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# GENERALIZED ANALYSIS OF TIME SCALE DYNAMIC LYAPUNOV'S INEQUALITIES BY USING SPECHT'S AND KANTOROVICH'S RATIOS

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ABSTRACT. In this paper, we present extensions of dynamic Lyapunov's inequalities and their reverse versions on time scales by using Specht's and Kantorovich's ratios. Our approach unifies and extends some continuous inequalities and their corresponding discrete and quantum analogues.

## 1. INTRODUCTION

The following reverse Rogers–Hölder's inequality by using Specht's ratio is proved in [18].

**Theorem 1.1.** *Let $\frac{1}{p}+\frac{1}{q}=1$ and $p>1$. If $f(x)$ and $h(x)$ are non-negative continuous functions and $f^{\frac{1}{p}}(x)h^{\frac{1}{q}}(x)$ is integrable on $[a,b]$, then*

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}}\left(\int_a^b h^q(x)dx\right)^{\frac{1}{q}} \leq \int_a^b S\left(\frac{\Omega f^p(x)}{\Lambda h^q(x)}\right) f(x)h(x)dx, \tag{1.1}$$

*where $\Lambda=\int_a^b f^p(x)dx$, $\Omega=\int_a^b h^q(x)dx$, and $S(.)$ is Specht's ratio.*

The following reverse Rogers–Hölder's inequality on time scales by using Specht's ratio is proved in [5].

**Theorem 1.2.** *Let $a,b\in\mathbb{T}_k^k$ and $w,f,h\in C\left([a,b]_{\mathbb{T}},\mathbb{R}_0^+\right)$ be such that neither $w\equiv 0$, $f\equiv 0$ nor $h\equiv 0$ and $f^p$ and $h^q$ are $\diamond_\alpha$-integrable on $[a,b]_{\mathbb{T}}$. If $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, then*

$$\left(\int_a^b w(x)f^p(x)\diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b w(x)h^q(x)\diamond_\alpha x\right)^{\frac{1}{q}} \leq \int_a^b S\left(\frac{\Omega f^p(x)}{\Lambda h^q(x)}\right) w(x)f(x)h(x)\diamond_\alpha x, \tag{1.2}$$

*where $\Lambda=\int_a^b w(x)f^p(x)\diamond_\alpha x$, $\Omega=\int_a^b w(x)h^q(x)\diamond_\alpha x$, and $S(.)$ is Specht's ratio.*

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We will merge and expand these results on time scales. Stefan Hilger [8] initiated the calculus of time scales in 1988. The three most popular branches of the theory of time scales are delta calculus, nabla calculus, and diamond-alpha calculus. Lyapunov's inequality and its various extensions, generalizations, and refinements play a vital role in mathematical analysis. Dynamic Lyapunov's inequality is equivalent to generalized Radon's inequality, Radon's inequality, the weighted power mean inequality, Schlömilch's inequality, Rogers–Hölder's inequality, and Bernoulli's inequality, see [10]. The theory of time scales is applied to arrange results in hybridization form, see [11–14].

In this research article, it is assumed that all noticeable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a,b \in \mathbb{T}$ with $a < b$ and an interval $[a,b]_{\mathbb{T}}$ means the intersection of real interval with the given time scale.

## 2. PRELIMINARIES

Here, we need the basic concepts of delta calculus. The results of delta calculus are adopted from monographs [3, 4].

A *time scale* is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0,+\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the *forward graininess function*. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0,+\infty)$ such that $\nu(t) := t - \rho(t)$ is called the *backward graininess function*. If $\sigma(t) > t$, we say that $t$ is *right-scattered*, while if $\rho(t) < t$, we say that $t$ is *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then $t$ is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then $t$ is called *left-dense*. If $\mathbb{T}$ has a left-scattered maximum $M$, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative $f^\Delta$ is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^\Delta(t) \in \mathbb{R}$ such that for all $\varepsilon > 0$, there is a neighborhood $U$ of $t$, such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in U$, then $f$ is said to be *delta differentiable* at $t$, and $f^\Delta(t)$ is called the *delta derivative* of $f$ at $t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* (*rd-continuous*), if it is continuous at each right-dense point and there exists a finite left-sided limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.

