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THE GENERALIZED QUANTUM DIFFERENCE OPERATOR IN c AND c_0 AND THEIR DUALS

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ABSTRACT. In this article, we present a generalization of the k^{th} order of forward difference operator Δ^k using its quantum analog Δ_q^k. We study its domains $c(\Delta_q^k)$ and $c_0(\Delta_q^k)$ in the spaces c and c_0 of convergent and null sequences, respectively. Additionally, we show that the domains $c(\Delta_q^k)$ and $c_0(\Delta_q^k)$ are BK-spaces and linearly isomorphic to c and c_0, respectively. Furthermore, we construct Schauder bases and examine the Köthe duals of the spaces $c(\Delta_q^k)$ and $c_0(\Delta_q^k)$. The final segment deals with the characterization of certain class of matrix mappings from the spaces $c(\Delta_q^k)$ and $c_0(\Delta_q^k)$ to the space $\nu \in \{c, c_0, \ell_\infty, \ell_1\}$.

1. q-ANALOG

A q-analog of a mathematical expression is a generalization of the expression that employs a new parameter q that returns the original expression in the limit as $q \to 1$. Mathematicians are generally interested in q-analogs that arise naturally rather than in arbitrarily constructing q-analogs of known results. The application of q-calculus while establishing the q-analog of the classical derivative and integral operators is given by Jackson [15]. Initially, the q-analog studied in detail is the hyper geometric series introduced in the 19th century. Since then, q-analogs have been studied most frequently in the mathematics fields of combinatorics and special functions. It finds many applications including the study of fractals and expressions for the entropy of a chaotic dynamical system.
The notation below is quite familiar in the q-analog.

Let $0 < q < 1$. Then the q-integer is defined as

$$[v]_q = \begin{cases} \sum_{t=0}^{v-1} q^t, & (v = 1, 2, \cdots), \\ 0, & (v = 0). \end{cases}$$

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It is obvious that if $q \to 1^-$, then $[v]_q = v$.
The q-analog $\binom{p}{v}_q$ of the binomial coefficient $\binom{p}{v}$ is defined as

$$\binom{p}{v}_q = \begin{cases} \frac{[p]_q!}{[p-v]_q![v]_q!}, & 0 \leq v \leq p, \\ 0, & otherwise, \end{cases}$$

where q-factorial of $[v]_q!$ of v is defined as

$$[v]_q! = \begin{cases} \prod_{p=1}^{v} [p]_q, & (v = 1, 2, \cdots), \\ 1, & (v = 0). \end{cases}$$

Also, $\binom{0}{0}_q = \binom{v}{0}_q = \binom{v}{v}_q = 1$. Moreover, $\binom{v}{v-k}_q = \binom{v}{k}_q$, which is the natural q-analog of its ordinary version $\binom{v}{v-k} = \binom{v}{k}$. For more basic terminologies of the q-analogs, readers can refer to [17, 22] and references therein.

Lemma 1.1. *The Gauss's q- binomial formula is given by*

$$(y+s)_q^n = \sum_{t=0}^{n} \binom{n}{t}_q q^{\binom{t}{2}} y^{n-t} s^t,$$

where $\binom{t}{2} = 0$ *for* $t < 2$.

2. q-Difference Operator

The forward difference operator Δ is defined by $(\Delta y)_t = y_t - y_{t+1}$. The forward difference operator plays an important role in the field of sequence space, summability theory, approximation theory etc. For instance, in the ordinary sense the sequence $(y_t) = (t)_{t=1}^{\infty}$ is divergent, but the sequence $(\Delta y)_t = -1$ is convergent for all t . In [16], Kızmaz studied and introduced the domains $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$. The forward operator Δ was later generalized to second-order Δ^2 defined as $(\Delta^2 y)_t = (\Delta y)_t - (\Delta y)_{t+1}$. In the literature, several generalizations of the difference operator Δ and their domains have been contributed and studied by several researchers and can be found in nice papers [7, 9, 10, 12–14]. The reader may also refer to the recent monographs [5] and [20], and references therein, devoted to summability theory and the sequence spaces generated by some triangle matrices with a new approach. In [8], Demiriz and Şahin studied the sequence space by means of a q-analog of the Cesàro matrix $C(q)$, where $C(q) = (c_{nt}^q)$ (see [2]) is defined as

$$c_{nt}^q = \begin{cases} \frac{q^t}{(n+1)[q]}, & 0 \leq t \leq n, \\ 0, & t > n. \end{cases}$$

