SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.21.01.08
Vol.21 (34), No.1 (2025), 89–102

SPIRAL-LIKE FUNCTIONS ASSOCIATED WITH BOREL DISTRIBUTION-TYPE MITTAG-LEFFLER FUNCTION

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ABSTRACT. The purpose of the present paper is to obtain some sufficient conditions for analytic functions, whose coefficients are probabilities of the Borel Distribution-Type Mittag-Leffler Function, to belong to the class of spiral-like univalent functions. Further, we discuss the geometric properties of an integral operator related to the Borel Distribution-type Mittag-Leffler Function. Moreover, we investigated and explored some applications of our main results for the error function.

1. INTRODUCTION

Let $\mathcal{A}$ represent the family of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

in the open unit disk given by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the normalized condition $f'(0) - 1 = 0 = f(0)$. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) that are also univalent in $\mathbb{D}$.

A function $f \in \mathcal{A}$ is said to be starlike of order γ, $(0 \leq \gamma < 1)$, if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in \mathbb{D}.$$

This function class is denoted by $S^*(\gamma)$. We write $S^*(0) := S^*$, where S^* denotes the class of functions $f \in \mathcal{A}$ such that $f(D)$ is a starlike domain with respect to the origin.

A function $f \in \mathcal{A}$ is said to be convex of order $\gamma(0 \leq \gamma < 1)$, if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, \quad z \in \mathbb{D}.$$

This function class is denoted by $\mathcal{K}(\gamma)$. We note $\mathcal{K}(0) := \mathcal{K}$, the well-known standard class of convex functions. The classes of starlike and convex functions of order

2020 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. Analytic function, Borel distribution, Mittag-Leffler function, spiral-like function, convolution.

γ were studied earlier by Robertson [18] and Silverman [19]. It is an established fact that

$$f \in \mathcal{K}(\gamma) \Leftrightarrow zf' \in S^*(\gamma).$$

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k, z \in \mathbb{D}$, we define the *Hadamard product* (or *convolution*) of f and g by

$$(f * g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in \mathbb{D}.$$

Let $E_\alpha(z)$ be the function defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (\alpha \in \mathbb{C}, Re(\alpha) > 0, z \in \mathbb{C}),$$

that was introduced by Mittag-Leffler [16] and commonly known as the Mittag-Leffler function. The study of the Mittag-Leffler function and its various generalizations has become a very popular topic in Geometric Function Theory. During the last two decades this function has come into prominence after about nine decades of its discovery by the Swedish Mathematician Mittag-Leffler, due to the vast potential of its applications in solving the problems of physical, biological, engineering, earth sciences, and other applications.The Mittag Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag-Leffler function plays an analogous role in the solution of non-integer order differential equations. A more general function of $E_\alpha(z)$ is $E_{\alpha,\beta}(z)$ introduced by Wiman [25] given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad (\alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, z \in \mathbb{C}).$$

When $\beta = 1$, it is abbreviated as $E_\alpha(z) = E_{\alpha,1}(z)$. We observe that the function $E_{\alpha,\beta}$ contains many well-known functions as its special case, for example,

$$E_0 = \frac{1}{1-z}, \quad E_1(\pm z) = E_{1,1}(\pm z) = e^{\pm z}, \quad E_2(z) = E_{2,1}(z) = cosh(\sqrt{z}),$$

$$E_2(-z^2) = cosz, \quad E_3(z) = \frac{1}{2}\left[e^{z^{1/3}} + 2e^{-(1/2)z^{1/3}} cos\left(\frac{\sqrt{3}}{2}\right) z^{1/3}\right],$$

$$E_4(z) = \frac{1}{2}\left[cosz^{1/4} + coshz^{1/4}\right], \quad E_{1,2}(z) = \frac{e^z - 1}{z}, \qquad E_{2,2}\left(-z^2\right) = \frac{sinz}{z},$$

$$E_{2,1}\left(z^2\right) = coshz, \quad E_{2,1}\left(-z^2\right) = cosz, \quad E_{2,2}(z) = \sqrt{z} sinh\sqrt{z}.$$

For a detailed account of various properties, generalizations, and applications of this function, the reader may refer to earlier important works of Attiya [4], Bansal and Prajapat [5], Frasin et al. [9], Gorenflo et al. [10], Noreen et al. [17], Luchko and Srivastava [13], Srivastava et al. [23], and others (refer [11, 12, 21]).

