SARAJEVO JOURNAL OF MATHEMATICS
Vol.15 (28), No.2, (2019), 227 – 237
DOI: 10.5644/SJM.15.02.07

STATISTICAL RELATIVE UNIFORM CONVERGENCE IN DUALLY RESIDUATED LATTICE TOTALLY ORDERED SEMIGROUPS

KAMIL DEMIRCI AND SEVDA YILDIZ

ABSTRACT. We define the notions of statistical relative uniform convergence and statistical relative uniform Cauchy in dually residuated lattice totally ordered semigroups (simply, DRlt-semigroups). Then, we give some basic properties for statistically relatively uniform convergent sequences. Also, we introduce statistical relative uniform limit points and cluster points in DRlt-semigroups, then the relations between these and limit points of the sequence are given.

1. INTRODUCTION AND PRELIMINARIES

The notion of a Dually Residuated Lattice Ordered Semigroup (simply, a *DRl-semigroup*), a broad generalization of Brouwerian Algebras and commutative l-groups, has been introduced and studied by Swamy [9–11]. Then, Jasem [6] introduced u-uniform convergence and relatively uniform convergence for DRl-semigroups. In the current work, we introduce the notion of statistical relative uniform convergence and also the definition of statistical relative uniform Cauchy in DRlt-semigroups is given. Then, we give some basic properties for statistically relatively uniform convergent sequences.

We now recall the notion of a DRl-semigroup that has been introduced by Swamy in [9] and related properties used in the paper.

A system $A=(A,+,\leq,-)$ is called a *dually residuated lattice ordered semigroup* (simply, a *DRl-semigroup*) if and only if

(1) $(A,+,\leq)$ is a commutative lattice ordered semigroup with zero element 0, i.e. $(A,+)$ is a commutative semigroup with zero 0 and $(A,\leq)$ is a lattice such that $a+(b\vee c)=(a+b)\vee(a+c)$ and $a+(b\wedge c)=(a+b)\wedge(a+c)$ for all $a,b,c\in A$,
(2) given a,b in A there exists a least x in A such that $b+x\geq a$, and we denote this x by $a-b$ (for a given pair a,b this x is uniquely determined),
(3) $(a-b)\vee 0+b\leq a\vee b$ for all $a,b\in A$, and

2010 *Mathematics Subject Classification.* 06F20, 40A35.
Key words and phrases. Dually Residuated Lattice Ordered Semigroup, Relatively Uniform Convergence, Statistical Convergence.

(4) $(a-a) \geq 0$.

The following theorem shows that any DRl-semigroup can be equationally defined:

Theorem 1.1. *[9] Any DRl-semigroup can be equationally defined as an algebra with binary operations* $+, \vee, \wedge, -$, *by replacing* (2) *by the equations:*

$$x+(y-x) \geq y, \;\; x-y \leq (x \vee z)-y, \;\; (x+y)-y \leq x.$$

Any abelian lattice ordered group is a DRl-semigroup. For any a and b in a DRl-semigroup A, we shall write $|a-b| = (a-b) \vee (b-a)$, ($|a-b|$ is called the symmetric difference of a and b). The symmetric difference satisfies the following conditions:

(1) $|a-b| \geq 0$, $|a-b| = 0$ if and only if $a = b$,
(2) $|a-b| = |b-a|$,
(3) $|a-c| \leq |a-b| + |b-c|$.

Any DRl-semigroup is an autometrized algebra with the symmetric difference ([9], Theorem 9).

Theorem 1.2. *[6] Let* A *be a DRl-semigroup,* $a,b,c,d \in A$. *Then*

(i) $(a-b)+(c-d) \geq (a+c)-(b+d)$,
(ii) $|a-b|+|c-d| \geq |(a+c)-(b+d)|$,
(iii) $(a-b)+(c-d) \geq (c-b)-(d-a)$.

In this paper, we will need the following assumptions in a DRl-semigroup A from [9]:

Let $a,b,c \in A$. Then

$(A1)$ $a \leq b$ implies $a-c \leq b-c$ and $c-b \leq c-a$,
$(A2)$ $(a \vee b)-c = (a-c) \vee (b-c)$,
$(A3)$ $a-(b \wedge c) = (a-b) \vee (a-c)$,
$(A4)$ $a-(b+c) = (a-b)-c = (a-c)-b$,
$(A5)$ $(a-b)+(b-c) \geq (a-c)$,
$(A6)$ $a-(b-c) \leq (a-b)+c$ and $(a+b)-c \leq (a-c)+b$.

We denote $A^+ = \{x \in A;\ x \geq 0\}$. A DRl-semigroup is said to be *Archimedean* if for each $x,y \in A^+$, $nx \leq y$ for each $n \in \mathbb{N}$ implies $x = 0$.

2. Statistical Relative Uniform Convergence

Jasem [6] introduced notions of a u-uniform convergence and a relatively uniform convergence in DRl-semigroup as follows.

