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# STRONG IDEAL CONVERGENCE IN TOPOLOGICAL SPACES

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ABSTRACT. We define two convergence techniques in this study: statistical $A_{\mathcal{T}}$-strong convergence and $A_{\mathcal{T}}^{I}$-strong convergence, both of which are generalizations of $A_{\mathcal{T}}$-strong convergence in Hausdorff topological spaces via a certain class of special functions. Similar to the classic scenario, we find some correlations between $A^{I}$-statistical convergence and $A_{\mathcal{T}}^{I}$-strong convergence. Additionally, we obtain a characterization of $A^{I}$-statistical convergence.

## 1. INTRODUCTION AND BACKGROUND

Summability theory is currently playing an intriguing role in a topological space. A scalar-valued or linear space valued sequence can be given limits using summability theory, especially if the sequence is non-convergent [2]. Summability theory in topological spaces had been addressed by certain authors under the assumption that topological space has a group structure or a linear structure. Additionally, there are some summability techniques that do not require a linear structure in the topological space, such as statistical convergence [6, 7] as well as $A$-statistical convergence [10] (see [3, 14, 19]). More recent works, in this line, can be found in [8, 11, 15] where many references are mentioned. The notion of $I$-convergence was introduced by Kostyrko et. al. in 2000-2001 [12] on metric spaces and the same notion was explored by Lahiri et. al. in 2005 [13] on topological spaces.

A class of pre-metrics on arbitrary Hausdorff topological spaces, having several characteristics that are quite similar to the topological base, had been used to introduce the study of strong convergence, namely, $A_{\mathcal{T}}$-strongly convergence, in [20]. In this article, in the line of [20], we have investigated strong convergence using the theory of $I$-convergence, and we have demonstrated that our findings are more robust than those of classical strong convergence.

We need some definitions to follow. A family $I \subset P(\mathbb{N})$ is known an ideal on $\mathbb{N}$ if it is closed under finite unions and satisfies the hereditary property (i.e., $A \in I, B \subset A$ implies $B \in I$. When $I$ contains all singleton subsets of $\mathbb{N}$, it is referred to as an

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admissible ideal. The family of sets $\mathcal{F}(I) = \{M \subset \mathbb{N} : \exists\, C \in I : M = \mathbb{N} \setminus C\}$ is called the filter associated with the ideal $I$. According to Lahiri et. al. [13] "A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a topological space $X$ is said to be $I$-convergent to $\xi \in X$ if for any nonempty open set $U$ containing $\xi$, the set $\{k \in \mathbb{N} : x_k \notin U\} \in I$."

The limits of the convergent sequences are preserved when a summability matrix $A = (b_{nk})$ is so called regular. Throughout the paper $I$ denotes a non-trivial proper (i.e., $\mathbb{N} \notin I,\ I \neq \{\emptyset\}$) admissible ideal on the set of natural numbers $\mathbb{N}$.

Now we recall some well-known notions from literature. According to Maio et. al.[14] "Let $A = (b_{nk})$ is a non-negative regular summability matrix. If $\{x_k\}_{k\in\mathbb{N}}$ is a sequence in a Hausdorff topological space $X$ such that for any open set $U$ that contains $\xi$

$$\lim_n \sum_{k:x_k\notin U} b_{nk} = 0,$$

then the sequence $\{x_k\}_{k\in\mathbb{N}}$ is called $A$-statistically convergent to $\xi \in X$".

In 1988-1989, J. S. Connor [4, 5] defined $A$-strong convergence as "For a non-negative regular summability matrix $A = (b_{nk})$, a sequence $\{x_k\}_{k\in\mathbb{N}}$ in a metric space $(X,d)$ is said to be $A$-strongly convergent to $\xi \in X$ if

$$\lim_n \sum_k d(x_k,\xi) b_{nk} = 0\text{''}.$$

In 2012-2013, Savas et. al. [16, 17] defined $A^I$-statistical convergence as "For a non-negative regular summability matrix $A = (b_{nk})$, a sequence $\{x_k\}_{k\in\mathbb{N}}$ is said to be $A^I$-statistically convergent to $L$ if for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{ n \in \mathbb{N} : \sum_{k\in K(\varepsilon)} b_{nk} \geq \delta \right\} \in I,$$

where $K(\varepsilon) = \{k \in \mathbb{N} : \mid x_k - L \mid \geq \varepsilon\}$".