Observe that Mittag-Leffler function $E_{\alpha,\beta}(z)$ does not belong to the family $\mathcal{A}$. Thus, it is natural to consider the following normalization of Mittag-Leffler functions

$$\mathbb{E}_{\alpha,\beta}(z) = z\Gamma(\beta)E_{\alpha,\beta}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)} z^k, \tag{1.2}$$

$$(\alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, z \in \mathbb{C}).$$

The celebrated and widely used error function erf [1] is given by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} z^{2k+1}. \tag{1.3}$$

The complement of the error function, $\operatorname{erfc}(z)$, is defined by

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} z^{2k+1}.$$

The normalized form of the error function, denoted by $\operatorname{Erf}(z)$, and normalized using the condition $\operatorname{Erf}'(0) = 1$, is given by

$$\operatorname{Erf}(z) = \frac{\sqrt{\pi z}}{2} \operatorname{erf}(z) = \sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{(k-1)!(2k-1)} z^k.$$

It is of interest to note that by fixing $\alpha = \frac{1}{2}$ and $\beta = 1$, we get

$$E_{\frac{1}{2},1}(z) = e^{z^2} \operatorname{erfc}(-z),$$

that is

$$E_{\frac{1}{2},1}(z) = e^{z^2} \left(1 + \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(2k+1)} z^{2k+1}\right).$$

A function $f \in \mathcal{A}$ is said to be a spiral-like function if

$$Re\left(e^{-i\xi} \frac{zf'(z)}{f(z)}\right) > 0, \quad z \in \mathbb{D}, \tag{1.4}$$

for some $\xi \in \mathbb{R}$ with $|\xi| < \dfrac{\pi}{2}$. This class of spiral-like function was introduced in [20]. Also, the function f is said to be convex spiral-like if $zf'(z)$ is spiral-like [2, 3].

Murugusundaramoorthy [14, 15] introduced certain subclasses of spiral-like functions as below.

Definition 1.1. *For* $0 \le \rho < 1, 0 \le \gamma < 1$ *and* $|\xi| < \dfrac{\pi}{2}$

$$\mathcal{S}(\xi, \rho, \gamma) = \left\{ f \in \mathcal{S} : Re\left(e^{i\xi} \frac{zf'(z)}{(1-\rho)f(z) + \rho z f'(z)}\right) > \gamma cos\xi,\ z \in \mathbb{D} \right\}. \tag{1.5}$$

By virtue of Alexander's relation (see [6]), we define the following subclass:

Definition 1.2. *For* $0 \leq \rho < 1, 0 \leq \gamma < 1$ *and* $|\xi| < \dfrac{\pi}{2}$

$$\mathcal{K}(\xi,\rho,\gamma) = \left\{ f \in \mathcal{S} : Re\left(e^{i\xi} \frac{zf''(z)+f'(z)}{f'(z)+\rho z f''(z)} \right) > \gamma cos\xi, \ z \in \mathbb{D} \right\}. \tag{1.6}$$

Recently, Wanas and Khuttar [24] developed the power series whose coefficients represent probabilities of the Borel distribution

$$\mathcal{M}(\mu,z) = z + \sum_{k=2}^{\infty} \frac{(\mu(k-1))^{k-2} e^{-\mu(k-1)}}{(k-1)!} z^k, \quad (0 < \mu \leq 1, z \in \mathbb{D}). \tag{1.7}$$

We note that, by using the ratio test we conclude that the radius of convergence of the above power series is infinity.

Thus by using (1.2) and (1.7) and by the convolution operator, we define the Mittag-Leffler-type Borel distribution series as below

$$\mathcal{B}^{\mu}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) * \mathcal{M}(\mu,z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\beta)\,(\mu(k-1))^{k-2} e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!} z^k, \tag{1.8}$$

where $0 < \mu \leq 1, \alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, z \in \mathbb{C}$.