The next definition is given in [3, 4].

**Definition 2.1.** *A function $F : \mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$, provided that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$. Then the delta integral of $f$ is defined by*

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

The following results of nabla calculus are taken from [2–4].

If $\mathbb{T}$ has a right-scattered minimum $m$, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$ and $\mathbb{T}_k^k = \mathbb{T}^k \cap \mathbb{T}_k$. A function $f : \mathbb{T}_k \to \mathbb{R}$ is called *nabla differentiable* at $t \in \mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t) \in \mathbb{R}$ such that given any $\varepsilon > 0$, there is a neighborhood $V$ of $t$, such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s|,$$

for all $s \in V$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *left-dense continuous* (*ld-continuous*), provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T}, \mathbb{R})$.

The next definition is given in [2–4].

**Definition 2.2.** *A function $G : \mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g : \mathbb{T} \to \mathbb{R}$, provided that $G^{\nabla}(t) = g(t)$ holds for all $t \in \mathbb{T}_k$. Then, the nabla integral of $g$ is defined by*

$$\int_a^b g(t)\nabla t = G(b) - G(a).$$

Now we present a short introduction of the diamond-$\alpha$ derivative as given in [1, 15].

**Definition 2.3.** *Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the $\Delta$ and $\nabla$ senses. For $t \in \mathbb{T}$, the diamond-$\alpha$ dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by*

$$f^{\diamond_\alpha}(t) = \alpha f^{\Delta}(t) + (1 - \alpha) f^{\nabla}(t), \; 0 \leq \alpha \leq 1.$$

*Thus $f$ is diamond-$\alpha$ differentiable if and only if $f$ is $\Delta$ and $\nabla$ differentiable.*

The diamond-$\alpha$ derivative reduces to the standard $\Delta$-derivative for $\alpha = 1$, or the standard $\nabla$-derivative for $\alpha = 0$. It represents a weighted dynamic derivative for $\alpha \in (0, 1)$.

**Definition 2.4** ([15]). *Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then, the diamond-$\alpha$ integral from $a$ to $t$ of $h$ is defined by*

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1 - \alpha) \int_a^t h(s)\nabla s, \; 0 \leq \alpha \leq 1,$$

*provided that there exist delta and nabla integrals of $h$ on $\mathbb{T}$.*

*Specht's ratio* [6, 16] is defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}} \ for \ h > 0, \ h \neq 1.$$

We present here some properties of Specht's ratio. See [6, 16, 17] for the proof and details:

(i) $S(1) = 1$ and $S(h) = S\left(\frac{1}{h}\right) > 1$ for all $h > 0$.

(ii) $S(h)$ is a monotone increasing function on $(1, +\infty)$ and monotone decreasing function on $(0, 1)$.

We also consider *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0.$$

The function $K$ is decreasing on $(0, 1)$ and increasing on $[1, +\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following first inequality is due to Furuichi [7] and provides a refinement for Young's inequality, and the second inequality [9] is given by

$$S\left(\left(\frac{a}{b}\right)^{\gamma}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q} \leq K^{\delta}\left(\frac{a}{b}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \tag{2.1}$$

for $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, $\gamma = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and $\delta = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.
The following multiplicative refinement [19] and a multiplicative reverse for Young's inequality [17] is given by

$$K^{\gamma}\left(\frac{a}{b}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q} \leq S\left(\frac{a}{b}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \tag{2.2}$$

for $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$ and $\gamma = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

## 3. Lyapunov's Inequality

In order to present our main results, first, we give a simple proof of an extension of dynamic Lyapunov's inequality and its reverse version on time scales.