Definition 2.1. *Let* A *be a DRl-semigroup,* (x_k) *a sequence in* A, $u \in A^+$. *It is said that a sequence* (x_k) *in* A *converges u-uniformly to an element* $x \in A$, *written* $x_k \xrightarrow{u} x$, *if the following condition is satisfied:*

for each $p \in \mathbb{N}$ *there exists* $k_p \in \mathbb{N}$, *such that* $p|x_k - x| \leq u$ *for each* $k \in \mathbb{N}, k \geq k_p$.

Definition 2.2. *Let A be a DRl-semigroup. We say that a sequence (x_k) in A relatively uniformly converges (briefly ru-converges) to an element $x \in A$, in symbols $x_k \to x$, whenever there exists $u \in A^+$ such that $x_k \xrightarrow{u} x$.*

Example 2.1. *Let A be the unit interval $[0,1]$ of real numbers, $a \oplus b = \min\{1, a+b\}$. With the usual ordering $(A, \oplus, \leq, -)$ is a DRl-semigroup. Let define (x_k) by*

$$x_k = \begin{cases} \frac{1}{2}, & k = p^2 \\ \frac{p^2}{2p^2+2}, & k \neq p^2 \end{cases}, \ p = 1, 2, 3, \ldots.$$

and $u \in \left[\frac{1}{4}, 1\right]$. Then

$$\begin{aligned} p\left|x_k - \tfrac{1}{2}\right| &= p\left[\left(x_k - \tfrac{1}{2}\right) \vee \left(\tfrac{1}{2} - x_k\right)\right] \\ &= p\left(\tfrac{1}{2} - x_k\right) \\ &= \underbrace{\left(\frac{1}{2} - x_k\right) \oplus \ldots \oplus \left(\frac{1}{2} - x_k\right)}_{p \text{ times}} \\ &= \begin{cases} 0, & k = p^2 \\ \frac{p}{2p^2+2}, & k \neq p^2 \end{cases}, p = 1, 2, 3, \ldots. \end{aligned}$$

and there exists $k_p \in \mathbb{N}$, such that $p\left|x_k - \frac{1}{2}\right| \leq u$ for each $k \in \mathbb{N}$, $k \geq k_p$. Hence, $x_k \to \frac{1}{2}$.

The statistical convergence was first introduced by Steinhaus [8] and after the papers of Connor [1] and Fridy [3] about this convergence method, the developments started and it was studied by other authors [2, 5, 7]. We use $\mathbb{N}$ for the set of all positive integers. Let $K \subseteq \mathbb{N}$, then the natural density of K denoted by $\delta(K)$, is given by

$$\delta(K) := \lim_n \frac{1}{n} \left|\{k \leq n \ : k \in K\}\right|$$

whenever the limit exists, where $|B|$ denotes the cardinality of the set B. It is known that the density may not exists for each set K. But the upper density $\overline{\delta}$ always exists and it is identified by $\overline{\delta}(K) := \limsup_n \frac{1}{n} \left|\{k \leq n \ : k \in K\}\right|$. Moreover, $\overline{\delta}(K) > 0$.

Now, we introduce the notion of the statistical relative uniform convergence with respect to the dually residuated lattice totally ordered semigroup A (simply, a *DRlt-semigroup A*) which is the DRl-semigroup such that A is totally ordered set.

Definition 2.3. *Let A be a DRlt-semigroup, (x_k) a sequence in A, $u \in A^+$. It is said that a sequence (x_k) in A is statistically u-uniform convergent to $x \in A$ provided that for each $p \in \mathbb{N}$*

$$\delta(\{k \in \mathbb{N} \ : p\left|x_k - x\right| \geq \ u\}) = 0$$

or equivalently

$$\lim_n \frac{1}{n} \left|\{k \leq n \ : p\left|x_k - x\right| \geq \ u\}\right| = 0.$$

In that case, we write $x_k \xrightarrow{st^u} x$.

Definition 2.4. *Let A be a DRlt-semigroup. We say that a sequence (x_k) in A is statistically relatively uniform convergent to $x \in A$ whenever there exists $u \in A^+$ such that $x_k \overset{st^u}{\to} x$. In that case, we write $st^r - \lim x_k = x$ and x is said to be st^u-limit.*

Example 2.2. *Let A be the unit interval $[0,1]$ of real numbers, $a \oplus b = \min\{1, a+b\}$. With the usual ordering $(A, \oplus, \leq, -)$ is a DRlt-semigroup. Let define (x_k) by*

$$x_k = \begin{cases} 1, & k = n^2 \\ 0, & k \neq n^2 \end{cases}, \; n = 1,2,3,....$$

and $u \neq 0$. Then

$$\begin{aligned} p|x_k - 0| &= p|x_k| = px_k \\ &= \underbrace{x_k \oplus ... \oplus x_k}_{p \text{ times}} \\ &= \begin{cases} 1, & k = n^2 \\ 0, & k \neq n^2 \end{cases}, n = 1,2,3,.... \end{aligned}$$

Hence,

$$\delta(\{k \in \mathbb{N} : p|x_k - 0| \geq u\}) = 0,$$

and we get $st^r - \lim x_k = 0$. However, $\{k \in \mathbb{N} : p|x_k - 0| \geq u\}$ is an infinite set. So, $x_k \nrightarrow 0$.

