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CERTAIN SUBCLASSES OF MULTIVALENT CLOSE-TO-CONVEX FUNCTIONS ASSOCIATED WITH GENERALIZED JANOWSKI FUNCTIONS

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ABSTRACT. Close-to-convex functions have a great importance in the field of Geometric function theory. Many researchers of this field have extensively established various subclasses of close-to-convex univalent functions and studied certain important properties of these subclasses. In this paper, we introduce a generalized subclass of multivalent close-to-convex functions in the open unit disc. We investigate several properties such as coefficient estimates, inclusion relation, distortion theorem,argument theorem and an important result for the defined class. Many known results follow as consequences of the results derived in this paper.

1. INTRODUCTION

Let us denote by $\mathcal{A}_p(p \in \mathbb{N})$, the class of functions f, which are analytic in the open unit disc $E = \{z : z \in \mathbb{C}, |z| < 1\}$ and have Taylor-Maclaurin series of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k.$$

For $p = 1$, the class $\mathcal{A}_p$ reduces to $\mathcal{A}_1$, which is the class of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ denote the class of functions in $\mathcal{A}_1$ which are univalent in E. A function w is said to be a Schwarz function if it has an expansion of the form $w(z) = \sum_{n=1}^{\infty} c_n z^n$ and satisfies the conditions $w(0) = 0$ and $|w(z)| \leq 1$. The class of Schwarz functions is denoted by $\mathcal{U}$.

An analytic function f is said to be subordinate to another analytic function g in E, if there exists a Schwarz function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$. If f is subordinate to g, then it is denoted by $f \prec g$. Moreover , if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

For $0 \leq \alpha < p$, let $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ denote the subclasses of $\mathcal{A}_p$ which are respectively the classes of multivalently starlike functions and multivalently convex

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functions of order α defined as

$$\mathcal{S}_p^*(\alpha) = \left\{ f : f \in \mathcal{A}_p, Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in E \right\}$$

and

$$\mathcal{K}_p(\alpha) = \left\{ f : f \in \mathcal{A}_p, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, z \in E \right\}.$$

The classes $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ were investigated by Goluzina [4]. It is obvious that $f \in \mathcal{K}_p(\alpha)$ if and only if $\dfrac{zf'}{p} \in \mathcal{S}_p^*(\alpha)$. For $0 \leq \alpha < 1$, $\mathcal{S}_1^*(\alpha) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{K}_1(\alpha) \equiv \mathcal{K}(\alpha)$, the classes of starlike functions of order α and convex functions of order α, respectively. Also $\mathcal{S}_p^*(0) \equiv \mathcal{S}_p^*$ and $\mathcal{K}_p(0) \equiv \mathcal{K}_p$, the classes of multivalent starlike functions and multivalent convex functions, respectively. Further $\mathcal{S}_1^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}_1(0) \equiv \mathcal{K}$, the well known classes of starlike functions and convex functions, respectively.

Umezawa [13] introduced the class $\mathcal{C}_p(\alpha)$ of multivalent close-to-convex functions defined as

$$\mathcal{C}_p(\alpha) = \left\{ f : f \in \mathcal{A}_p, Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha, g \in \mathcal{S}_p^*, z \in E \right\}.$$

For $p = 1, \alpha = 0$, the class $\mathcal{C}_p(\alpha)$ reduces to $\mathcal{C}$, the class of close-to-convex functions introduced by Kaplan [6].

For $0 \leq \gamma \leq p$, Bulut [2] established the class $\mathcal{K}_{\mathcal{S}}^{(k)}(\gamma,\mathrm{p})$ consisting of the functions $f \in \mathcal{A}_p$ which satisfy the condition

$$Re\left(\frac{z^{(k-1)p+1}f'(z)}{g_k(z)}\right) > \gamma,$$

where

$$g_k(z) = \Pi_{\nu=0}^{k-1}\varepsilon^{-\nu p}g(\varepsilon^\nu z)(\varepsilon^k = 1; k \geq 1), \tag{1.1}$$

and $g \in \mathcal{S}_p^*\left(\frac{(k-1)p}{k}\right)$.

Further, Vyas and Kant [14] introduced the class $\mathcal{K}_p^{(k)}(\alpha,\beta)$ which consists of the functions $f \in \mathcal{A}_p$ that satisfy the condition

$$Re\left(\frac{z^{(k-1)p+1}f'(z)}{g_k(z)}\right) \prec \frac{1+\beta z}{1-\alpha\beta z},$$

where $g_k(z)$ is defined in (1) and $0 \leq \alpha \leq 1, 0 < \beta \leq 1$.