**Theorem 3.1.** *Let $w, f, g, h \in C\left([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\}\right)$ be $\diamond_\alpha$-integrable functions. Assume further that $|f|^{\beta_1(\beta_3-\beta_2)}|g|^{\beta_2(\beta_1-\beta_3)}|h|^{\beta_3(\beta_2-\beta_1)} = M$ for $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$, where $M$ is a positive real number.*

*If $\beta_1 < \beta_2 < \beta_3$, then the following inequalities hold true:*

$$\left(\int_a^b |w(x)||f(x)|^{\beta_1} \diamond_\alpha x\right)^{\beta_3-\beta_2} \left(\int_a^b S\left(\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)^{\gamma}\right) |w(x)||g(x)|^{\beta_2} \diamond_\alpha x\right)^{\beta_1-\beta_3}$$
$$\times \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\beta_2-\beta_1} \geq M \tag{3.1}$$

*and*

$$\left(\int_a^b |w(x)||f(x)|^{\beta_1} \diamond_\alpha x\right)^{\beta_3-\beta_2} \left(\int_a^b K^\delta\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||g(x)|^{\beta_2} \diamond_\alpha x\right)^{\beta_1-\beta_3} \times \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\beta_2-\beta_1} \leq M, \tag{3.2}$$

*where* $\Lambda = \int_a^b |w(x)||f(x)|^{\beta_1} \diamond_\alpha x$, $\Omega = \int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x$, $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$, $\gamma = \min\left\{\frac{1}{p},\frac{1}{q}\right\}$ *and* $\delta = \max\left\{\frac{1}{p},\frac{1}{q}\right\}$.

*Proof.* Let $p = \frac{\beta_3-\beta_1}{\beta_3-\beta_2} > 1$ and $q = \frac{\beta_3-\beta_1}{\beta_2-\beta_1} > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Setting

$$\Phi(x) = \frac{|w(x)||f(x)|^{\beta_1}}{\Lambda} \text{ and } \Psi(x) = \frac{|w(x)||h(x)|^{\beta_3}}{\Omega}$$

on $[a,b]_{\mathbb{T}}$. Young's inequalities from (2.1) become

$$\frac{S\left(\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)^\gamma\right)|w(x)||f(x)|^{\frac{\beta_1}{p}}|h(x)|^{\frac{\beta_3}{q}}}{\Lambda^{\frac{1}{p}}\Omega^{\frac{1}{q}}} \leq \frac{|w(x)||f(x)|^{\beta_1}}{p\Lambda} + \frac{|w(x)||h(x)|^{\beta_3}}{q\Omega} \leq \frac{K^\delta\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||f(x)|^{\frac{\beta_1}{p}}|h(x)|^{\frac{\beta_3}{q}}}{\Lambda^{\frac{1}{p}}\Omega^{\frac{1}{q}}}. \tag{3.3}$$

Integrating inequality (3.3) over $x$ from $a$ to $b$, we get

$$\int_a^b S\left(\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)^\gamma\right)|w(x)||f(x)|^{\frac{\beta_1}{p}}|h(x)|^{\frac{\beta_3}{q}} \diamond_\alpha x \leq \Lambda^{\frac{1}{p}}\Omega^{\frac{1}{q}} \leq \int_a^b K^\delta\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||f(x)|^{\frac{\beta_1}{p}}|h(x)|^{\frac{\beta_3}{q}} \diamond_\alpha x. \tag{3.4}$$

Therefore, we obtain

$$\begin{aligned}\int_a^b S\left(\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)^\gamma\right)|w(x)||f(x)|^{\beta_1\left(\frac{\beta_3-\beta_2}{\beta_3-\beta_1}\right)}|h(x)|^{\beta_3\left(\frac{\beta_2-\beta_1}{\beta_3-\beta_1}\right)} \diamond_\alpha x \\ \leq \left(\int_a^b |w(x)||f(x)|^{\beta_1} \diamond_\alpha x\right)^{\frac{\beta_3-\beta_2}{\beta_3-\beta_1}} \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\frac{\beta_2-\beta_1}{\beta_3-\beta_1}} \\ \leq \int_a^b K^\delta\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||f(x)|^{\beta_1\left(\frac{\beta_3-\beta_2}{\beta_3-\beta_1}\right)}|h(x)|^{\beta_3\left(\frac{\beta_2-\beta_1}{\beta_3-\beta_1}\right)} \diamond_\alpha x.\end{aligned} \tag{3.5}$$