Theorem 2.1. *Let A be an Archimedean DRlt-semigroup, (x_k) a sequence in A, $x, y \in A$. Then $st^r - \lim$ is unique.*

Proof. Assume that $st^r - \lim x_k = x$ and $st^r - \lim x_k = y$. Then there exist $u, v \in A^+$ such that $x_k \overset{st^u}{\to} x$, $x_k \overset{st^v}{\to} y$. Let $p \in \mathbb{N}$ and define the following sets:

$$\begin{aligned} K_1(p,u) &: = \{k \in \mathbb{N} : p|x_k - x| \geq u\}, \\ K_2(p,v) &: = \{k \in \mathbb{N} : p|x_k - y| \geq v\}. \end{aligned}$$

Since $x_k \overset{st^u}{\to} x$, $\delta(K_1(p,u)) = 0$. Furthermore, using $x_k \overset{st^v}{\to} y$, we get $\delta(K_2(p,v)) = 0$. Now let $K(p, u+v) := K_1(p,u) \cap K_2(p,v)$. Then we observe that $\delta(K(p,u+v)) = 0$ which implies that $\delta(\mathbb{N} \backslash K(p,u+v)) = 1$. If $k \in \mathbb{N} \backslash K(p,u+v)$, then we have

$$\begin{aligned} u + v &\geq p|x_k - x| + p|x_k - y| \\ &= p(|x_k - x| + |x_k - y|) = p(|x - x_k| + |x_k - y|) \\ &\geq p|x - y|. \end{aligned}$$

So Archimedeanicity of A implies $|x - y| = 0$. Hence $x = y$. □

Theorem 2.2. *Let A be an Archimedean DRlt-semigroup and (x_k) be a sequence in A. If (x_k) ru-converges to x, then $st^r - \lim x_k = x$.*

Proof. By hypothesis, for each $p \in \mathbb{N}$ there exists $k_p \in \mathbb{N}$, such that $p|x_k - x| \leq u$ for all $k \geq k_p$. From this, it can be said that the set $\{k \in \mathbb{N} : p|x_k - x| \geq u\}$ has at most finitely many terms. It is known that the density of every finite subset of the natural numbers is zero, hence we can see that $\delta(\{k \in \mathbb{N} : p|x_k - x| \geq u\}) = 0$, whence the result. □

Theorem 2.3. *Let A be an Archimedean DRlt-semigroup, (x_k) and (y_k) be sequences in A, $x,y \in A$. Let $st^r - \lim x_k = x$ and $st^r - \lim y_k = y$. Then*

(i) $st^r - \lim (x_k + y_k) = x + y$,
(ii) $st^r - \lim (x_k - y_k) = x - y$,
(iii) $st^r - \lim (x_k \vee y_k) = x \vee y$,
(iv) $st^r - \lim (x_k \wedge y_k) = x \wedge y$.

Proof. Let $st^r - \lim x_k = x$ and $st^r - \lim y_k = y$. Then there exist $u,v \in A^+$, such that $x_k \overset{st^u}{\to} x$, $y_k \overset{st^v}{\to} y$. Now let $p \in \mathbb{N}$ and define the following sets:

$$\begin{aligned} K_1(p,u) &: = \{k \in \mathbb{N} : p|x_k - x| \geq u\}, \\ K_2(p,v) &: = \{k \in \mathbb{N} : p|y_k - y| \geq v\}. \end{aligned}$$

Since $x_k \overset{st^u}{\to} x$ and $y_k \overset{st^v}{\to} y$, we get $\delta(K_1(p,u)) = 0$ and $\delta(K_2(p,v)) = 0$. Now let $K(p,u+v) := K_1(p,u) \cap K_2(p,v)$. Then we observe that $\delta(K(p,u+v)) = 0$ which implies that $\delta(\mathbb{N} \backslash K(p,u+v)) = 1$.

(i) If $k \in \mathbb{N} \backslash K(p,u+v)$, then in view of Theorem 1.2 (ii), we have

$$\begin{aligned} u+v &\geq p|x_k - x| + p|y_k - y| \\ &= p(|x_k - x| + |y_k - y|) \\ &\geq p|(x_k + y_k) - (x+y)|. \end{aligned}$$

This shows that

$$\delta(\{k \in \mathbb{N} : p|(x_k + y_k) - (x+y)| \geq u+v\}) = 0.$$

So $x_k + y_k \overset{st^{u+v}}{\to} x + y$ and hence $st^r - \lim (x_k + y_k) = x + y$.

(ii) Similar to (i), we can prove that $st^r - \lim (x_k - y_k) = x - y$.