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < p$, Aouf [1] introduced the class $\mathcal{P}(A,B;p;\alpha)$, the subclass of $\mathcal{A}_p$ which consists of the functions of the form $p(z) = p + \sum_{k=1}^{\infty} p_k z^k$ such that $p(z) \prec \dfrac{p + [pB + (A-B)(p-\alpha)]z}{1+Bz}$. Also for $p = 1, \alpha = 0$, the class $\mathcal{P}(A,B;p;\alpha)$ agrees with $\mathcal{P}(A,B)$, which is a subclass of $\mathcal{A}_1$ introduced by Janowski [5].

Motivated by the above mentioned work, now we define the following generalized subclass of $\mathcal{A}_p$.

Definition 1.1. *Let $\mathcal{K}_s^{(k)}(p;A,B;\eta)$ denote the class of functions $f \in \mathcal{A}_p$ which satisfy the conditions,*

$$\frac{z^{(k-1)p+1}f'(z)}{g_k(z)} \prec \frac{p+[pB+(A-B)(p-\eta)]z}{1+Bz},$$

where $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in \mathcal{S}_p^*\left(\frac{(k-1)p}{k}\right)$, $0 \leq \eta < 1, -1 \leq B < A \leq 1$, $z \in E$ and $g_k(z)$ is defined in (1).
The following observations are obvious.
(i) $\mathcal{K}_s^{(k)}(1;\beta[1-(1+\alpha)\gamma],-\alpha\beta;0) \equiv \mathcal{K}_s(\gamma,\alpha,\beta)$, the class established by Seker and Cho [12].
(ii) $\mathcal{K}_s^{(k)}(1;1-2\gamma,-1;0) \equiv \mathcal{K}_s^{(k)}(\gamma)$, the class studied by Seker [11].
(iii) $\mathcal{K}_s^{(2)}(1;1,-1;0) \equiv \mathcal{K}_s$, the class introduced by Gao and Zhou [3].
(iv) $\mathcal{K}_s^{(2)}(1;1-2\gamma,-1;0) \equiv \mathcal{K}_s(\gamma)$, the class established by Kowalczyk and Les Bomba [8].
(v) $\mathcal{K}_s^{(k)}(p;\beta,-\alpha\beta;0) \equiv \mathcal{K}_p^k(\alpha,\beta)$, the class established by Vyas and Kant [14].
(vi) $\mathcal{K}_s^{(k)}(p;1-2\gamma,-1;0) \equiv \mathcal{K}_s^{(k)}(\gamma,p)$, the class studied by Bulut [2].

By definition of subordination, it follows that $f \in \mathcal{K}s^{(k)}(p;A,B;\eta)$ implies

$$\frac{z^{(k-1)p+1}f'(z)}{g_k(z)} = \frac{p+[pB+(A-B)(p-\eta)]w(z)}{1+Bw(z)}, w \in \mathcal{U}. \tag{1.2}$$

We study various properties such as coefficient estimates, inclusion relationship, distortion theorem and argument theorem for the functions in the class $\mathcal{K}_s^{(k)}(p;A,B;\eta)$. The results proved by various authors follow as special cases.

Throughout this paper, we assume that $-1 \leq B < A \leq 1, 0 \leq \eta < 1, k \geq 1, p \in \mathbb{N}, z \in E$.

2. Preliminary Results

For the derivation of our main results, we require the following lemmas:
Lemma 2.1. *[2][Lemma 1, p. 3] If*

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \in \mathcal{S}_p^*\left(\frac{(k-1)p}{k}\right),$$

then

$$G_k(z) = \frac{g_k(z)}{z^{(k-1)p}} = z^p + \sum_{n=p+1}^{\infty} d_n z^n \in \mathcal{S}_p^*.$$

Lemma 2.2. *[1][Theorem 3, p. 6] Let,*

$$\frac{p+[pB+(A-B)(p-\eta)]w(z)}{1+Bw(z)} = P(z) = p+\sum_{n=1}^{\infty} p_n z^n, \tag{2.1}$$

then

$$|p_n| \leq (p-\eta)(A-B), n \geq 1.$$

Lemma 2.3. *[9][Lemma 2.12, p. 10] Let* $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$*, then*

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z}.$$

Lemma 2.4. *[1][Theorem 1, p. 3] If* $g \in \mathcal{S}_p^*$*, then for* $|z| = r, 0 < r < 1$*, we have*

$$\frac{r^p}{(1+r)^{2p}} \leq |g(z)| \leq \frac{r^p}{(1-r)^{2p}}.$$

Lemma 2.5. *[10][Theorem 4.1, p. 13] If* $\psi(z)$ *is regular in* E*,* $\phi(z)$ *and* $h(z)$ *are convex univalent in* E *such that* $\psi(z) \prec \phi(z)$*, then* $\psi(z) * h(z) \prec \phi(z) * h(z)$*,* $z \in E$*.*

3. Main Results

The first theorem of this paper provides the coefficient bounds for the functions in the defined class.