Taking power $(\beta_3-\beta_1)>0$ of inequality (3.5), we get

$$\left(\int_a^b S\left(\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)^{\gamma}\right)|w(x)||f(x)|^{\beta_1\left(\frac{\beta_3-\beta_2}{\beta_3-\beta_1}\right)}|h(x)|^{\beta_3\left(\frac{\beta_2-\beta_1}{\beta_3-\beta_1}\right)}\diamond_\alpha x\right)^{\beta_3-\beta_1}$$
$$\le\left(\int_a^b|w(x)||f(x)|^{\beta_1}\diamond_\alpha x\right)^{\beta_3-\beta_2}\left(\int_a^b|w(x)||h(x)|^{\beta_3}\diamond_\alpha x\right)^{\beta_2-\beta_1}$$
$$\le\left(\int_a^b K^{\delta}\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||f(x)|^{\beta_1\left(\frac{\beta_3-\beta_2}{\beta_3-\beta_1}\right)}|h(x)|^{\beta_3\left(\frac{\beta_2-\beta_1}{\beta_3-\beta_1}\right)}\diamond_\alpha x\right)^{\beta_3-\beta_1}. \tag{3.6}$$

Using the condition that $|f|^{\beta_1(\beta_3-\beta_2)}|g|^{\beta_2(\beta_1-\beta_3)}|h|^{\beta_3(\beta_2-\beta_1)}=M$ for $\beta_1<\beta_2<\beta_3$, where $M$ is a positive real number, inequality (3.6) reduces to

$$\left(\int_a^b S\left(\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)^{\gamma}\right)|w(x)|M^{\frac{1}{\beta_3-\beta_1}}|g(x)|^{\beta_2}\diamond_\alpha x\right)^{\beta_3-\beta_1}$$
$$\le\left(\int_a^b|w(x)||f(x)|^{\beta_1}\diamond_\alpha x\right)^{\beta_3-\beta_2}\left(\int_a^b|w(x)||h(x)|^{\beta_3}\diamond_\alpha x\right)^{\beta_2-\beta_1}$$
$$\le\left(\int_a^b K^{\delta}\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)|M^{\frac{1}{\beta_3-\beta_1}}|g(x)|^{\beta_2}\diamond_\alpha x\right)^{\beta_3-\beta_1}. \tag{3.7}$$

The first inequality of (3.7) directly yields (3.1), and the second inequality of (3.7) directly yields (3.2). Thus, the proof of Theorem 3.1 is now complete. □

**Theorem 3.2.** *Let $w,f,g,h\in C([a,b]_{\mathbb{T}},\mathbb{R}-\{0\})$ be $\diamond_\alpha$-integrable functions. Assume further that $|f|^{\beta_1(\beta_3-\beta_2)}|g|^{\beta_2(\beta_1-\beta_3)}|h|^{\beta_3(\beta_2-\beta_1)}=M$ for $\beta_1,\beta_2,\beta_3\in\mathbb{R}$, where $M$ is a positive real number.*

*If $\beta_1<\beta_2<\beta_3$, then the following inequalities hold true:*

$$\left(\int_a^b|w(x)||f(x)|^{\beta_1}\diamond_\alpha x\right)^{\beta_3-\beta_2}\left(\int_a^b K^{\gamma}\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||g(x)|^{\beta_2}\diamond_\alpha x\right)^{\beta_1-\beta_3}$$
$$\times\left(\int_a^b|w(x)||h(x)|^{\beta_3}\diamond_\alpha x\right)^{\beta_2-\beta_1}\ge M, \tag{3.8}$$

*and*

$$\left(\int_a^b|w(x)||f(x)|^{\beta_1}\diamond_\alpha x\right)^{\beta_3-\beta_2}\left(\int_a^b S\left(\frac{\Omega|f(x)|^{\beta_1}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||g(x)|^{\beta_2}\diamond_\alpha x\right)^{\beta_1-\beta_3}$$
$$\times\left(\int_a^b|w(x)||h(x)|^{\beta_3}\diamond_\alpha x\right)^{\beta_2-\beta_1}\le M, \tag{3.9}$$