## 2. $A^I_{\mathcal{T}}$-Strong Convergence in a Topological Space

The idea of strong convergence cannot be examined in arbitrary topological spaces because of the dependence of the strong convergence on the metric functions. The well-known connection between strong convergence and statistical convergence cannot, therefore, be easily extended to topological spaces. In this section, we define $A^I$-statistical convergence in a topological space and introduce the notion of $A^I_{\mathcal{T}}$-strong convergence using a particular function defined on the topological space. We determine how these convergences relate to one another. Throughout the paper $M(NN-R-S)$ will denote the class of all non-negative regular summability matrices.

**Definition 2.1.** *Let $A = (b_{nk}) \in M(NN-R-S)$. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a Hausdorff topological space $X$ is said to be $A^I$-statistically convergent to $\xi \in X$ if for any $\delta > 0$*

*and for any open set $U$ containing $\xi$,*

$$\left\{ n \in \mathbb{N} : \sum_{k: x_k \notin U} b_{nk} \geq \delta \right\} \in I.$$

It is simple to observe that, in place of open sets, the members of the topology's base can be used to define $A^I$-statistical convergence.

Using the concept of pre-metric [1] and the following properties in topological space we define $A^I_{\mathcal{T}}$-strong convergence in a topological space. Let $(X, \tau)$ be a Hausdorff topological space and $\xi \in X$. Throughout the paper $\mathcal{B}_\xi$ denotes the family of elements of the base of $\tau$ that contains $\xi$ and $B_T(\xi, \varepsilon) := \{\beta \in X : T(\beta, \xi) < \varepsilon\}$, where $T$ is a function. Also we denote $\mathcal{L}(X)$ as the set of functions $T : X \times X \to [0, \infty)$ that satisfy the condition "For any $\varepsilon > 0$ and for any $\xi \in X$ there exists $U_\varepsilon \in \mathcal{B}_\xi$ such that $U_\varepsilon \subset B_T(\xi, \varepsilon)$" [20]. Though any function from $\mathcal{L}(X)$ is a pre-metric (see, for instance, [1]) on $X$, the topology need not to be pre-metrizable. These pre-metrics are here partly compatible with the topology because they satisfy that certain condition.

**Definition 2.2.** *Let $(X, \tau)$ be a Hausdorff topological space, $A = (b_{nk}) \in M(NN-R-S)$ and $\mathcal{T} \subset \mathcal{L}(X)$. A sequence $\{x_k\}_{k \in \mathbb{N}}$ in $X$ is said to be statistically $A_{\mathcal{T}}$-strongly convergent to $\xi \in X$ if for any $T \in \mathcal{T}$, the sequence $\left\{ \sum_k T(x_k, \xi) b_{nk} \right\}_{n \in \mathbb{N}}$ is statistically convergent to zero, provided the series is convergent for each $n$.*

**Definition 2.3.** *Let $(X, \tau)$ be a Hausdorff topological space, $A = (b_{nk}) \in M(NN-R-S)$ and $\mathcal{T} \subset \mathcal{L}(X)$. A sequence $\{x_k\}_{k \in \mathbb{N}}$ in $X$ is said to be $A^I_{\mathcal{T}}$-strongly convergent to $\xi \in X$ if for any $\varepsilon > 0$ and for any $T \in \mathcal{T}$*

$$\left\{ n \in \mathbb{N} : \sum_k T(x_k, \xi) b_{nk} \geq \varepsilon \right\} \in I,$$

*provided the series is convergent for each $n$.*

For $I = I_d$, the ideal consisting of all the subsets of $\mathbb{N}$ whose natural density are zero, the above convergences coincide. Here we are giving an example of $A^I_{\mathcal{T}}$-strongly convergent sequence in a non-metrizable Hausdorff space.

**Example 2.1.** *Consider the topological space $(\mathbb{R}, \tau_L)$, where $\tau_L$ be the lower limit topology. It is a non-metrizable Hausdorff space [18]. Let us take the sequence $\{x_k\}_{k \in \mathbb{N}}$ defined by*

$$x_k = \begin{cases} 0, & \text{if } k \text{ is even} \\ 1, & \text{otherwise} \end{cases}$$