Theorem 3.1. *If* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{K}_s^{(k)}(p;A,B;\eta)$*, then*

$$|a_n| \leq \left(\frac{p}{n}\right) \frac{(p+n-1)!}{(n-p)!(2p-1)!} + \frac{(A-B)(p-\eta)}{n}\left[1+\sum_{m=p+1}^{n-1} \frac{(p+m-1)!}{(m-p)!(2p-1)!}\right]. \tag{3.1}$$

Proof. As $f \in \mathcal{K}_s^{(k)}(p;A,B;\eta)$, (2) can be written as

$$\frac{z^{(k-1)p+1} f'(z)}{g_k(z)} = P(z),$$

which can be further expressed as

$$\frac{zf'(z)}{G_k(z)} = P(z), \tag{3.2}$$

where

$$G_k(z) = \frac{g_k(z)}{z^{(k-1)p}} = z^p + \sum_{n=p+1}^{\infty} d_n z^n. \tag{3.3}$$

By Lemma 2.1, we have $G_k \in \mathcal{S}_p^*$.

Using the expansions of $f(z)$, $G_k(z)$ and $P(z)$ in (3.2), we obtain
$$p+(p+1)a_{p+1}z+(p+2)a_{p+2}z^2+...+na_nz^{n-p}+...$$
$$=[1+d_{p+1}z+d_{p+2}z^2+...+d_nz^{n-p}+...][p+p_1z+p_2z^2+...+p_nz^n+...]. \quad (3.4)$$
As $G_k(z)=z^p+\sum_{n=p+1}^{\infty}d_nz^n\in\mathcal{S}_p^*$, it is well known [1] that $|d_n|\le\frac{(p+n-1)!}{(n-p)!(2p-1)!}$.
Comparing the coefficients of z^{n-p} in (3.4), we have
$$na_n=pd_n+d_{n-1}p_1+d_{n-2}p_2+...+d_{p+1}p_{n-p-1}+p_{n-p}. \quad (3.5)$$
Applying the triangle inequality, using Lemma 2.2 and the inequality $|d_n|\le\frac{(p+n-1)!}{(n-p)!(2p-1)!}$ in (3.5), the result (3.1) can be easily obtained. □

For $p=1, A=\beta[1-(1+\alpha)\gamma], B=-\alpha\beta, \eta=0$, Theorem 3.1 gives the following result.

Corollary 3.1. *If $f\in\mathcal{K}_s^{(k)}(\gamma;\alpha;\beta)$, then*
$$|a_n|\le1+\frac{\beta(n-1)(1+\alpha)(1-\gamma)}{2}.$$

Putting $p=1, A=1-2\gamma$, $B=-1$, and $\eta=0$ in Theorem 3.1, the following result is obvious.

Corollary 3.2. *If $f\in\mathcal{K}_s^{(k)}(\gamma)$, then*
$$|a_n|\le n-(n-1)\gamma.$$

Substituting $p=1, k=2, A=1-2\gamma$, $B=-1$, $\eta=0$ in Theorem 3.1, we can easily obtain the following result.

Corollary 3.3. **Corollary 3** *If $f\in\mathcal{K}_s(\gamma)$, then*
$$|a_n|\le n-(n-1)\gamma.$$

Taking $p=1, k=2, A=1$, $B=-1$, $\eta=0$, Theorem 3.1 yields the following result.

Corollary 3.4. *If $f\in\mathcal{K}_s$, then*
$$|a_n|\le n.$$

The following theorem gives an inclusion relation of the functions.