(iii) If $k \in \mathbb{N} \backslash K(p,u+v)$, then according to $(A2)$ we obtain

$$\begin{aligned} u+v &\geq p|x_k - x| + p|y_k - y| = p(|x_k - x| + |y_k - y|) \\ &\geq p(|x_k - x| \vee |y_k - y|) \\ &= p[(x_k - x) \vee (x - x_k) \vee (y_k - y) \vee (y - y_k)] \\ &= p[(x_k - x) \vee (y_k - y) \vee (x - x_k) \vee (y - y_k)] \\ &\geq p[(x_k - (x \vee y)) \vee (y_k - (x \vee y)) \vee (x - (x_k \vee y_k)) \vee (y - (x_k \vee y_k))] \\ &= p[((x_k \vee y_k) - (x \vee y)) \vee ((x \vee y) - (x_k \vee y_k))] \\ &= p|(x_k \vee y_k) - (x \vee y)|. \end{aligned}$$

Then,

$$\delta(\{k \in \mathbb{N} : p|(x_k \vee y_k) - (x \vee y)| \geq u+v\}) = 0.$$

So $x_k \vee y_k \overset{st^{u+v}}{\to} x \vee y$ and we have $st^r - \lim (x_k \vee y_k) = x \vee y$.

(iv) If $k \in \mathbb{N} \backslash K(p,u+v)$, then in view of $(A3)$ we get

$$\begin{aligned} u+v &\geq p|x_k - x| + p|y_k - y| = p(|x_k - x| + |y_k - y|) \\ &\geq p(|x_k - x| \vee |y_k - y|) \\ &= p[(x_k - x) \vee (x - x_k) \vee (y_k - y) \vee (y - y_k)] \\ &= p[(x_k - x) \vee (y_k - y) \vee (x - x_k) \vee (y - y_k)] \\ &\geq p[((x_k \wedge y_k) - x) \vee ((x_k \wedge y_k) - y) \vee ((x \wedge y) - x_k) \vee ((x \wedge y) - y_k)] \end{aligned}$$

$$= p\left[\left(\left(x_k \wedge y_k \right) - \left(x \wedge y \right) \right) \vee \left(\left(x \wedge y \right) - \left(x_k \wedge y_k \right) \right) \right]$$
$$= p\left| \left(x_k \wedge y_k \right) - \left(x \wedge y \right) \right| .$$

This shows that

$$\delta(\{k \in \mathbb{N} \ : p\left| \left(x_k \wedge y_k \right) - \left(x \wedge y \right) \right| \geq \ u + v\}) = 0.$$

Hence $x_k \wedge y_k \overset{st^{u+v}}{\to} x \wedge y$ *and we obtain* $st^r - \lim \left(x_k \wedge y_k \right) = x \wedge y$. □

Theorem 2.4. *Let A be an Archimedean DRlt-semigroup, (x_k) and (y_k) be sequences in A. If $x_k \leq y_k$ for all $k \in K \subset \mathbb{N}$ with $\delta(K) = 1$ and $st^r - \lim x_k = x$, $st^r - \lim y_k = y$, then $x \leq y$.*

Proof. By hypothesis, since $st^r - \lim x_k = x$ and $st^r - \lim y_k = y$ we get from Theorem 2.3 (iii) that $st^r - \lim \left(x_k \vee y_k \right) = x \vee y$. Therefore, there exists $u \in A^+$ such that $x_k \vee y_k \overset{st^u}{\to} x \vee y$. Let $p \in \mathbb{N}$ and define the following set:

$$K_1\left(p,u \right) := \{k \in \mathbb{N} \ : p\left| \left(x_k \vee y_k \right) - \left(x \vee y \right) \right| \geq \ u\} .$$

Since $x_k \vee y_k \overset{st^u}{\to} x \vee y$, $\delta\left(K_1\left(p,u \right) \right) = 0$. This implies that

$$\delta\left(\mathbb{N} \backslash K_1\left(p,u \right) \right) = \delta\left(\{k \in \mathbb{N} \ : u \geq p\left| \left(x_k \vee y_k \right) - \left(x \vee y \right) \right| \} \right) = 1.$$

Let $k \in K \cap \left(\mathbb{N} \backslash K_1\left(p,u \right) \right)$. It is clear that $\delta\left(K \cap \left(\mathbb{N} \backslash K_1\left(p,u \right) \right) \right) = 1$ and $u \geq p\left| \left(x_k \vee y_k \right) - \left(x \vee y \right) \right| = p\left| y_k - \left(x \vee y \right) \right|$.

So $\delta\left(\{k \in \mathbb{N} \ : p\left| y_k - \left(x \vee y \right) \right| \geq u\} \right) = 0$. Then from this, we have that $st^r - \lim y_k = x \vee y$. Because of $st^r - \lim$ is unique, $x \vee y = y$. This yields $x \leq y$. □

Theorem 2.5. *Let A be an Archimedean DRlt-semigroup, (x_k), (y_k) and (z_k) be sequences in A. If*

(i) $x_k \leq y_k \leq z_k$ *for all* $k \in K \subset \mathbb{N}$ *with* $\delta(K) = 1$ *and*

(ii) $st^r - \lim x_k = st^r - \lim z_k = x$,

then $st^r - \lim y_k = x$.