Now, using the linear operator $\mathcal{L}^{\mu}_{\alpha,\beta} : \mathcal{A} \to \mathcal{A}$ defined by the convolution

$$\begin{aligned} \mathcal{L}^{\mu}_{\alpha,\beta} f(z) = \mathcal{B}^{\mu}_{\alpha,\beta}(z) * f(z) &= z + \sum_{k=2}^{\infty} \frac{\Gamma(\beta)\,(\mu(k-1))^{k-2} e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!} a_k z^k, \qquad (1.9) \\ &= z + \sum_{k=2}^{\infty} \phi_k a_k z^k, \end{aligned}$$

where $\alpha, \beta \in \mathbb{C}, Re(\alpha) > 0, Re(\beta) > 0, z \in \mathbb{C}, 0 < \mu \leq 1$ and

$$\phi_k = \sum_{k=2}^{\infty} \frac{\Gamma(\beta)\,(\mu(k-1))^{k-2} e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}. \tag{1.10}$$

To establish our main results, we need to recall the following lemmas.

Lemma 1.1. *A function $f(z)$ given by (1.1) is a member of $\mathcal{S}(\xi,\rho,\gamma)$ if*

$$\sum_{k=2}^{\infty} [(1-\rho)(k-1)sec\xi + (1-\gamma)(1+k\rho-\rho)]\,|a_k| \leq 1-\gamma, \tag{1.11}$$

for some $|\xi| < \dfrac{\pi}{2}, 0 \leq \rho < 1, 0 \leq \gamma < 1$.

Lemma 1.2. *A function $f(z)$ given by (1.1) is a member of $\mathcal{K}(\xi,\rho,\gamma)$ if*

$$\sum_{k=2}^{\infty} k\,[(1-\rho)(k-1)sec\xi + (1-\gamma)(1+k\rho-\rho)]\,|a_k| \leq 1-\gamma, \tag{1.12}$$

for some $|\xi| < \dfrac{\pi}{2}, 0 \leq \rho < 1, 0 \leq \gamma < 1.$

Definition 1.3. *A function* $f \in \mathcal{A}$ *is said to in the class* $\mathcal{R}_{\nu}^{\tau}(\delta)$*, if it satisfies the inequality*

$$\left| \frac{(1-\delta)\frac{f(z)}{z} + \nu f'(z) - 1}{2\tau(1-\delta) + (1-\nu)\frac{f(z)}{z} + \nu f'(z) - 1} \right| < 1 \ (z \in \mathbb{U}),$$

where $\tau \in \mathbb{C} \setminus \{0\}, \delta < 1, 0 < \nu \leq 1.$

This class was introduced by Swaminathan [22]. for $\nu = 1$ the class is reduces to familiar class introduced by Dixit and Pal [7].

Lemma 1.3. *For* $f \in \mathcal{R}_{\nu}^{\tau}(\delta)$ *is of the form (1.1), then*

$$|a_k| = \frac{2|\tau|(1-\delta)}{1+\nu(k-1)}, \ k \in \mathbb{N} \setminus \{1\}. \tag{1.13}$$

The bounds given in (1.13) are sharp.

Lemma 1.4. *A function* $f(z)$ *given by (1.1) is a member of* $\mathcal{S}(\xi, \gamma)$ *if*

$$\sum_{k=2}^{\infty} [(k-1)sec\xi + (1-\gamma)]\,|a_k| \leq 1-\gamma, \tag{1.14}$$

for some $|\xi| < \dfrac{\pi}{2}, 0 \leq \gamma < 1.$

Lemma 1.5. *A function* $f(z)$ *given by (1.1) is a member of* $\mathcal{K}(\lambda, \gamma)$ *if*

$$\sum_{k=2}^{\infty} k\,[(k-1)sec\xi + (1-\gamma)]\,|a_k| \leq 1-\gamma, \tag{1.15}$$

for some $|\xi| < \dfrac{\pi}{2}, 0 \leq \gamma < 1.$

Motivated by the earlier work Deeb et al. [8], we define the subclasses of spiral like functions $\mathcal{S}(\xi, \rho, \gamma)$ and $\mathcal{K}(\xi, \rho, \gamma)$ related to the Borel distribution-type Mittag Leffler function as in Definitions 1.1 and 1.2. We obtain inclusion results, image properties of $L_{\alpha,\beta}^{\mu}$ operator and geometric properties of integral operator $\Phi_{\alpha,\beta}^{\mu}$ for these subclasses of the spiral-like function classes which we introduce here.