*where* $\Lambda = \int_a^b |w(x)||f(x)|^{\beta_1} \diamond_\alpha x$, $\Omega = \int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x$, $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$, $\gamma = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

*Proof.* We apply the inequalities given in (2.2). The rest is similar to the proof of Theorem 3.1. □

*Remark* 3.1. Let $\mathbb{T} = \mathbb{R}$, $w \equiv 1$, $f(x), g(x), h(x) > 0$ on the set $[a,b]$, $p = \frac{\beta_3 - \beta_1}{\beta_3 - \beta_2} > 1$ and $q = \frac{\beta_3 - \beta_1}{\beta_2 - \beta_1} > 1$. If we replace $f$ with $f^{\frac{p}{\beta_1}}$, $h$ with $h^{\frac{q}{\beta_3}}$ and $g$ with $(fh)^{\frac{1}{\beta_2}}$ in inequality (3.9), then we get $M = 1$ and inequality (3.9) reduces to (1.1).

*Remark* 3.2. Let $w(x), f(x), g(x), h(x) > 0$ on the set $[a,b]_\mathbb{T}$, $p = \frac{\beta_3 - \beta_1}{\beta_3 - \beta_2} > 1$ and $q = \frac{\beta_3 - \beta_1}{\beta_2 - \beta_1} > 1$. If we replace $f$ with $f^{\frac{p}{\beta_1}}$, $h$ with $h^{\frac{q}{\beta_3}}$ and $g$ with $(fh)^{\frac{1}{\beta_2}}$ in inequality (3.9), then we get $M = 1$ and inequality (3.9) reduces to inequality (1.2).

Next, we give other dynamic Lyapunov's inequalities on time scales by using Specht's and Kantorovich's ratios.

**Theorem 3.3.** *Let* $w, f, g, h \in C([a,b]_\mathbb{T}, \mathbb{R} - \{0\})$ *be* $\diamond_\alpha$*-integrable functions. Assume further that* $|f|^{\beta_1(\beta_3 - \beta_2)}|g|^{\beta_2(\beta_1 - \beta_3)}|h|^{\beta_3(\beta_2 - \beta_1)} = M$ *for* $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$, *where* $M$ *is a positive real number.*

*If* $\beta_2 < \beta_1 < \beta_3$, *then the following inequalities hold true:*

$$\left(\int_a^b S\left(\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)^\gamma\right)|w(x)||f(x)|^{\beta_1} \diamond_\alpha x\right)^{\beta_3 - \beta_2} \left(\int_a^b |w(x)||g(x)|^{\beta_2} \diamond_\alpha x\right)^{\beta_1 - \beta_3} \times \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\beta_2 - \beta_1} \leq M, \tag{3.10}$$

*and*

$$\left(\int_a^b K^\delta\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||f(x)|^{\beta_1} \diamond_\alpha x\right)^{\beta_3 - \beta_2} \left(\int_a^b |w(x)||g(x)|^{\beta_2} \diamond_\alpha x\right)^{\beta_1 - \beta_3} \times \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\beta_2 - \beta_1} \geq M, \tag{3.11}$$

*where* $\Lambda = \int_a^b |w(x)||g(x)|^{\beta_2} \diamond_\alpha x$, $\Omega = \int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x$, $\frac{1}{p} + \frac{1}{q} = 1$ *with* $p > 1$, $\gamma = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$ *and* $\delta = \max\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