Proof. Since $st^r - \lim x_k = st^r - \lim z_k = x$, there exist $u, v \in A^+$, such that $x_k \overset{st^u}{\to} x$, $z_k \overset{st^v}{\to} x$. Let $p \in \mathbb{N}$ and define the following sets:

$$K_1\left(p,u \right) := \{k \in \mathbb{N} : p\left| x_k - x \right| \geq \ u\} ,$$
$$K_2\left(p,v \right) := \{k \in \mathbb{N} : p\left| z_k - x \right| \geq \ v\} ,$$

then $\delta\left(K_1\left(p,u \right) \right) = 0$ and $\delta\left(K_2\left(p,v \right) \right) = 0$. Let $k \in K \cap \left(\mathbb{N} \backslash K_1\left(p,u \right) \right) \cap \left(\mathbb{N} \backslash K_2\left(p,v \right) \right)$. It is clear that $\delta\left(K \cap \left(\mathbb{N} \backslash K_1\left(p,u \right) \right) \cap \left(\mathbb{N} \backslash K_2\left(p,v \right) \right) \right) = 1$ and

$$\begin{aligned} u + v &\geq p\left| x_k - x \right| + p\left| z_k - x \right| = p\left(\left| x_k - x \right| + \left| z_k - x \right| \right) \\ &\geq p\left(\left| x_k - x \right| \vee \left| z_k - x \right| \right) \\ &= p\left[\left(x_k - x \right) \vee \left(x - x_k \right) \vee \left(z_k - x \right) \vee \left(x - z_k \right) \right] \\ &\geq p\left[\left(z_k - x \right) \vee \left(x - x_k \right) \right] . \end{aligned}$$

In view of $(A1)$ from $x_k \leq y_k \leq z_k$ we get $y_k - x \leq z_k - x, \ x - y_k \leq x - x_k$. This yields $\left(y_k - x \right) \vee \left(x - y_k \right) \leq \left(z_k - x \right) \vee \left(x - x_k \right)$. Therefore $u + v \geq p\left[\left(z_k - x \right) \vee \left(x - x_k \right) \right] \geq p\left[\left(y_k - x \right) \vee \left(x - y_k \right) \right] = p\left| y_k - x \right|$. Let

$$K_3\left(p,u + v \right) := \{k \in \mathbb{N} \ : p\left| y_k - x \right| \geq u + v\} .$$

It is clear that the set

$$K_3(p,u+v) \subseteq K_1(p,u) \cup K_2(p,v) \cup (\mathbb{N} \backslash K)$$

and $\delta(K_3(p,u+v)) = 0$. Hence $st^r - \lim y_k = x$. □

We now introduce the notion of $st^r -$Cauchy sequence and give a characterization.

Definition 2.5. *Let A be an Archimedean DRlt-semigroup and (x_k) be a sequence in A. We say that a sequence (x_k) is a $st^r -$Cauchy with respect to the statistical relative uniform convergence, if for some $u \in A^+$ and each $p \in \mathbb{N}$, there exists a positive integer $m \in \mathbb{N}$ satisfying*

$$\delta(\{k \in \mathbb{N} : p|x_k - x_m| \geq u\}) = 0.$$

Theorem 2.6. *Let A be an Archimedean DRlt-semigroup and (x_k) be a sequence in A. If (x_k) is statistically relatively uniform convergent, then it is a $st^r -$Cauchy.*

Proof. Assume that $st^r - \lim x_k = x$. Then there exists $u \in A^+$, such that $x_k \overset{st^u}{\to} x$. Therefore the set $K_1(p,u) := \{k \in \mathbb{N} : p|x_k - x| \geq u\}$ has density zero. This implies that the set $\mathbb{N} \backslash K_1(p,u)$ has density one and therefore non empty. So we can choose $m \in \mathbb{N}$ with $m \notin K_1(p,u)$, but then we have $u \geq p|x_m - x|$. Let $K(p,2u) := \{k \in \mathbb{N} : p|x_k - x_m| \geq 2u\}$. We prove that $\delta(K(p,2u)) = 0$. Since

$$\begin{aligned} 2u &\geq p|x_k - x| + p|x_m - x| = p(|x_k - x| + |x - x_m|) \\ &\geq p|x_k - x_m| \end{aligned}$$

we can write

$$\{k \in \mathbb{N} : 2u \geq p|x_k - x_m|\} \supset \{k \in \mathbb{N} : u \geq p|x_k - x|\} \cap \{k \in \mathbb{N} : u \geq p|x_m - x|\}.$$

From this we get

$$\begin{aligned} \delta(\{k \in \mathbb{N} : p|x_k - x_m| \geq 2u\}) &\leq \delta(\{k \in \mathbb{N} : p|x_k - x| \geq u\}) \\ &+ \delta(\{k \in \mathbb{N} : p|x_m - x| \geq u\}). \end{aligned}$$

Hence

$$\delta(\{k \in \mathbb{N} : p|x_k - x_m| \geq 2u\}) = 0$$

which shows that our claim $\delta(K(p,2u)) = 0$ holds true. □

3. LIMIT POINTS AND CLUSTER POINTS IN DRLT-SEMIGROUPS

Fridy defined statistical limit points and statistical cluster points of a number sequence (x_k) in 1993 [4]. Now we study the concepts of statistical relative uniform limit points and cluster points in DRlt-semigroups. Also, we give relations between them and the set of limit points in DRlt-semigroups.

