log}(\frac{r}{\ell}))\right).$$

Now, since $u > 0$, there obviously exists $h \in [\ell, R[$ such that $b(R-r)^{-u} \geq 1 \forall r \in [h, R[$, therefore

$$\text{Log}(\text{Log}(|f|(r))) \leq \text{Log}\left(\text{Log}(|f|(\ell)) + \text{Log}(b(R-r)^{-u} \text{Log}(\frac{r}{\ell}))\right). \quad (2.1)$$

Consequently, by (2.1), we obtain

$$\frac{\text{Log}(\text{Log}(|f|(r)))}{-\text{Log}(R-r)} \leq \frac{\text{Log}(\text{Log}(|f|(\ell)))}{-\text{Log}(R-r)} + \frac{\text{Log}(b)}{-\text{Log}(R-r)} + u + \frac{\text{Log}(\text{Log}(\frac{r}{\ell}))}{-\text{Log}(R-r)}.$$

We can check that

$$\lim_{r \rightarrow R^-} \frac{\text{Log}(\text{Log}(|f|(\ell))) + \text{Log}(b)}{-\text{Log}(R-r)} = \lim_{r \rightarrow R^-} \frac{\text{Log}(\text{Log}(\frac{r}{\ell}))}{-\text{Log}(R-r)} = 0,$$

and hence $\limsup_{r \rightarrow R^-} \frac{\text{Log}(\text{Log}(|f|(r)))}{-\text{Log}(R-r)} \leq u$. Consequently, choosing $\varepsilon > 0$, there exists

$u' \in [\ell, R[$ such that $\frac{\text{Log}(\text{Log}(|f|(r)))}{-\text{Log}(R-r)} \leq u' + \varepsilon \forall r \in [u, R[$ and hence $\rho(f) \leq u' + \varepsilon$.

But since that holds for every $u > \theta(f)$ and for every $\varepsilon > 0$, we have $\rho(f) \leq u$ and hence $\rho(f) \leq \theta(f)$. Let us now show that $\rho(f) \geq \theta(f) - 1$. By Theorem 2.2, we have

$$\begin{aligned} \text{Log}(|f|(r)) - \text{Log}(|f|(\frac{r^2}{R})) &\geq s(r, f) \left(\frac{r^2}{R}\right) (\text{Log}(r) - \text{Log}(\frac{r^2}{R})) \\ &= s(r, f) \left(\frac{r^2}{R}\right) (\text{Log}(R) - \text{Log}(r)). \end{aligned} \quad (2.2)$$

Consider now a number $t < \theta(f)$ and a sequence $(r_n)_{n \in \mathbb{N}}$ of $]0, R[$ such that

$\lim_{n \rightarrow +\infty} r_n = R$ and such that $\limsup_{n \rightarrow +\infty} s(r_n, f) \left(R - \frac{r_n^2}{R}\right)^t \geq b > 0$. Then by (2) we have

$$\text{Log}(|f|(r_n)) \geq \frac{b(\text{Log}(R) - \text{Log}(r_n))}{\left(R - \frac{r_n^2}{R}\right)^t}$$

Consequently,

$$\begin{aligned} \text{Log}(\text{Log}(|f|(r_n))) &\geq \text{Log}(b) + \text{Log}(\text{Log}(R) - \text{Log}(r_n)) - t(\text{Log}(R - r_n) + \\ &\quad \text{Log}(R + r_n)) + 2t\text{Log}(R), \end{aligned}$$

and therefore,

$$\frac{\text{Log}(\text{Log}(|f|(r_n)))}{-\text{Log}(R - r_n)} \geq \frac{\text{Log}(b)}{-\text{Log}(R - r_n)} + \frac{\text{Log}(\text{Log}(R) - \text{Log}(r_n))}{-\text{Log}(R - r_n)} + t \left(1 + \frac{\text{Log}(R + r_n) + 2\text{Log}(R)}{-\text{Log}(R - r_n)} \right).$$