*Proof.* Let $p = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1} > 1$ and $q = \frac{\beta_3 - \beta_2}{\beta_1 - \beta_2} > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Setting

$$\Phi(x) = \frac{|w(x)||g(x)|^{\beta_2}}{\Lambda} \text{ and } \Psi(x) = \frac{|w(x)||h(x)|^{\beta_3}}{\Omega}$$

on $[a,b]_\mathbb{T}$. Young's inequalities from (2.1) become

$$\frac{S\left(\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)^{\gamma}\right)|w(x)||g(x)|^{\frac{\beta_2}{p}}|h(x)|^{\frac{\beta_3}{q}}}{\Lambda^{\frac{1}{p}}\Omega^{\frac{1}{q}}} \leq \frac{|w(x)||g(x)|^{\beta_2}}{p\Lambda}+\frac{|w(x)||h(x)|^{\beta_3}}{q\Omega}$$
$$\leq \frac{K^{\delta}\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||g(x)|^{\frac{\beta_2}{p}}|h(x)|^{\frac{\beta_3}{q}}}{\Lambda^{\frac{1}{p}}\Omega^{\frac{1}{q}}}. \quad (3.12)$$

Integrating inequality (3.12) over $x$ from $a$ to $b$, we get

$$\int_a^b S\left(\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)^{\gamma}\right)|w(x)||g(x)|^{\frac{\beta_2}{p}}|h(x)|^{\frac{\beta_3}{q}}\diamond_\alpha x \leq \Lambda^{\frac{1}{p}}\Omega^{\frac{1}{q}}$$
$$\leq \int_a^b K^{\delta}\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||g(x)|^{\frac{\beta_2}{p}}|h(x)|^{\frac{\beta_3}{q}}\diamond_\alpha x. \quad (3.13)$$

Therefore, we obtain

$$\int_a^b S\left(\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)^{\gamma}\right)|w(x)||g(x)|^{\beta_2\left(\frac{\beta_3-\beta_1}{\beta_3-\beta_2}\right)}|h(x)|^{\beta_3\left(\frac{\beta_1-\beta_2}{\beta_3-\beta_2}\right)}\diamond_\alpha x$$
$$\leq \left(\int_a^b |w(x)||g(x)|^{\beta_2}\diamond_\alpha x\right)^{\frac{\beta_3-\beta_1}{\beta_3-\beta_2}}\left(\int_a^b |w(x)||h(x)|^{\beta_3}\diamond_\alpha x\right)^{\frac{\beta_1-\beta_2}{\beta_3-\beta_2}}$$
$$\leq \int_a^b K^{\delta}\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||g(x)|^{\beta_2\left(\frac{\beta_3-\beta_1}{\beta_3-\beta_2}\right)}|h(x)|^{\beta_3\left(\frac{\beta_1-\beta_2}{\beta_3-\beta_2}\right)}\diamond_\alpha x. \quad (3.14)$$

Taking power $(\beta_3-\beta_2)>0$ of inequality (3.14), we get

$$\left(\int_a^b S\left(\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)^{\gamma}\right)|w(x)||g(x)|^{\beta_2\left(\frac{\beta_3-\beta_1}{\beta_3-\beta_2}\right)}|h(x)|^{\beta_3\left(\frac{\beta_1-\beta_2}{\beta_3-\beta_2}\right)}\diamond_\alpha x\right)^{\beta_3-\beta_2}$$
$$\leq \left(\int_a^b |w(x)||g(x)|^{\beta_2}\diamond_\alpha x\right)^{\beta_3-\beta_1}\left(\int_a^b |w(x)||h(x)|^{\beta_3}\diamond_\alpha x\right)^{\beta_1-\beta_2}$$
$$\leq \left(\int_a^b K^{\delta}\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)|w(x)||g(x)|^{\beta_2\left(\frac{\beta_3-\beta_1}{\beta_3-\beta_2}\right)}|h(x)|^{\beta_3\left(\frac{\beta_1-\beta_2}{\beta_3-\beta_2}\right)}\diamond_\alpha x\right)^{\beta_3-\beta_2}. \quad (3.15)$$