Theorem 3.2. *If $-1\le B_2=B_1<A_1\le A_2\le1$ and $0\le\eta_2\le\eta_1<1$, then*
$$\mathcal{K}_s^{(k)}(p;A_1,B_1;\eta_1)\subset\mathcal{K}_s^{(k)}(p;A_2,B_2;\eta_2).$$

Proof. As $f \in \mathcal{K}_s^{(k)}(p;A_1,B_1;\eta_1)$,
$$\frac{z^{(k-1)p+1}f'(z)}{g_k(z)} \prec \frac{p+[pB_1+(A_1-B_1)(p-\eta_1)]z}{1+B_1z}.$$
As $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \eta_2 \leq \eta_1 < 1$, we have
$$-1 \leq B_1 + \frac{(p-\eta_1)(A_1-B_1)}{p} \leq B_2 + \frac{(p-\eta_2)(A_2-B_2)}{p} \leq 1.$$
Thus, by Lemma 2.3, we have
$$\frac{z^{(k-1)p+1}f'(z)}{g_k(z)} \prec \frac{p+[pB_2+(A_2-B_2)(p-\eta_2)]z}{1+B_2z},$$
which implies $f \in \mathcal{K}_s^{(k)}(p;A_2,B_2;\eta_2)$. □

The following result gives the distortion and growth theorems.

Theorem 3.3. *If $f \in \mathcal{K}_s^{(k)}(p;A,B;\eta)$, then for $|z| = r, 0 < r < 1$, we have*
$$\left(\frac{p-[pB+(A-B)(p-\eta)]r}{1-Br}\right)\left(\frac{r^{p-1}}{(1+r)^{2p}}\right) \leq |f'(z)|$$
$$\leq \left(\frac{p+[pB+(A-B)(p-\eta)]r}{1+Br}\right)\left(\frac{r^{p-1}}{(1-r)^{2p}}\right) \tag{3.6}$$
and
$$\int_0^r \left(\frac{p-[pB+(A-B)(p-\eta)]t}{1-Bt}\right)\left(\frac{t^{p-1}}{(1+t)^{2p}}\right)dt \leq |f(z)|$$
$$\leq \int_0^r \left(\frac{p+[pB+(A-B)(p-\eta)]t}{1+Bt}\right)\left(\frac{t^{p-1}}{(1-t)^{2p}}\right)dt. \tag{3.7}$$

Proof. From (3.2), we have
$$|f'(z)| = \frac{|G_k(z)|}{|z|}P(z). \tag{3.8}$$
Aouf [1] proved that
$$\frac{p-[pB+(A-B)(p-\eta)]r}{1-Br} \leq |P(z)| \leq \frac{p+[pB+(A-B)(p-\eta)]r}{1+Br}. \tag{3.9}$$
Since $G_k \in \mathcal{S}_p^*$, by Lemma 2.4, we have
$$\frac{r^p}{(1+r)^{2p}} \leq |G_k(z)| \leq \frac{r^p}{(1-r)^{2p}}. \tag{3.10}$$
Using (3.8) together with (3.9) in (3.7), the result (3.10) can be easily obtained. By integrating (3.10) from 0 to r, (3.6) follows. □

For $p=1, A=\beta[1-(1+\alpha)\gamma], B=-\alpha\beta, \eta=0$, Theorem 3.3 gives the following result due to Seker and Cho [12].

Corollary 3.5. *If $f \in \mathcal{K}_s^{(k)}(\gamma;\alpha;\beta)$, then for $|z|=r, 0<r<1$, we have*

$$\left(\frac{1-\beta[1-(1+\alpha)\gamma]r}{1+\alpha\beta r}\right)\cdot\frac{1}{(1+r)^2} \leq |f'(z)|$$

$$\leq \left(\frac{1+\beta[1-(1+\alpha)\gamma]r}{1-\alpha\beta r}\right)\cdot\frac{1}{(1-r)^2}$$

and

$$\int_0^r \left(\frac{1-\beta[1-(1+\alpha)\gamma]t}{1+\alpha\beta t}\right)\cdot\frac{1}{(1+t)^2}dt \leq |f(z)|$$

$$\leq \int_0^r \left(\frac{1+\beta[1-(1+\alpha)\gamma]t}{1-\alpha\beta t}\right)\cdot\frac{1}{(1-t)^2}dt.$$

Putting $p=1, A=1-2\gamma, B=-1, \eta=0$ in Theorem 3.3, the following result due to Seker [11] is obvious.

Corollary 3.6. *If $f \in \mathcal{K}_s^{(k)}(\gamma)$, then for $|z|=r, 0<r<1$, we have*

$$\frac{2\gamma r}{(1+r)^3} \leq |f'(z)| \leq \frac{2(1-\gamma)r}{(1-r)^3}$$

and

$$\int_0^r \left(\frac{2\gamma t}{(1+t)^3}\right)dt \leq |f(z)| \leq \int_0^r \left(\frac{2(1-\gamma)t}{(1-t)^3}\right)dt.$$

Substituting $k=2, A=1-2\gamma, B=-1, \alpha=0$ and $\beta=1$ in Theorem 3.3, we can easily obtain the following result due to Kowalczyk and Les Bomba [8].