Definition 3.1. *Let A be a DRlt-semigroup and (x_k) be a sequence in A, $u \in A^+$. $L \in A$ is said to be a limit point of the sequence (x_k) providing that there is a subsequence of (x_k) that ru-converges to x. Let the set of all limit points of the sequence denoted by $L_{ru}[(x_k)]$.*

Definition 3.2. *Let A be a DRlt-semigroup and (x_k) be a sequence in A, $u \in A^+$. If $\{x_{k(n)}\}$ is a subsequence of (x_k) and $K := \{k(n) \in \mathbb{N} : n \in \mathbb{N}\}$ then $\{x\}_K$ in which case $\delta(K) = 0$ is the abbreviation of $\{x_{k(n)}\}$ and $\{x\}_K$ is said to be thin subsequence or a subsequence of density zero. In other words, if K does not have density zero then $\{x\}_K$ is a nonthin subsequence of (x_k).*

Definition 3.3. *Let A be a DRlt-semigroup and (x_k) be a sequence in A, $u \in A^+$. $x \in A$ is said to be a statistical relative uniform limit point of the sequence (x_k) on the condition that there is a nonthin subsequence of (x_k) that ru-converges to x. Then we say x is a st^r-limit point of sequence (x_k). Let $\Lambda_{st^r}[(x_k)]$ denote the set of all st^r-limit points of the sequence.*

Example 3.1. *Let A and (x_k) be the same as in Example 2.2, then $L_{ru}[(x_k)] = \{0,1\}$ and $\Lambda_{st^r}[(x_k)] = \{0\}$.*

It is clear that $\Lambda_{st^r}[(x_k)] \subset L_{ru}[(x_k)]$ for any sequence (x_k). To show that $\Lambda_{st^r}[(x_k)]$ and $L_{ru}[(x_k)]$ can be very different, we give an example as follows.

Example 3.2. *Let A be the unit interval $[0,1]$ of real numbers, $a \oplus b = \min\{1, a+b\}$. With the usual ordering $(A, \oplus, \leq, -)$ is a DRlt-semigroup. Let (r_k) be a sequence whose range is the set of all rational numbers of $[0,1]$ and define (x_k) by*

$$x_k = \begin{cases} r_j, & k = j^2 \\ \frac{1}{2}, & k \neq j^2 \text{ and } k \text{ is odd} \\ \frac{1}{3}, & k \neq j^2 \text{and } k \text{ is even} \end{cases}, \quad j = 1,2,3,....$$

Then for $k \neq j^2$ and k is odd, $p|x_k - x| = p\left|\frac{1}{2} - \frac{1}{2}\right| = 0 \leq u$. Therefore $x_k \to \frac{1}{2}$. For $k \neq j^2$and k is even, $p|x_k - x| = p\left|\frac{1}{3} - \frac{1}{3}\right| = 0 \leq u$. Hence $x_k \to \frac{1}{3}$. We get $\Lambda_{st^r}[(x_k)] = \left\{\frac{1}{2}, \frac{1}{3}\right\}$. However, for every $x \in [0,1]$ and $k = j^2$,

$$\begin{aligned} p|x_k - x| &= p\left|r_j - x\right| \\ &= \left|r_j - x\right| \oplus ... \oplus \left|r_j - x\right| \\ &= \min\left\{1, \left|r_j - x\right| + ... + \left|r_j - x\right|\right\} \\ &\leq 1 \end{aligned}$$

$\{r_k : k \in \mathbb{N}\}$ is dense in $[0,1]$ implies that $L_{ru}[(x_k)] = [0,1]$.

Definition 3.4. *Let A be a DRlt-semigroup and (x_k) be a sequence in A, $u \in A^+$. Then $x \in A$ is said to be a statistical relative uniform cluster point of the sequence (x_k) provided that for each $p \in \mathbb{N}$,*

$$\overline{\delta}(\{k \in \mathbb{N} : p|x_k - x| \leq u\}) > 0.$$

In that case we say that x is an st^r-cluster point of sequence (x_k). Let $\Gamma_{st^r}[(x_k)]$ denote the set of all st^r-cluster points of sequence (x_k).