Clearly,

$$\lim_{n \rightarrow +\infty} \left(\frac{\text{Log}(b)}{\text{Log}(R - r_n)} \right) = \lim_{n \rightarrow +\infty} \frac{\text{Log}(R + r_n) + 2\text{Log}(R)}{\text{Log}(R - r_n)} = 0,$$

and by elementary reasonings, we can check that

$$\lim_{x \rightarrow R^-} \frac{\text{Log}(\text{Log}(R) - \text{Log}(x))}{\text{Log}(R - x)} = 1,$$

therefore

$$\lim_{n \rightarrow +\infty} \frac{\text{Log}(\text{Log}(R) - \text{Log}(r_n))}{\text{Log}(R - r_n)} = 1.$$

Consequently,

$$\limsup_{n \rightarrow +\infty} \frac{\text{Log}(\text{Log}(|f|(r_n)))}{-\text{Log}(R - r_n)} \geq t - 1,$$

and therefore

$$\limsup_{r \rightarrow R^-} \frac{\text{Log}(\text{Log}(|f|(r)))}{-\text{Log}(R - r)} \geq t - 1.$$

That holds for every $t < \theta(f)$ and shows that if $\theta(f) < +\infty$, then $\rho(f) \geq \theta(f) - 1$. Next, if $\theta(f) = +\infty$, then we have $\rho(f) = +\infty$, which is excluded by hypothesis since $f \in \mathcal{A}_{ub}(E)$. Consequently, the inequality $\rho(f) \geq \theta(f) - 1$ is established.

Let us now show that $\rho(f) \geq \theta(f)$ when $\psi(f) < +\infty$. Suppose $\theta(f) > \rho(f)$ and let $z \in]\rho(f), \theta(f)[$. Then by Remark 1.2 we have $\limsup_{r \rightarrow R^-} s(r, f)(R - r)^z = +\infty$, but then $\limsup_{r \rightarrow R^-} s(r, f)(R - r)^{\rho(f)} = +\infty$, i.e. $\psi(f) = +\infty$, a contradiction. Therefore, $\theta(f) \leq \rho(f)$ and hence whenever $\psi(f) < +\infty$, we have $\theta(f) = \rho(f)$. $\square$

Theorem 2.3 obviously suggests the following conjecture

Conjecture 2.1. *Let $f \in \mathcal{A}(E)$. Then $\rho(f) = \theta(f)$.*

By Corollary 1 in [3] which is also Theorem C.9.19 in [4] we can state the following theorem which is an easy consequence of the Nevanlinna Theorem on 3 small functions for entire functions.

Theorem 2.4. *Let $f \in \mathcal{A}_{ub}(E)$ and let $w_1, w_2 \in \mathcal{A}_f(E)$ be distinct. Let $S(r) = \max(Z(r, w_1), Z(r, w_2))$. Then, $Z(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + 7S(r) + O(1)$.*

Theorem 2.5. *Let $f \in \mathcal{A}_{ub}(E)$ and let $w_j \in \mathcal{A}_f(E)$ ($j = 1, \dots, q$) be q distinct small functions with respect to f . Then,*

$$qZ(r, f) \leq 2 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(Z(r, f)).$$

Proof. By Theorem 2.4, for every pair (i, j) such that $1 \leq i < j \leq q$, we have

$$Z(r, f) \leq \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + o(Z(r, f)).$$

The number of such inequalities is then C_q^2 . Summing up, we now obtain

$$C_q^2 Z(r, f) \leq \sum_{(i, j), 1 \leq i < j \leq q} \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + o(Z(r, f)). \quad (2.3)$$

In this sum, for each index i , the number of terms $\bar{Z}(r, f - w_i)$ is clearly $C_{q-1}^1 = q - 1$. Consequently, by (2.3) we obtain

$$C_q^2 Z(r, f) \leq (q - 1) \sum_{i=1}^q \bar{Z}(r, f - w_i) + o(Z(r, f)),$$

and hence

$$\frac{q}{2} Z(r, f) \leq \sum_{i=1}^q \bar{Z}(r, f - w_i) + o(Z(r, f)). \quad \square$$