Corollary 3.7. *If $f \in \mathcal{K}_s(\gamma)$, then for $|z|=r, 0<r<1$, we have*

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+(1-2\gamma)r}{(1-r)^3}$$

and

$$\int_0^r \left(\frac{1-(1-2\gamma)t}{(1+t)^3}\right)dt \leq |f(z)| \leq \int_0^r \left(\frac{1+(1-2\gamma)t}{(1-t)^3}\right)dt.$$

Taking $k=2, A=1, B=-1, \alpha=0$ and $\beta=1$, Theorem 3.3 yields the following result due to Gao and Zhou [3].

Corollary 3.8. *If $f \in \mathcal{K}_s$, then for $|z| = r, 0 < r < 1$, we have*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$$

and

$$\int_0^r \left(\frac{1-t}{(1+t)^3}\right) dt \leq |f(z)| \leq \int_0^r \left(\frac{1+t}{(1-t)^3}\right) dt.$$

For $A = 1-2\gamma$, $B = -1$, and $\eta = 0$, Theorem 3.3 agrees with the following result due to Bulut [2].

Corollary 3.9. *If $f \in \mathcal{K}_s^k(\gamma, p)$, then for $|z| = r, 0 < r < 1$, we have*

$$\frac{[p-(p-2\gamma)r]r^{p-1}}{(1+r)^{2p+1}} \leq |f'(z)| \leq \frac{[p+(p-2\gamma)r]r^{p-1}}{(1-r)^{2p+1}}$$

and

$$\int_0^r \left(\frac{[p-(p-2\gamma)t]t^{p-1}}{(1+t)^{2p+1}}\right) dt \leq |f(z)| \leq \int_0^r \left(\frac{[p+(p-2\gamma)t]t^{p-1}}{(1-t)^{2p+1}}\right) dt.$$

Putting $A = \beta$, $B = -\alpha\beta$, and $\eta = 0$ in Theorem 3.3, the following result due to Vyas and Kant [14] is obvious.

Corollary 3.10. *If $f \in \mathcal{K}_p^k(\alpha, \beta)$, then for $|z| = r, 0 < r < 1$, we have*

$$\frac{p(1-\beta r)r^{p-1}}{(1+\alpha\beta r)(1+r)^{2p}} \leq |f'(z)| \leq \frac{p(1+\beta r)r^{p-1}}{(1-\alpha\beta r)(1-r)^{2p}}$$

and

$$\int_0^r \left(\frac{p(1-\beta t)t^{p-1}}{(1+\alpha\beta t)(1+t)^{2p}}\right) dt \leq |f(z)| \leq \int_0^r \left(\frac{p(1+\beta t)t^{p-1}}{(1-\alpha\beta t)(1-t)^{2p}}\right) dt.$$

The following theorem is the argument theorem which gives the upper bound of the argument of functions.

Theorem 3.4. *If $f \in \mathcal{K}_s^{(k)}(p; A, B; \eta)$ and $F(z) = zf'(z)$, then for $|z| = r, 0 < r < 1$, we have*

$$\left|arg \frac{F(z)}{z^p}\right| \leq 2p(sin^{-1}(r)) + sin^{-1}\left(\frac{(A-B)(p-\eta)r}{p-[pB+(A-B)(p-\eta)]Br^2}\right). \qquad (3.11)$$

Proof. From (3.2), we have

$$F(z) = G_k(z)P(z),$$

which implies

$$\left|arg\frac{F(z)}{z^p}\right| \leq |argP(z)| + \left|arg\frac{G_k(z)}{z^p}\right|. \tag{3.12}$$

It was proved by Aouf [1] that

$$|argP(z)| \leq sin^{-1}\left(\frac{(A-B)(p-\eta)r}{p-[pB+(A-B)(p-\eta)]Br^2}\right). \tag{3.13}$$

Also Aouf [1] established that for $G_k \in \mathcal{S}_p^*$,

$$\left|arg\frac{G_k(z)}{z^p}\right| \leq 2p(sin^{-1}r). \tag{3.14}$$

By using (3.12) and (3.13) in (3.11), the result (3.10) can be easily obtained. □

The following theorem provides the estimates for various coefficients of the functions in the class $\mathcal{K}_s^{(k)}(p;A,B;\eta)$.