2. Inclusion Results

We begin this section by finding the inclusion results for the function that belongs to the classes $\mathcal{S}(\xi, \rho, \gamma)$ and $\mathcal{K}(\xi, \rho, \gamma)$. For convenience throughout the following, we use the following identities

$$E_{\alpha,\beta}(1) - 1 = \sum_{k=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}, \tag{2.1}$$

$$E'_{\alpha,\beta}(1) - 1 = \sum_{k=2}^{\infty} \frac{k\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}, \tag{2.2}$$

$$E''_{\alpha,\beta}(1) = \sum_{k=2}^{\infty} \frac{k(k-1)\Gamma(\beta)}{\Gamma(\alpha(k-1)+\beta)}, \tag{2.3}$$

$$\mathcal{M}(\mu,1) = \sum_{k=2}^{\infty} \frac{(\mu(k-1))^{k-2} e^{-\mu(k-1)}}{(k-1)!}. \tag{2.4}$$

Theorem 2.1. *If*

$$\left\{[(1-\rho)sec\xi + \rho(1-\gamma)]E'_{\alpha,\beta}(1) + [(1-\rho)(1-\gamma-sec\xi)]E_{\alpha,\beta}(1)\right\}(\mathcal{M}(\mu,1)) \leq 2(1-\gamma), \tag{2.5}$$

then $\mathcal{B}^{\mu}_{\alpha,\beta} \in \mathcal{S}(\xi,\rho,\gamma)$.

Proof. According to Lemma1.1 it is sufficient to show that

$$\sum_{k=2}^{\infty} [(1-\rho)sec\xi(k-1) + (1-\gamma)(1+k\rho-k)] \frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!} \leq 1-\gamma. \tag{2.6}$$

Put $k = (k-1)+1$.
Let

$$\begin{aligned}
\varphi_1(\xi,\rho,\gamma) &= \sum_{k=2}^{\infty} [(1-\rho)sec\xi(k-1) + (1-\gamma)(1+k\rho-k)] \\
&\qquad \times \frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!} \\
&= [(1-\rho)sec\xi + \rho(1-\gamma)] \sum_{k=2}^{\infty} \frac{k\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!} \\
&\qquad + (1-\rho)(1-\gamma-sec\xi) \sum_{k=2}^{\infty} \frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!} \\
&= [(1-\rho)sec\xi + \rho(1-\gamma)] \left[E'_{\alpha,\beta} - 1\right] \sum_{k=2}^{\infty} \frac{(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{(k-1)!} \\
&\qquad + (1-\rho)(1-\gamma-sec\xi) \left[E_{\alpha,\beta} - 1\right] \sum_{k=2}^{\infty} \frac{(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{(k-1)!} \\
&= \left\{[(1-\rho)sec\xi + \rho(1-\gamma)]E'_{\alpha,\beta}(1) + [(1-\rho)(1-\gamma-sec\xi)]E_{\alpha,\beta}(1)\right\} \\
&= (\mathcal{M}(\mu,1)) - (1-\gamma).
\end{aligned}$$

Thus, from the assumption (2.5) it follows that $\varphi_1(\xi,\gamma,\rho) \leq 1-\gamma$, that is (2.6) holds, therefore $\mathcal{B}^{\mu}_{\alpha,\beta} \in \mathcal{S}(\xi,\rho,\gamma)$. $\square$

Theorem 2.2. *If*

$$\left\{[(1-\rho)sec\xi+\rho(1-\gamma)]E''_{\alpha,\beta}(1)+(1-\gamma)E'_{\alpha,\beta}(1)\right\}\mathcal{M}(\mu,1)) \leq (1-\gamma), \tag{2.7}$$

then $\mathcal{B}^{\mu}_{\alpha,\beta} \in \mathcal{K}(\xi,\rho,\gamma)$.