Theorem 2.6. *Let $f, g \in \mathcal{A}_{ub}(E)$ be distinct and share q distinct small functions ignoring multiplicity $w_j \in \mathcal{A}_f(E) \cap \mathcal{A}_g(E)$ ($j = 1, \dots, q$). Then,*

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g) + o(Z(r, f)) + o(Z(r, g)).$$

Proof. Let b be a zero of $f - w_i$ for a certain index i . Then, it is also a zero of $g - w_i$. Suppose that b is counted several times in the sum $\sum_{j=1}^q \bar{Z}(r, f - w_j)$, which

means that it is a zero of another function $f - w_h$ for a certain index $h \neq i$. Then, we have $w_i(b) = w_h(b)$ and hence b is a zero of the function $w_i - w_h$ which belongs to $\mathcal{A}_f(E)$. Now, put $\tilde{Z}(r, f - w_1) = \bar{Z}(r, f - w_1)$ and for each $j > 1$, let $\tilde{Z}(r, f - w_j)$ be the counting function of zeros of $f - w_j$ in the disk $d(0, r^-)$ ignoring multiplicity and avoiding the zeros already counted as zeros of $f - w_h$ for some $h < j$. Consider now the sum $\sum_{j=1}^q \tilde{Z}(r, f - w_j)$. Since the functions $w_i - w_j$ belong to $\mathcal{A}_f(E)$, clearly,

we have

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) = \sum_{j=1}^q \tilde{Z}(r, f - w_j) = o(Z(r, f)).$$

It is clear, from the assumption, that $f(x) - w_j(x) = 0$ implies $g(x) - w_j(x) = 0$ and hence $f(x) - g(x) = 0$. Since $f - g$ is not the identically zero function, it follows that

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g).$$

Consequently,

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g) + o(Z(r, f)) + o(Z(r, g)). \quad \square$$

Theorem 2.7. *Let $f, g \in \mathcal{A}_{ub}(E)$ be distinct and share q distinct small functions ignoring multiplicity $w_j \in \mathcal{A}_f(E) \cap \mathcal{A}_g(E)$ ($j = 1, \dots, q$). Then,*

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g) + o(Z(r, f)) + o(Z(r, g)).$$

Proof. Suppose that f and g belong to $\mathcal{A}_u(E)$, are distinct and share q distinct small functions I.M. $w_j \in \mathcal{A}_f(E) \cap \mathcal{A}_g(E)$ ($j = 1, \dots, q$).

Let b be a zero of $f - w_i$ for a certain index i . Then it is also a zero of $g - w_i$.

Suppose that b is counted several times in the sum $\sum_{j=1}^q \bar{Z}(r, f - w_j)$, which means

that it is a zero of another function $f - w_h$ for a certain index $h \neq i$. Then we have $w_i(b) = w_h(b)$ and hence b is a zero of the function $w_i - w_h$ which belongs to $\mathcal{A}_f(E)$. Now, put $\tilde{Z}(r, f - w_1) = \bar{Z}(r, f - w_1)$ and for each $j > 1$, let $\tilde{Z}(r, f - w_j)$ be the counting function of zeros of $f - w_j$ in the disk $d(0, r^-)$ ignoring multiplicity and avoiding the zeros already counted as zeros of $f - w_h$ for some $h < j$. Consider now the sum $\sum_{j=1}^q \tilde{Z}(r, f - w_j)$. Since the functions $w_i - w_j$ belong to $\mathcal{A}_f(E)$, clearly, we have

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) = \sum_{j=1}^q \tilde{Z}(r, f - w_j) = o(Z(r, f)).$$

It is clear, from the assumption, that $f(x) - w_j(x) = 0$ implies $g(x) - w_j(x) = 0$ and hence $f(x) - g(x) = 0$. Since $f - g$ is not the identically zero function, it follows that

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g).$$

Consequently,

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g) + o(Z(r, f)) + o(Z(r, g)). \quad \square$$

By Theorem 2.7, we can now deduce the following theorem.