Theorem 3.5. *If* $f \in \mathcal{K}_s^{(k)}(p;A,B;\eta)$*, then*

$$|a_{p+1}| \leq \frac{1}{(p+1)^2}[2p^2+(p+1)(p-\eta)(A-B)] \tag{3.15}$$

and

$$|a_{p+2}| \leq \frac{1}{(p+1)(p+2)}\left[p^2(2p+1)+2p(A-B)(p-\eta)+(p+1)(A-B)(p-\eta)\right]. \tag{3.16}$$

Proof. From Definition 1.1, using the principle of subordination, we have that

$$\frac{zf'(z)}{G_k(z)} = \frac{p+[pB+(A-B)(p-\eta)]w(z)}{1+Bw(z)}, w(z) \in \mathcal{U}.$$

By expanding and comparing the coefficients, we obtain

$$a_{p+1} = \frac{p}{p+1}d_{p+1} + \frac{(A-B)(p-\eta)}{(p+1)}c_1 \tag{3.17}$$

and

$$a_{p+2} = \frac{p}{p+2}d_{p+2} + \frac{1}{(p+2)}[(A-B)(p-\eta)]d_{p+1}c_1 + \frac{(A-B)(p-\eta)}{(p+2)}[c_2 - Bc_1^2]. \tag{3.18}$$

Also, for $G_k \in \mathcal{S}_p^*$,

$$|d_{p+1}| \leq \frac{2p}{p+1} \tag{3.19}$$

and

$$|d_{p+2}| \leq \frac{p(2p+1)}{p+1}. \tag{3.20}$$

Also, it was proved in [7], that for any complex number γ,

$$|c_2 - \gamma c_1^2| \leq max\{1, |\gamma|\}. \tag{3.21}$$

Applying the triangle inequality and using (3.19), (3.20) and (3.21) in (3.17) and (3.18), along with the inequality $|c_1| \leq 1$, the results (3.15) and (3.16) are obvious. □

The following theorem gives an important result.

Theorem 3.6. *If $f \in \mathcal{K}_s^{(k)}(p;A,B;\eta)$, then there exists $P(z) \in \mathcal{P}(A,B;p;\eta)$ such that for all s and t with $|s| \leq 1$, $|t| \leq 1$ $(s \neq t)$,*

$$\frac{f'(sz)P(tz)t^{p-1}}{f'(tz)P(sz)s^{p-1}} \prec \left(\frac{1-tz}{1-sz}\right)^{2p}.$$

Proof. From definition, we have

$$zf'(z) = P(z)G_k(z).$$

By differentiating logarithmically, we get

$$\frac{zf''(z)}{f'(z)} - \frac{zP'(z)}{P(z)} - p + 1 = \frac{zG_k'(z)}{G_k(z)} - p.$$

As $G_k \in S_p^*$,

$$\frac{zf''(z)}{f'(z)} - \frac{zP'(z)}{P(z)} - p + 1 \prec \frac{2pz}{1-z}.$$

For $|s| \leq 1$, $|t| \leq 1$ $(s \neq t)$,

$$h(z) = \int_0^z \left(\frac{s}{1-su} - \frac{t}{1-tu}\right) du$$

is convex univalent in E. Using Lemma 2.5, we have

$$\left(\frac{zf''(z)}{f'(z)} - \frac{zP'(z)}{P(z)} - p + 1\right) * h(z) \prec \frac{2pz}{1-z} * h(z).$$

For any function $q(z)$ analytic in E with $q(0) = 0$, we obtain

$$(q * h)(z) = \int_{tz}^{sz} q(u)\frac{du}{u}, z \in E.$$

Therefore, we have

$$\int_{tz}^{sz} \left(\frac{uf''(u)}{f'(u)} - \frac{uP'(u)}{P(u)} - p + 1\right) \frac{du}{u} \prec 2pz \int_{tz}^{sz} \frac{du}{1-u},$$

which implies the result. □

4. Conclusion

This paper is concerned with a new and generalized subclass of multivalent close-to-convex functions. The class is defined using the concept of subordination. Various properties of this class such as coefficient estimates, inclusion relation, distortion theorem, growth theorem, argument theorem and an important result, have been established. By giving particular values to the parameters, some earlier known results have been derived. This paper will motivate other researchers in this field to study some more classes of multivalent functions using subordination, associated with the Janowski function.

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