*and let us consider the family of functions $\mathcal{T} = \{T_r\}_{r\geq 0}$ defined for any $r \geq 0$ such that*

$$T_r(x,y) = \begin{cases} x-y, & x \geq y \\ r, & x < y. \end{cases}$$

*Then $T_r \in \mathcal{T}$. Let $I$ be a non-trivial admissible ideal of $\mathbb{N}$ such that $I \neq I_{fin}$ (the ideal containing all finite subsets of $\mathbb{N}$). Consider an infinite set $\mathcal{D} = \{q_1 < q_2 < q_3 < \cdots\}$ from $I$ and take the infinite matrix $A = (b_{nk})$ given by,*

$$b_{nk} = \begin{cases} 1 & \text{if } n = q_i, k = 2q_i \text{ for some } i \in \mathbb{N} \\ 1 & \text{if } n \neq q_i, \text{for any } i, k = 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

*For any $\varepsilon > 0$ and $\xi \in X$ we consider $U_\varepsilon = [\xi, \xi+\varepsilon) \in \mathcal{B}_\xi$, then for all $\beta \in U_\varepsilon$ we have $T(\beta,\xi) = \beta - \xi < \varepsilon$. So $\mathcal{T} \subset \mathcal{L}(X)$ and $\{x_k\}_{k\in\mathbb{N}}$ is not convergent in the corresponding topology. But we have*

$$T_r(x_k,1) = \begin{cases} r, & k \text{ is even} \\ 0, & \text{otherwise.} \end{cases}$$

*Now, for any $r \geq 0$ and $\delta > 0$ we get that*

$$\sum_k T(x_k,1) b_{nk} = \sum_{k\in 2\mathbb{N}} b_{nk} = \sum_{k\in 2\mathbb{N}} r \geq \delta$$

*for $n = q_i$. Thus for any $\delta > 0$, $\left\{ n \in \mathbb{N} : \sum_k T(x_k,1) b_{nk} \geq \delta \right\} = \mathcal{D} \in I$ showing that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is $A^I_{\mathcal{T}}$-strongly convergent to 1 but not $A_{\mathcal{T}}$-strongly convergent to 1. So $A^I_{\mathcal{T}}$ strong convergence is stronger than $A_{\mathcal{T}}$-strong convergence.*

*In particular, if we choose $I \neq I_{fin}$ and $I \neq I_d$ and the infinite set $\mathcal{D} \in I \setminus I_d$ then the sequence $\{x_k\}_{k\in\mathbb{N}}$ is $A^I_{\mathcal{T}}$-strongly convergent to 1 but not statistically $A_{\mathcal{T}}$-strongly convergent to 1.*

Connor [4], Khan and Orhan [9] established the relation between $A$-statistical convergence and $A$-strong convergence. Also in the field of $A^I$-statistical convergence Savas, Das and Dutta [16, 17] generalized its relation with $A^I$-summability.
We denote $\mathcal{Q}(X)$ as the set of functions $T : X \times X \to [0,\infty)$ that satisfy the condition: " For all $\xi \in X$ and for all $B \in \mathcal{B}_\xi$ there exists $M > 0$ such that for all $\beta \notin B$ , $T(\beta,\xi) > M$ " [20].

**Theorem 2.1.** *Let $X$ be a Hausdorff topological space and $A = (b_{nk}) \in M(NN-R-S)$ and let $\mathcal{T} \subset \mathcal{L}(X)$. Then,*
*(i) $A^I_{\mathcal{T}}$-strong convergence with $\mathcal{T} \cap \mathcal{Q}(X) \neq \varnothing$ implies $A^I$-statistical convergence.*

*(ii) $A^I$-statistical convergence with the condition*

$$\sup_{T\in\mathcal{T}}\sup_k T(x_k,\xi)<\infty \tag{2.1}$$

*implies $A^I_{\mathcal{T}}$-strong convergence.*

*Proof.* $(i)$ Let a sequence $\{x_k\}_{k\in\mathbb{N}}$ be $A^I_{\mathcal{T}}$-strongly convergent to $\xi$ in $X$. Let $B\in\mathcal{B}_\xi$ and $T\in\mathcal{T}\cap Q(X)$. Then there exists $M>0$ such that for all $\beta\notin B$, $T(\beta,\xi)>M$. From non-negativity of $T$ we get

$$\begin{aligned}\sum_k T(x_k,\xi)b_{nk} &= \sum_{k:x_k\in B} T(x_k,\xi)b_{nk}+\sum_{k:x_k\notin B} T(x_k,\xi)b_{nk}\\ &\geq \sum_{k:x_k\notin B} T(x_k,\xi)b_{nk}\\ &\geq M\sum_{k:x_k\notin B} b_{nk}.\end{aligned}$$