Theorem 3.1. *Let A be an Archimedean DRlt-semigroup, (x_k) be a sequence in A, $x \in A$. For any sequence (x_k), $\Lambda_{st^r}[(x_k)] \subset \Gamma_{st^r}[(x_k)]$.*

Proof. Suppose $x \in \Lambda_{st^r}[(x_k)]$. So, there is a nonthin subsequence $\{x_{k(n)}\}$ of (x_k) that ru-converges to x, i.e.

$$\delta\left(\left\{k(n) \in \mathbb{N} : p\left|x_{k(n)} - x\right| \leq u\right\}\right) = d > 0.$$

Since

$$\{k \in \mathbb{N} : p|x_k - x| \leq u\} \supset \left\{k(n) \in \mathbb{N} : p\left|x_{k(n)} - x\right| \leq u\right\}$$

for each $p \in \mathbb{N}$, we have

$$\{k \in \mathbb{N} : p|x_k - x| \leq u\} \supset \{k(n) \in \mathbb{N} : n \in \mathbb{N}\} \setminus \left\{k(n) \in \mathbb{N} : p\left|x_{k(n)} - x\right| \geq u\right\}.$$

Since $\{x_{k(n)}\}$ ru-converges to x, the set

$$\left\{k(n) \in \mathbb{N} : p\left|x_{k(n)} - x\right| \geq u\right\}$$

is finite for any $p \in \mathbb{N}$. Therefore,

$$\overline{\delta}(\{k \in \mathbb{N} : p|x_k - x| \leq u\}) \geq \overline{\delta}(\{k(n) \in \mathbb{N} : n \in \mathbb{N}\}) - \overline{\delta}\left(\left\{k(n) \in \mathbb{N} : p\left|x_{k(n)} - x\right| \geq u\right\}\right).$$

Hence

$$\overline{\delta}(\{k \in \mathbb{N} : p|x_k - x| \leq u\}) > 0$$

that means $x \in \Gamma_{st^r}[(x_k)]$. □

Example 3.3. *Define the sequence (x_k) by*

$$x_k = \frac{1}{m}, k = 2^{m-1}(2q+1),$$

i.e., $m-1$ is the number of factors of 2 in the prime factorization of k. For each $p \in \mathbb{N}$, $u = 1$,

$$px_k = \underbrace{x_k \oplus \ldots \oplus x_k}_{ptimes} = \underbrace{\frac{1}{m} \oplus \ldots \oplus \frac{1}{m}}_{ptimes} \leq 1,$$

it is easy to see that for each m, $\delta(\{k \in \mathbb{N} : px_k \leq 1\}) = 2^{-m} > 0$, whence $\frac{1}{m} \in \Lambda_{st^r}[(x_k)]$. Also, $\delta(\{k \in \mathbb{N} : p|x_k - 0| \leq 1\}) = \delta(\{k \in \mathbb{N} : 0 \leq px_k \leq 1\}) = 2^{-m}$, so $0 \in \Gamma_{st^r}[(x_k)]$ and we have $\Gamma_{st^r}[(x_k)] = \{0\} \cup \left\{\frac{1}{m} : m = 1, 2, \ldots\right\}$. Now we claim that $0 \notin \Lambda_{st^r}[(x_k)]$; for, if $\{x\}_K$ is a subsequence that has limit zero, then we obtain that $\delta(K) = 0$ by observing that for each m, there exists $M > 0$ such that

$$\begin{aligned} |K_n| &= |\{k \in K_n : px_k \geq 1\}| + |\{k \in K_n : px_k \leq 1\}| \\ &\leq M + |\{k \in \mathbb{N} : px_k \leq 1\}| \\ &\leq M + \tfrac{n}{2^m}. \end{aligned}$$

Hence $\delta(K) \leq 2^{-m}$, and since m is arbitrary this implies that $\delta(K) = 0$.

Theorem 3.2. *Let A be an Archimedean DRlt-semigroup, (x_k) be a sequence in A, $x \in A$. For any sequence (x_k), $\Gamma_{st^r}[(x_k)] \subset L_{st^r}[(x_k)]$.*

Proof. $x \in \Gamma_{st^r}[(x_k)]$, then

$$\overline{\delta}(\{k \in \mathbb{N} : p|x_k - x| \leq u\}) > 0$$

for each $p \in \mathbb{N}$. We set $\{x\}_K$ a nonthin subsequence of (x_k) such that

$$K = K(p,u) := \left\{k(n) \in \mathbb{N} : p\left|x_{k(n)} - x\right| \geq u\right\}$$

for each $p \in \mathbb{N}$ and $\delta(K) \neq 0$. Hence there are infinitely many elements in K, $x \in L_{st^r}[(x_k)]$. □

Theorem 3.3. *Let A be an Archimedean DRlt-semigroup, (x_k) be a sequence in A. For a sequence (x_k), $st^r - \lim x_k = x_0$ then $\Lambda_{st^r}[(x_k)] = \Gamma_{st^r}[(x_k)] = \{x_0\}$.*

Proof. Assume that $st^r - \lim x_k = x_0$, then there exists $u \in A^+$ such that $x_k \overset{st^u}{\to} x_0$. First we show that $\Lambda_{st^r}[(x_k)] = \{x_0\}$. We suppose that $\Lambda_{st^r}[(x_k)] = \{x_0, y_0\}$ such that $p|x_0 - y_0| \geq 2u$. Hence, there exist $\left\{x_{k(n)}\right\}$ and $\left\{x_{j(i)}\right\}$ nonthin subsequences of (x_k) that ru-converge to x_0, y_0, respectively. Since $\left\{x_{j(i)}\right\}$ ru-converge to y_0, for each $p \in \mathbb{N}$

$$K := K(p,u) := \left\{j(i) \in \mathbb{N} : p\left|x_{j(i)} - y_0\right| \geq u\right\}$$

is a finite set so $\delta(K) = 0$. Then observe that

$$\{j(i) \in \mathbb{N} : i \in \mathbb{N}\} = \left\{j(i) \in \mathbb{N} : p\left|x_{j(i)} - y_0\right| \geq u\right\} \cup \left\{j(i) \in \mathbb{N} : p\left|x_{j(i)} - y_0\right| \leq u\right\}$$

which implies

$$\delta\left(\left\{j(i) \in \mathbb{N} : p\left|x_{j(i)} - y_0\right| \leq u\right\}\right) \neq 0. \tag{3.1}$$