Proof. According to Lemma 1.2 it is sufficient to show that

$$\sum_{k=2}^{\infty} k\left[(1-\rho)sec\xi(k-1)+(1-\gamma)(1+k\rho-k)\right]\frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!} \leq 1-\gamma. \tag{2.8}$$

Writing $k=(k-1)+1$ and $k^2=(k-1)(k-2)+3(k-1)+1$, let

$$\begin{aligned}
\varphi_2(\xi,\rho,\gamma) &= \sum_{k=2}^{\infty} k\left[(1-\rho)sec\xi(k-1)+(1-\gamma)(1+k\rho-k)\right] \\
&\qquad \times \frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!} \\
&= [(1-\rho)sec\xi+\rho(1-\gamma)]\sum_{k=2}^{\infty}\frac{k(k-1)\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!} \\
&\quad +(1-\rho)(1-\gamma-sec\xi)\sum_{k=2}^{\infty}\frac{k\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!} \\
&= [(1-\rho)sec\xi+\rho(1-\gamma)]\left[E''_{\alpha,\beta}(1)\right]\sum_{k=2}^{\infty}\frac{(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{(k-1)!} \\
&\quad +(1-\rho)(1-\gamma-sec\xi)\left[E'_{\alpha,\beta}(1)-1\right]\sum_{k=2}^{\infty}\frac{(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{(k-1)!} \\
&= \left\{[(1-\rho)sec\xi+\rho(1-\gamma)]E''_{\alpha,\beta}(1)+(1-\gamma)\left[E'_{\alpha,\beta}(1)-1\right]\right\}(\mathcal{M}(\mu,1)).
\end{aligned}$$

Thus, from the assumption (2.7) it follows that $\varphi_2(\xi,\gamma,\rho) \leq 1-\gamma$, that is (2.8) holds, therefore $\mathcal{B}^{\mu}_{\alpha,\beta} \in \mathcal{K}(\xi,\rho,\gamma)$. $\square$

3. Image Properties of $\mathcal{L}^{\mu}_{\alpha,\beta}$ Operator

In this section, we study the effect of the $\mathcal{L}^{\mu}_{\alpha,\beta}$ operator on the function class $\mathcal{R}^{\tau}_{\nu}(\delta)$. Making use of Lemma 1.3, we will focus on the influence of the Borel distribution-type Mittag-Leffler function on the classes $\mathcal{S}(\xi,\rho,\gamma)$ and $\mathcal{K}(\xi,\rho,\gamma)$.

Theorem 3.1. *If*

$$\frac{2|\tau|(1-\delta)}{\nu}\Bigg[[(1-\rho)\sec\xi+\rho(1-\gamma)]\left[E_{\alpha,\beta}(1)-1\right]+(1-\rho)(1-\gamma+\sec\xi)$$
$$\times\int_0^1\left(\frac{E_{\alpha,\beta}(t)}{t}-1\right)dt\Bigg]\mathcal{M}(\mu,1)<(1-\gamma), \tag{3.1}$$

then $L^{\mu}_{\alpha,\beta}(f)\in\mathcal{S}(\xi,\rho,\gamma)$.

Proof. According to Lemma 1.1, it is sufficient to show that

$$\sum_{k=2}^{\infty}[(1-\rho)\sec\xi(k-1)+(1-\gamma)(1+k\rho-k)]\times\frac{\Gamma(\beta)(\mu(k-1))(k-2)e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}<1-\gamma. \tag{3.2}$$

Let $f\in\mathcal{R}^{\tau}_{\nu}(\delta)$. Then by Lemma 1.3, we have

$$|a_k|=\frac{2|\tau|(1-\delta)}{1+\nu(k-1)},\quad k\in\mathbb{N}\setminus\{1\},$$

and $1+\nu(k-1)\geq\nu k$. Thus, we have

$$\begin{aligned}\varphi_3(\xi,\rho,\gamma)&=\frac{2|\tau|(1-\delta)}{\nu}\sum_{k=2}^{\infty}\frac{1}{k}[(1-\rho)\sec\xi(k-1)+(1-\gamma)(1+k\rho-k)]\\&\quad\times\frac{\Gamma(\beta)(\mu(k-1))(k-2)e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\\&=[(1-\rho)\sec\xi+(1-\gamma)]\frac{2|\tau|(1-\delta)}{\nu}\sum_{k=2}^{\infty}\frac{\Gamma(\beta)(\mu(k-1))^{(k-2)}e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\\&\quad+(1-\rho)(1-\gamma\sec\xi)\frac{2|\tau|(1-\delta)}{\nu}\sum_{k=2}^{\infty}\frac{1}{k}\frac{\Gamma(\beta)(\mu(k-1))(k-2)e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}.\end{aligned}$$