Let $\delta>0$ be given. Therefore

$$\left\{n\in\mathbb{N}:\sum_{k:x_k\notin B} b_{nk}\geq\delta\right\}\subseteq\left\{n\in\mathbb{N}:\sum_k T(x_k,\xi)b_{nk}\geq\delta\cdot M\right\}.$$

As the sequence $\{x_k\}_{k\in\mathbb{N}}$ is $A^I_{\mathcal{T}}$-strongly convergent to $\xi$ in $X$ then the right side set belongs to $I$ and this implies

$$\left\{n\in\mathbb{N}:\sum_{k:x_k\notin B} b_{nk}\geq\delta\right\}\in I.$$

As $B$ is arbitrary, the sequence $\{x_k\}_{k\in\mathbb{N}}$ is $A^I$-statistically convergent to $\xi$. Hence the proof of $(i)$ is completed.

$(ii)$ Let us suppose $\{x_k\}_{k\in\mathbb{N}}$ is $A^I$-statistically convergent to $\xi\in X$ and (1) holds and let $T\in\mathcal{T}$. Then for any $\varepsilon>0$ there exists $B\in\mathcal{B}_\xi$ such that for all $\beta\in B$, $T(\beta,\xi)<\varepsilon$. Hence non-negativity of $T$ gives

$$\begin{aligned}\sum_k T(x_k,\xi)b_{nk} &= \sum_{k:x_k\in B} T(x_k,\xi)b_{nk}+\sum_{k:x_k\notin B} T(x_k,\xi)b_{nk}\\ &\leq \varepsilon\sum_{k:x_k\in B} b_{nk}+\sup_{T\in\mathcal{T}}\sup_k T(x_k,\xi)\sum_{k:x_k\notin B} b_{nk}\\ &\leq \varepsilon+M\sum_{k:x_k\notin B} b_{nk},\end{aligned}$$

where $M=\sup_{T\in\mathcal{T}}\sup_k T(x_k,\xi)$. For a given $\delta>0$ choose $\varepsilon>0$ such that $\varepsilon<\delta$. Then,

$$\left\{n\in\mathbb{N}:\sum_k T(x_k,\xi)b_{nk}\geq\delta\right\}\subseteq\left\{n\in\mathbb{N}:\sum_{k:x_k\notin B} b_{nk}\geq\frac{\delta-\varepsilon}{M}\right\}.$$

Since $\{x_k\}_{k\in\mathbb{N}}$ is $A^I$-statistically convergent to $\xi$ in $X$, the set on the right side belongs to $I$ and consequently this implies that $\{x_k\}_{k\in\mathbb{N}}$ is $A^I_{\mathcal{T}}$-strongly convergent to $\xi$. $\square$

*Remark* 2.1. In Example 2.1 we have given an example of an $A^I_{\mathcal{T}}$-strongly convergent sequence and we can see that the family $\mathcal{T}$ is a subfamily of $Q(X)$. For $T_r \in \mathcal{T}$, $\xi \in \mathbb{R}$ and $U \in \mathcal{B}_\xi$ we have $U$ in the most general form of $[\xi,\gamma)$. Then for any $\beta \notin U$, $T_r(\beta,\xi) = \beta - \xi$ for $\beta > \xi$; otherwise $T_r(\beta,\xi) = r$. Also $T(\gamma,\xi) = \gamma - \xi$ and $\beta > \xi \Rightarrow \beta > \gamma$ so, $T_r(\beta,\xi) > T(\gamma,\xi)$. Hence $T_r(\beta,\xi) > \min\{\frac{r}{2}, T(\gamma,\xi)\}$. Therefore, by Theorem 2.1 the given sequence in Example 2.1 converges $A^I$-statistically to 1 with respect to the given ideal.

On the other hand in the preceding example as the given sequence is $A^I$-statistically convergent to 1 and we can define a family of functions $\mathcal{T} = \{T_r\}_{r\geq 0}$ for any $r \geq 0$ as

$$T_r(x,y) = \begin{cases} x-y, & x \geq y \\ \frac{1}{r+1}, & x < y \end{cases}$$

that satisfies (1) i.e., $\sup_{T\in\mathcal{T}} \sup_k T(x_k,1) = 1 < \infty$ and also $\mathcal{T} \subset \mathcal{L}(X) \cap Q(X)$. Hence it justifies part (ii) of Theorem 2.1.