Since $st^r - \lim x_k = x_0$,

$$\delta(\{k \in \mathbb{N} : p|x_k - x_0| \geq u\}) = 0 \tag{3.2}$$

for each $p \in \mathbb{N}$. Therefore, we can write

$$\delta(\{k \in \mathbb{N} : p|x_k - x_0| \leq u\}) \neq 0.$$

For every $p|x_0 - y_0| \geq 2u$,

$$\left\{j(i) \in \mathbb{N} : p\left|x_{j(i)} - y_0\right| \leq u\right\} \cap \{k \in \mathbb{N} : p|x_k - x_0| \leq u\} = \varnothing.$$

Hence, we can write

$$\left\{j(i) \in \mathbb{N} : p\left|x_{j(i)} - y_0\right| \leq u\right\} \subset \{k \in \mathbb{N} : p|x_k - x_0| \geq u\}.$$

Therefore

$$\overline{\delta}\left(\left\{j(i) \in \mathbb{N} : p\left|x_{j(i)} - y_0\right| \leq u\right\}\right) \leq \overline{\delta}(\{k \in \mathbb{N} : p|x_k - x_0| \geq u\}) = 0.$$

This contradicts (3.1). Hence $\Lambda_{st^r}[(x_k)] = \{x_0\}$.

Now we suppose that $\Gamma_{st^r}[(x_k)] = \{x_0, z_0\}$ such that $p|x_0 - y_0| \geq 2u$. Then

$$\overline{\delta}(\{k \in \mathbb{N} : p|x_k - z_0| \leq u\}) \neq 0. \tag{3.3}$$

Since

$$\{k \in \mathbb{N} : p|x_k - x_0| \leq u\} \cap \{k \in \mathbb{N} : p|x_k - z_0| \leq u\} = \varnothing$$

for every $p|x_0 - y_0| \geq 2u$, so

$$\{k \in \mathbb{N} : p|x_k - x_0| \geq u\} \supset \{k \in \mathbb{N} : p|x_k - z_0| \leq u\}.$$

Therefore

$$\overline{\delta}\left(\{k \in \mathbb{N} : p\,|x_k - x_0| \geq u\}\right) \geq \overline{\delta}\left(\{k \in \mathbb{N} : p\,|x_k - z_0| \leq u\}\right). \tag{3.4}$$

From (3.3), the right hand-side of (3.4) is greater than zero and from (3.2), the left hand-side of (3.4) equals zero. This is a contradiction. Hence $\Gamma_{st^r}\left[(x_k)\right] = \{x_0\}$. □

REFERENCES

[1] J. Connor: *The statistical and strong p-Cesaro convergence of sequences*, Analysis **8** (1988) 47-63.

[2] J. Connor, J. Kline: *On Statistical Limit Points and the Consistency of Statistical Convergence*, J. Math. Anal. Appl. **197** (1996) 392–399.

[3] J.A. Fridy: *On statistical convergence*, Analysis **5** (1985), 301-313.

[4] J.A. Fridy: *Statistical limit points*, Proc. Amer. Math. Soc. **118(4)** (1993) 1187-1192.

[5] J.A. Fridy and C. Orhan: *Lacunary statistical convergence*, Pacific J. Math. **160** (1993) 43–51.

[6] M. Jasem: *Relatively Uniform Convergence In Dually Residuated Lattice Ordered Semigroups*, Aplimat (2011) 129-136.

[7] E. Kolk: *Matrix Summability of Statistically Convergent Sequences*, Analysis **13** (1993) 77-83.

[8] H. Steinhaus: *Sur la convergence ordinaire et la convergence asymptotique*, Collog. Math. **2** (1951) 73-74.

[9] K. L. N. Swamy: *Dually Residuated Lattice Ordered Semigroups*, Math. Annalen, **159** (1965) 105-114.

[10] K. L. N. Swamy: *Dually Residuated Lattice Ordered Semigroups, II*, Math. Annalen, **160** (1965) 64-71.

[11] K. L. N. Swamy: *Dually Residuated Lattice Ordered Semigroups, III*, Math. Annalen, **167** (1966) 71-74.

(Received: April 13, 2019)
(Revised: July 22, 2019)

Kamil Demirci and Sevda Yıldız
Sinop University
Department of Mathematics
Sinop, Turkey
e-mail: *kamild@sinop.edu.tr*
e-mail: *sevdaorhan@sinop.edu.tr*