Theorem 2.8. *Let $f, g \in \mathcal{A}_{ub}(E)$. share 3 distinct small functions with respect to f and g , ignoring multiplicity. Then, $f = g$.*

Proof. We put $V(r) = \max(Z(r, f), Z(r, g))$. Suppose that f and g are distinct and share q small functions I.M. w_j , ($1 \leq j \leq q$). By Theorem 2.5, we have

$$qZ(r, f) \leq 2 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(Z(r, f)).$$

But thanks to Theorem 2.7, we can derive

$$qZ(r, f) \leq 2Z(r, f - g) + o(Z(r, f)),$$

and similarly

$$qZ(r, g) \leq 2Z(r, f - g) + o(Z(r, g))$$

hence $qV(r) \leq 2Z(r, f - g) + o(V(r))$, and hence $qV(r) \leq 2V(r) + o(V(r))$, which is obviously absurd whenever $q \geq 3$ and proves that $f = g$ when f and g . $\square$

The following theorem now is just a consequence of Theorems 1.6 and 2.8.

Theorem 2.9. *Let $f, g \in \mathcal{A}_{ub}(E)$ be such that $0 < \tilde{\rho}(f) < +\infty$, $0 < \tilde{\rho}(g) < +\infty$ and share 3 distinct functions, ignoring multiplicity, $h_j \in \mathcal{A}(E)$, $j = 1, 2, 3$ such that $\rho(h_j) < \min(\tilde{\rho}(f), \tilde{\rho}(g))$. Then, $f = g$ and each h_j is a small function with respect to f .*

Corollary 2.3. *Let $f, g \in \mathcal{A}_{ub}(E)$ be such that $0 < \tilde{\rho}(f) < +\infty$, $0 < \tilde{\rho}(g) < +\infty$ and share 3 distinct functions, ignoring multiplicity, $h_j \in \mathcal{A}(E)$, $j = 1, 2, 3$ such that $\theta(h_j) < \min(\tilde{\rho}(f), \tilde{\rho}(g))$. Then, $f = g$ and each h_j is a small function with respect to f .*

Also, by Theorem 1.3, we have the following corollary.

Corollary 2.4. *Let $f, g \in \mathcal{A}_{ub}(E)$ be regular, such that $0 < \rho(f) < +\infty$, $0 < \rho(g) < +\infty$ and share 3 distinct functions, ignoring multiplicity, $h_j \in \mathcal{A}(E)$, $j = 1, 2, 3$ and such that $\max(\rho(h_j)) < \min(\rho(f), \rho(g))$. Then, $f = g$ and each h_j is a small function with respect to f .*

By Theorem 2.3, we have Corollaries 2.5 and 2.6.

Corollary 2.5. *Let $f, g \in \mathcal{A}_{ub}(E)$ be regular, such that $0 < \rho(f) < +\infty$, $0 < \rho(g) < +\infty$ and share 3 distinct functions, ignoring multiplicity, $h_j \in \mathcal{A}(E)$, $j = 1, 2, 3$ and such that $\max(\rho(h_j)) < \min((\theta(f), \theta(g)) - 1)$. Then, $f = g$ and each h_j is a small function with respect to f .*

Corollary 2.6. *Let $f, g \in \mathcal{A}_{ub}(E)$ be regular, such that $0 < \rho(f) < +\infty$, $0 < \rho(g) < +\infty$ and share 3 distinct functions, ignoring multiplicity, $h_j \in \mathcal{A}(E)$, $j = 1, 2, 3$ and such that $\max(\theta(h_j)) < \min((\theta(f), \theta(g)) - 1)$. Then, $f = g$ and each h_j is a small function with respect to f .*