Let $(X,d)$ be a metric space and $d \in \mathcal{L}(X) \cap Q(X)$. If $\mathcal{T} = \{d\}$ then any bounded sequence satisfies (1) and we get the following corollary for $I = I_{fin}$, ideal containing all finite subsets of $\mathbb{N}$.

**Corollary 2.1.** *[4] Let $A \in M(NN-R-S)$. A sequence in $X$ which is A-strongly convergent to $\xi \in X$, is also A-statistically convergent to $\xi$. For a bounded sequence in $X$, A-statistical convergence and A-strong convergence are equivalent.*

**Theorem 2.2.** *Let $A = (b_{nk}) \in M(NN-R-S)$ and let $\mathcal{T} \subset \mathcal{L}(X)$ where $X$ be a Hausdorff topological space. If a sequence $\{x_k\}_{k\in\mathbb{N}}$ is $A^I$-statistically convergent to $\xi$ in $X$ and satisfies*
*$(i)$ for any $\varepsilon > 0$ and $T \in \mathcal{T}$ there exists a compact subset $F \subset X$ such that*

$$\sup_n \sum_{x_k \notin K} T(x_k,\xi) b_{nk} < \varepsilon;$$

*$(ii)$ for any compact set $C \subset X$ and $T \in \mathcal{T}$ there exists $M > 0$ such that $\sup\limits_{x_k\in C} T(x_k,\xi) < M$,*
*then $\{x_k\}_{k\in\mathbb{N}}$ is $A^I_{\mathcal{T}}$-strongly convergent to $\xi$.*

*Proof.* Let $\{x_k\}_{k\in\mathbb{N}}$ be a sequence in $X$ and $A^I$-statistically convergent to $\xi$. Let $\varepsilon > 0$ and $T \in \mathcal{T}$ Also, from the hypothesis there exists a compact set $F \subset X$ such that

$$\sup_n \sum_{k:x_k \notin F} T(x_k,\xi) b_{nk} < \frac{\varepsilon}{2}$$

and there exists $M > 0$ such that

$$\sup_{k:x_k\in F} T(x_k,\xi) < M.$$

As $T \in \mathcal{L}(X)$ there exists $U_\varepsilon \in \mathcal{B}_\xi$ such that $U_\varepsilon \subset B_T(\xi, \frac{\varepsilon}{2})$. Now for any positive integer $n$

$$\begin{aligned}
\sum_k T(x_k,\xi)b_{nk} &= \sum_{k:x_k\in F} T(x_k,\xi)b_{nk} + \sum_{k:x_k\notin F} T(x_k,\xi)b_{nk} \\
&= \sum_{k:x_k\in F\cap U_\varepsilon} T(x_k,\xi)b_{nk} + \sum_{k:x_k\in F\setminus U_\varepsilon} T(x_k,\xi)b_{nk} + \sum_{k:x_k\notin F} T(x_k,\xi)b_{nk} \\
&\leq \frac{\varepsilon}{2}\sum_k b_{nk} + \sup_{k:x_k\in F} T(x_k,\xi) \sum_{k:x_k\notin U_\varepsilon} b_{nk} + \sup_n \sum_{k:x_k\notin F} T(x_k,\xi)b_{nk} \\
&\leq \frac{\varepsilon}{2}\sum_k b_{nk} + M\sum_{k:x_k\notin U_\varepsilon} b_{nk} + \frac{\varepsilon}{2}.
\end{aligned}$$

As $A = (a_{nk})$ is a regular summability matrix then for any given $\delta > 0$ choosing $\varepsilon < \delta$,

$$\left\{n\in\mathbb{N} : \sum_k T(x_k,\xi)b_{nk} \geq \delta\right\} \subseteq \left\{n\in\mathbb{N} : \sum_{k:x_k\notin U_\varepsilon} b_{nk} \geq \frac{\delta-\varepsilon}{M}\right\}.$$

Since the right hand side set belongs to $I$, $\left\{n\in\mathbb{N} : \sum_k T(x_k,\xi)b_{nk} \geq \delta\right\} \in I$ for any $\delta > 0$. Hence the proof is completed. □

## 3. Characterization of $A^I$-statistical convergence

The next theorem determines a characterization of $A^I$-statistical convergence.