Using equation (2.1) and the above inequality, we get

$$\varphi_3(\xi,\rho,\gamma)=\frac{2|\tau|(1-\delta)}{\nu}\Bigg[[(1-\rho)\sec\xi+\rho(1-\gamma)]\left[E_{\alpha,\beta}(1)-1\right]$$
$$+(1-\rho)(1-\gamma+\sec\xi)\int_0^1\left(\frac{E_{\alpha,\beta}(t)}{t}-1\right)dt\Bigg]\mathcal{M}(\mu,1)<(1-\gamma). \tag{3.3}$$

Thus, from the assumption (3.1), it follows that $\varphi_3(\xi,\rho,\gamma)<1-\gamma$, that is, inequality (3.2) holds. Therefore, $L^{\mu}_{\alpha,\beta}(f)\in\mathcal{S}(\xi,\rho,\gamma)$. □

Theorem 3.2. *If*

$$\begin{aligned}\Big\{[(1-\rho)sec\xi+\rho(1-\gamma)]\,E^{'}_{\alpha,\beta}(1)+[(1-\rho)(1-\gamma-sec\xi)]\,E_{\alpha,\beta}(1)\Big\}\\ \frac{2|\tau|(1-\delta)}{\nu}(\mathcal{M}(\mu,1))\leq(1-\gamma),\end{aligned}\tag{3.4}$$

then $\mathcal{L}^{\mu}_{\alpha,\beta}(f)\in\mathcal{K}(\xi,\rho,\gamma)$.

Proof. According to Lemma 1.2 it is sufficient to show that

$$\begin{aligned}\sum_{k=2}^{\infty}k\,[(1-\rho)sec\xi(k-1)+(1-\gamma)(1+k\rho-k)]\\ \times\frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\leq 1-\gamma.\end{aligned}\tag{3.5}$$

Let $f\in\mathcal{R}^{\tau}_{\nu}(\delta)$ then by Lemma 1.3, we have

$$|a_k|=\frac{2|\tau|(1-\delta)}{1+\nu(k-1)},\ k\in\mathbb{N}\setminus\{1\},$$

and $1+\nu(k-1)\geq\nu k$. Thus, we have

$$\begin{aligned}\varphi_2(\xi,\rho,\gamma)&=\frac{2|\tau|(1-\delta)}{\nu}\sum_{k=2}^{\infty}[(1-\rho)sec\xi(k-1)+(1-\gamma)(1+k\rho-k)]\\ &\qquad\times\frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\\ &=[(1-\rho)sec\xi+\rho(1-\gamma)]\,\frac{2|\tau|(1-\delta)}{\nu}\sum_{k=2}^{\infty}\frac{k\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\\ &\quad+(1-\rho)(1-\gamma-sec\xi)\frac{2|\tau|(1-\delta)}{\nu}\sum_{k=2}^{\infty}\frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\\ &=[(1-\rho)sec\xi+\rho(1-\gamma)]\left[E^{'}_{\alpha,\beta}(1)-1\right]\frac{2|\tau|(1-\delta)}{\nu}\sum_{k=2}^{\infty}\frac{(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{(k-1)!}\\ &\quad+(1-\rho)(1-\gamma-sec\xi)\left[E_{\alpha,\beta}(1)-1\right]\frac{2|\tau|(1-\delta)}{\nu}\sum_{k=2}^{\infty}\frac{(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{(k-1)!}\\ &=\Big\{[(1-\rho)sec\xi+\rho(1-\gamma)]\,E^{'}_{\alpha,\beta}(1)+[(1-\rho)(1-\gamma-sec\xi)]\,E_{\alpha,\beta}(1)-(1-\gamma)\Big\}\\ &\quad\times\frac{2|\tau|(1-\delta)}{\nu}\mathcal{M}(\mu,1).\end{aligned}$$

Thus, from the assumption (3.4) it follows that $\varphi_3(\xi,\gamma,\rho)\leq 1-\gamma$, that is (3.5) holds, therefore $\mathcal{L}^{\mu}_{\alpha,\beta}(f)\in\mathcal{K}(\xi,\rho,\gamma)$. □

4. INTEGRAL OPERATOR

Here we introduce, an integral operator $\Phi^{\mu}_{\alpha,\beta}$ as follows:

$$\Phi^{\mu}_{\alpha,\beta}(z) = \int_0^z \frac{\mathcal{B}^{\mu}_{\alpha,\beta}(t)}{t} dt, \tag{4.1}$$

where $\mathcal{B}^{\mu}_{\alpha,\beta}(t)$ is defined by (1.8) and we get a necessary and sufficient condition for belonging to the class $\mathcal{S}(\xi,\rho,\gamma)$ and $\mathcal{K}(\xi,\rho,\gamma)$.