Corollary 2.7. *Let $f, g \in \mathcal{A}_u(E)$ be regular, such that $0 < \rho(f) < +\infty$, $0 < \rho(g) < +\infty$, and share 3 distinct functions, ignoring multiplicity, $h_j \in \mathcal{A}(E)$, $j = 1, 2, 3$ and such that $\rho(h_1) < \min(\rho(f), \rho(g))$ and $\text{Log}(\max(|h_2|(r), |h_3|(r))) \leq O(\text{Log}(|h_1|(r)))$. Then, $f = g$ and each h_j is a small function with respects to f .*

Theorem 2.10. *Let $f, g \in \mathcal{A}_{ub}(E)$ be clean, such that $0 < \rho(f) < +\infty$, $0 < \rho(g) < +\infty$ and share 3 distinct functions, ignoring multiplicity, $h_j \in \mathcal{A}(E)$, $j = 1, 2, 3$ such that $\rho(f) = \rho(g) = \rho(h_j)$, $j = 1, 2, 3$ and such that $\sigma(h_j) = 0$, $j = 1, 2, 3$. Then, $f = g$ and each h_j is a small function with respect to f .*

Proof. Indeed by Corollary 1.6, each h_j is a small function with respect to f and g . Consequently by Theorem 2.8, $f = g$. $\square$

Corollary 2.8. *Let $f, g \in \mathcal{A}_{ub}(E)$ be clean, such that $0 < \rho(f) < +\infty$, $0 < \rho(g) < +\infty$ and share 3 distinct functions, ignoring multiplicity, $h_j \in \mathcal{A}(E)$, $j = 1, 2, 3$ such that $\rho(f) = \rho(g) = \rho(h_1)$, and such that $\sigma(h_1) = 0$, and $\text{Log}(\max(|h_2|(r), |h_3|(r))) \leq O(\text{Log}(|h_1|(r)))$. Then, $f = g$ and each h_j is a small function with respect to f .*

Proof. Indeed, since $\psi(h_1) = 0$ and $\rho(h_1) \neq 0$, by Theorem 2.2 we have $\sigma(h_1) = 0$. Next, h_2, h_3 also are small functions. $\square$

By Theorem 1.7 we have the following corollary.

Corollary 2.9. *Let $f, g \in \mathcal{A}_{ub}(E)$ be clean, such that $0 < \rho(f) < +\infty$, $0 < \rho(g) < +\infty$ and share 3 distinct functions, ignoring multiplicity, $h_j \in \mathcal{A}(E)$, $j = 1, 2, 3$ such that $\rho(f) = \rho(g) = \rho(h_j)$, $j = 1, 2, 3$ and such that $\psi(h_j) < +\infty$, $j = 1, 2, 3$. Then, $f = g$ and each h_j is a small function with respect to f .*

Proof. Indeed, since $\psi(h_j) = 0$ and $\rho(h_j) \neq 0$, by Theorem 2.2 we have $\sigma(h_j) = 0$. $\square$

Corollary 2.10. *Let $f, g \in \mathcal{A}_{ub}(E)$ be clean, such that $0 < \rho(f) < +\infty$, $0 < \rho(g) < +\infty$ and share 3 distinct functions, ignoring multiplicity, $h_j \in \mathcal{A}(E)$, $j = 1, 2, 3$ such that $\rho(f) = \rho(g) = \rho(h_1)$, and $\psi(h_1) < +\infty$, and that $\text{Log}(\max(|h_2|(r), |h_3|(r))) \leq O(\text{Log}(|h_1|(r)))$. Then, $f = g$ and each h_j is a small function with respect to f .*

Proof. Indeed, since $\psi(h_1) = 0$ and $\rho(h_1) \neq 0$, by Theorem 2.2 we have $\sigma(h_1) = 0$. Next, h_2, h_3 also are small functions. $\square$

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