Using the condition that $|f|^{\beta_1(\beta_3-\beta_2)}|g|^{\beta_2(\beta_1-\beta_3)}|h|^{\beta_3(\beta_2-\beta_1)}=M$ for $\beta_2<\beta_1<\beta_3$, where $M$ is a positive real number, inequality (3.15) reduces to

$$\left(\int_a^b S\left(\left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right)^{\gamma}\right)|w(x)|M^{\frac{1}{\beta_2-\beta_3}}|f(x)|^{\beta_1}\diamond_\alpha x\right)^{\beta_3-\beta_2}$$

$$\leq \left(\int_a^b |w(x)||g(x)|^{\beta_2} \diamond_\alpha x\right)^{\beta_3-\beta_1} \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\beta_1-\beta_2}$$

$$\leq \left(\int_a^b K^\delta \left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right) |w(x)| M^{\frac{1}{\beta_2-\beta_3}} |f(x)|^{\beta_1} \diamond_\alpha x\right)^{\beta_3-\beta_2}. \quad (3.16)$$

The first inequality of (3.16) directly yields (3.10), and the second inequality of (3.16) directly yields (3.11). Thus, the proof of Theorem 3.3 is now complete. □

**Theorem 3.4.** *Let $w, f, g, h \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\diamond_\alpha$-integrable functions. Assume further that $|f|^{\beta_1(\beta_3-\beta_2)}|g|^{\beta_2(\beta_1-\beta_3)}|h|^{\beta_3(\beta_2-\beta_1)} = M$ for $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$, where $M$ is a positive real number.*

*If $\beta_2 < \beta_1 < \beta_3$, then the following inequalities hold true:*

$$\left(\int_a^b K^\gamma \left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right) |w(x)||f(x)|^{\beta_1} \diamond_\alpha x\right)^{\beta_3-\beta_2} \left(\int_a^b |w(x)||g(x)|^{\beta_2} \diamond_\alpha x\right)^{\beta_1-\beta_3}$$
$$\times \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\beta_2-\beta_1} \leq M, \quad (3.17)$$

*and*

$$\left(\int_a^b S \left(\frac{\Omega|g(x)|^{\beta_2}}{\Lambda|h(x)|^{\beta_3}}\right) |w(x)||f(x)|^{\beta_1} \diamond_\alpha x\right)^{\beta_3-\beta_2} \left(\int_a^b |w(x)||g(x)|^{\beta_2} \diamond_\alpha x\right)^{\beta_1-\beta_3}$$
$$\times \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\beta_2-\beta_1} \geq M, \quad (3.18)$$

*where $\Lambda = \int_a^b |w(x)||g(x)|^{\beta_2} \diamond_\alpha x$, $\Omega = \int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x$, $\frac{1}{p}+\frac{1}{q} = 1$ with $p > 1$, $\gamma = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.*

*Proof.* We apply the inequalities given in (2.2). The rest is similar to the proof of Theorem 3.3. □

*Remark* 3.3. Let $P = \frac{\beta_1-\beta_2}{\beta_1-\beta_3} < 0$ and $Q = \frac{\beta_1-\beta_2}{\beta_3-\beta_2}$, $Q \in (0,1)$ with $\frac{1}{P}+\frac{1}{Q} = 1$, $w \equiv 1$ and $f(x), g(x), h(x) > 0$ on the set $[a,b]_{\mathbb{T}}$. If we replace $f$ by $h^{\frac{Q}{\beta_1}}$, $g$ by $g^{\frac{P}{\beta_2}}$ and $h$ by $(gh)^{\frac{1}{\beta_3}}$ in inequality (3.18), simultaneously, then we get $M = 1$ and inequality (3.18) reduces to

$$\int_a^b g(x)h(x) \diamond_\alpha x \leq \left(\int_a^b S\left(\frac{\Omega g^{P-1}(x)}{\Lambda h(x)}\right) h^Q(x) \diamond_\alpha x\right)^{\frac{1}{Q}} \left(\int_a^b g^P(x) \diamond_\alpha x\right)^{\frac{1}{P}}, \quad (3.19)$$

where $\Lambda = \int_a^b g^P(x) \diamond_\alpha x$, $\Omega = \int_a^b g(x)h(x) \diamond_\alpha x$ and $S(.)$ is Specht's ratio. The inequality (3.19) may be found in [5].

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