Theorem 4.1. *If the function $\Phi^{\mu}_{\alpha,\beta}$ is given by (4.1) and*

$$\begin{aligned}\Big\{[(1-\rho)sec\xi+\rho(1-\gamma)]E'_{\alpha,\beta}+[(1-\rho)(1-\gamma-sec\xi)]E_{\alpha,\beta}(1)\Big\}\\ (\mathcal{M}(\mu,1)) \le 2(1-\gamma),\end{aligned} \tag{4.2}$$

then $\Phi^{\mu}_{\alpha,\beta} \in \mathcal{K}(\xi,\rho,\gamma)$.

Proof. Since

$$\Phi^{\mu}_{\alpha,\beta}(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\frac{z^k}{k},$$

according to Lemma 1.1 it is sufficient to show that

$$\begin{aligned}\sum_{k=2}^{\infty}[(1-\rho)sec\xi(k-1)+(1-\gamma)(1+k\rho-k)]\frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\\ \le 1-\gamma.\end{aligned} \tag{4.3}$$

Let

$$\begin{aligned}\varphi_4(\xi,\rho,\gamma) &= \sum_{k=2}^{\infty}[(1-\rho)sec\xi(k-1)+(1-\gamma)(1+k\rho-k)]\\ &\qquad\times\frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\\ &= [(1-\rho)sec\xi+\rho(1-\gamma)]\sum_{k=2}^{\infty}\frac{k\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\\ &\qquad+(1-\rho)(1-\gamma-sec\xi)\sum_{k=2}^{\infty}\frac{\Gamma(\beta)\,(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{\Gamma(\alpha(k-1)+\beta)(k-1)!}\\ &= [(1-\rho)sec\xi+\rho(1-\gamma)]\Big[E'_{\alpha,\beta}(1)-1\Big]\sum_{k=2}^{\infty}\frac{(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{(k-1)!}\\ &\qquad+(1-\rho)(1-\gamma-sec\xi)\big[E_{\alpha,\beta}(1)-1\big]\sum_{k=2}^{\infty}\frac{(\mu(k-1))^{k-2}\,e^{-\mu(k-1)}}{(k-1)!}\\ &= \Big\{[(1-\rho)sec\xi+\rho(1-\gamma)]E'_{\alpha,\beta}(1)+[(1-\rho)(1-\gamma-sec\xi)]E_{\alpha,\beta}(1)\Big\}\\ &\qquad(\mathcal{M}(\mu,1))-(1-\gamma).\end{aligned}$$

Thus, from the assumption (4.2) it follows that $\varphi_4(\xi,\gamma,\rho) \leq 1-\gamma$, that is (4.3) holds, therefore $\Phi^{\mu}_{\alpha,\beta}(z) \in \mathcal{K}(\xi,\rho,\gamma)$. □

Theorem 4.2. *Let the function $\Phi^{\mu}_{\alpha,\beta}$ be given by (4.2). If*

$$\Bigg[[(1-\rho)\sec\xi+\rho(1-\gamma)]\left[E_{\alpha,\beta}(1)-1\right]+(1-\rho)(1-\gamma+\sec\xi) \times \int_0^1 \left(\frac{E_{\alpha,\beta}(t)}{t}-1\right)dt\Bigg]\mathcal{M}(\mu,1) < (1-\gamma), \tag{4.4}$$

then $\Phi^{\mu}_{\alpha,\beta} \in \mathcal{S}(\xi,\rho,\gamma)$.