**Theorem 3.1.** *Let $X$ be a Hausdorff topological space. Let $A = (b_{nk}) \in M(NN - R - S)$ and let $\mathcal{T} \subset \mathcal{L}(X) \cap Q(X)$. Then a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $X$ is $A^I$-statistically convergent to $\xi \in X$ if and only if*

$$I - \lim_n \sum_k \frac{T(x_k,\xi)}{1+T(x_k,\xi)} b_{nk} = 0. \tag{3.1}$$

*Proof.* First, assume that $\{x_k\}_{k\in\mathbb{N}}$ is $A^I$-statistically convergent to $\xi \in X$ and let $\varepsilon > 0$. As $T \in \mathcal{L}(X)$ there exists $U_\varepsilon \in \mathcal{B}_\xi$ such that $U_\varepsilon \subset B_T(\xi,\varepsilon)$. Then we have

$$\begin{aligned}\sum_k \frac{T(x_k,\xi)}{1+T(x_k,\xi)}b_{nk} &= \sum_{x_k\notin U_\varepsilon} \frac{T(x_k,\xi)}{1+T(x_k,\xi)}b_{nk} + \sum_{x_k\in U_\varepsilon} \frac{T(x_k,\xi)}{1+T(x_k,\xi)}b_{nk}\\ &\leq \sum_{x_k\notin U_\varepsilon} b_{nk} + \varepsilon \sum_{x_k\in U_\varepsilon} b_{nk}\\ &\leq \sum_{x_k\notin U_\varepsilon} b_{nk} + \varepsilon.\end{aligned}$$

For any given $\delta > 0$ choose $\varepsilon > 0$ such that $\varepsilon < \delta$. Then

$$\left\{n\in\mathbb{N} : \sum_k \frac{T(x_k,\xi)}{1+T(x_k,\xi)}b_{nk} \geq \delta\right\} \subseteq \left\{n\in\mathbb{N} : \sum_{x_k\notin U_\varepsilon} b_{nk} \geq \delta-\varepsilon\right\}.$$

Since the sequence $\{x_k\}_{k\in\mathbb{N}}$ is $A^I$-statistically convergent to $\xi \in X$, the right hand side set belongs to $I$. Thus (3.1) holds.

Conversely, assume that (3.1) holds and let $B \in \mathcal{B}_\xi$. Since $T \in Q(X)$ there exists $M > 0$ such that for all $\beta \notin B, T(\beta,\xi) > M$. Then we have

$$\begin{aligned}\sum_{x_k\notin B} b_{nk} &\leq \frac{1+M}{M}\sum_{x_k\notin B} \frac{T(x_k,\xi)}{1+T(x_k,\xi)}b_{nk}\\ &\leq \frac{1+M}{M}\sum_k \frac{T(x_k,\xi)}{1+T(x_k,\xi)}b_{nk}.\end{aligned}$$

Therefore for any $\delta > 0$ choose $\varepsilon = \frac{\delta\cdot M}{1+M}$ and then

$$\left\{n\in\mathbb{N} : \sum_{x_k\notin B} b_{nk} \geq \delta\right\} \subseteq \left\{n\in\mathbb{N} : \sum_k \frac{T(x_k,\xi)}{1+T(x_k,\xi)}b_{nk} \geq \varepsilon\right\}.$$

Now (3.1) yields that the right hand side set belongs to $I$ and hence the sequence $\{x_k\}_{k\in\mathbb{N}}$ is $A^I$-statistically convergent to $\xi \in X$. □

We can characterize $A^I$-statistical convergence of real sequences as follows using Theorem 3.1.

**Corollary 3.1.** *Let $A = (b_{nk}) \in M(NN-R-S)$ and let $\{x_k\}_{k\in\mathbb{N}}$ be a real sequence in $\mathbb{R}$ endowed with the usual topology. Then $\{x_k\}_{k\in\mathbb{N}}$ is $A^I$-statistically convergent to a real number L if and only if*

$$I-\lim_n \sum_k \frac{|x_k-L|}{1+|x_k-L|}b_{nk} = 0.$$

In particular for $I = I_{fin}$, the above corollary implies Corollary 3 in [20].

## 4. Conclusion

A new form of a strong convergence method, namely, $A_T^I$-strong convergence, is introduced here by utilizing a class of pre-metrics having characteristics similar to those on arbitrary Hausdorff topological spaces to overcome the lack of linearity. Considering this new type of strong convergence, some interconnections with $A^I$-statistical convergence have been investigated. Finally, we characterize $A^I$-statistical convergence to some extent.

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