Proof. The enduring part of the proof of Theorem 4.2 is parallel to that of Theorem 4.1, and so we omit the details. □

5. APPLICATIONS TO THE ERROR FUNCTION

In this section, we establish the relationship between the Borel distribution-type Mittag Leffler function and the error function. For the special case α=1/2 and β=1, we can derive certain results based on the error function. Thus, a simple computation shows that if

$$\Psi(z) = E_{\frac{1}{2},1}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\frac{k+1}{2})},$$

then

$$\Psi(1) = \sum_{k=1}^{\infty} \frac{1}{\Gamma(\frac{k+1}{2})},\ \Psi'(1) = \sum_{k=1}^{\infty} \frac{k}{\Gamma(\frac{k+1}{2})},\ \Psi''(1) = \sum_{k=2}^{\infty} \frac{k(k-1)}{\Gamma(\frac{k+1}{2})}$$

$$\mathcal{P} := \mathcal{L}_1^{\frac{1}{2}}(z) = f(z) * \Psi(z) = z + \sum_{k=2}^{\infty} \frac{a_k z^k}{\Gamma(\frac{k+1}{2})}.$$

Also

$$I := \Phi_1^{\frac{1}{2}}(z) = \int_0^z \frac{\Psi(t)}{t}dt = z + \sum_{k=2}^{\infty} \frac{z^k}{k\Gamma(\frac{k+1}{2})}. \tag{5.1}$$

Using the above relations, from Theorem 2.1 and Theorem 2.2 we get,

Remark 5.1. If

$$\left\{[(1-\rho)sec\xi+\rho(1-\gamma)]\sum_{k=1}^{\infty}\frac{k}{\Gamma(\frac{k+1}{2})}+[(1-\rho)(1-\gamma-sec\xi)]\sum_{k=1}^{\infty}\frac{1}{\Gamma(\frac{k+1}{2})}\right\} \times(\mathcal{M}(\mu,1)) \leq 2(1-\gamma), \tag{5.2}$$

then $\Psi \in \mathcal{S}(\xi,\rho,\gamma)$.

Remark 5.2. If

$$\left\{[(1-\rho)sec\xi+\rho(1-\gamma)]\sum_{k=1}^{\infty}\frac{k(k-1)}{\Gamma(\frac{k+1}{2})}+[(1-\rho)(1-\gamma-sec\xi)]\sum_{k=1}^{\infty}\frac{k}{\Gamma(\frac{k+1}{2})}\right\} \times(\mathcal{M}(\mu,1))\leq 2(1-\gamma), \quad (5.3)$$

then $\Psi\in\mathcal{K}(\xi,\rho,\gamma)$.

Similarly, using Theorem 3.2 leads to the next remark.

Remark 5.3. If

$$\left\{[(1-\rho)sec\xi+\rho(1-\gamma)]\sum_{k=1}^{\infty}\frac{k}{\Gamma(\frac{k+1}{2})}+[(1-\rho)(1-\gamma-sec\xi)]\sum_{k=1}^{\infty}\frac{1}{\Gamma(\frac{k+1}{2})}\right\} \times\frac{2|\tau|(1-\delta)}{\nu}(\mathcal{M}(\mu,1))\leq(1-\gamma), \quad (5.4)$$

then $\Psi\in\mathcal{S}(\xi,\rho,\gamma)$.

Finally, from Theorem 4.1, we have the following remark.

Remark 5.4. If

$$\left\{[(1-\rho)sec\xi+\rho 1-\gamma]\sum_{k=1}^{\infty}\frac{k}{\Gamma(\frac{k+1}{2})}+[(1-\rho)(1-\gamma-sec\xi)]\sum_{k=1}^{\infty}\frac{1}{\Gamma(\frac{k+1}{2})}\right\} \times(\mathcal{M}(\mu,1))\leq 2(1-\gamma), \quad (5.5)$$

then $\Psi\in\mathcal{K}(\xi,\rho,\gamma)$ where I is defined by(5.1).

6. CONCLUSION

Throughout the paper, we defined new subclasses of spiral-like functions $\mathcal{S}(\xi,\rho,\gamma)$ and $\mathcal{K}(\xi,\rho,\gamma)$ involving the Borel distribution-type Mittag-Leffler function in the open unit disk. We obtained the sufficient condition, inclusion result, image property, and geometric properties of integral operators for these classes. We also discussed the connection between the Borel distribution-type Mittag-Leffler function and the error function.

7. ACKNOWLEDGEMENTS

The author thanks to the referees for the valuable suggestions which improved the presentation of the paper.

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(Received: May 15, 2023)
(Revised: July 13, 2025)

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