Recently in [25, 26], the authors studied the (p', q)-analog of Euler spaces and the q-analog of Catalan spaces. More recently, the q-difference spaces of the second order have been studied and introduced by Alotaibi et al. [3] and Yaying et al. [27, 28]. For detailed studies on the q-analog, the reader may strictly refer to [21, 29]. Following [6], the q-difference operator is defined as

$$\Delta_q y = (y_0 - y_1, q(y_1 - y_2), q^2(y_2 - y_3), q^3(y_3 - y_4), \cdots) \tag{2.1}$$

where $y = (y_0, y_1, y_2, \cdots)$. As a consequence, the operator induces immediately the q-binomial coefficients via iteration, $\Delta_q^k = \Delta_q(\Delta_q^{k-1})$ as follows

$$(\Delta_q^k y)_{nt} = q^{kt} \sum_{n=t}^{k} (-1)^{n-t} q^{\binom{n-t}{2}} \binom{k}{n-t}_q y_n = q^{kt} \sum_{n=0}^{k} (-1)^n q^{\binom{n}{2}} \binom{k}{n}_q y_{n+t}.$$

Furthermore, the q-difference matrix $\Delta_q^k = ((\delta_q^k)_{nt})$ is defined as

$$(\delta_q^k)_{nt} = (-1)^n q^{\binom{n}{2}} \binom{k}{n}_q,$$

where δ_q being lower triangular with non-zero diagonal entries is invertible. Equivalently,

$$\Delta_q^k = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -[k] & 1 & 0 & 0 & \cdots \\ q\binom{k}{2} & -[k] & 1 & 0 & \cdots \\ -q^3\binom{k}{2} & q\binom{k}{2} & -[k] & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The inverse of Δ_q^k is written as $(\delta_q^k)_{nt}^{-1}$ given by

$$(\delta_q^k)_{nt}^{-1} = (-1)^n q^{n(n-k)-\binom{n}{2}} \binom{k}{n}_q = \binom{k+t-n-1}{t-n}_q.$$

Quite recently, Ellidokuzoğlu and Demiriz [11] studied and introduced the domain $\ell_p(\Delta_q^k) = \{\ell\}_{\Delta_q^k}$ by using the Δ_q^k operator and exhibited this sequence space. For more interesting literature, we refer to the studies (see [4, 19, 30]).

Motivation: In the literature, several authors can be found associated with the quantum generalization of familiar operators including the difference operator, Hausdorff operators, etc. Recently, Yaying et al. [26], Alotaibi et al. [3] studied the sequence spaces of quantum generalization. However, no research has been carried out on the domain of generalized q-difference sequence spaces in c and c_0, respectively. Inspired by the above research, we present the generalized q-difference forward operator, and its domain in the spaces c and c_0 of convergent and null sequences, respectively. We also show topological properties and construct the Schauder basis for $c(\Delta_q^k)$ and $c_0(\Delta_q^k)$. Furthermore, we calculate the duals of the newly defined spaces $c(\Delta_q^k)$ and $c_0(\Delta_q^k)$ and finally characterize the classes $(c(\Delta_q^k), \nu)$ and $(c_0(\Delta_q^k), \nu)$ of certain matrix mappings, where $\nu \in \{\ell_\infty, c, c_0, \ell_1\}$.

3. Sequence Spaces

Let ω represent the space of all complex-valued sequences. The well-known classical sequence spaces are the set of all bounded sequences ℓ_∞, the null sequences c_0, the convergent sequences c, and the p-absolutely summable sequences ℓ_p, where $1 \leq p < \infty$. We also indicate the spaces of all the convergent and bounded series by cs and bs, respectively. A Banach sequence space with continuous coordinates is a BK-space. For instance, the space ℓ_p is a BK-space furnished by the norm $\|y\|_{\ell_p} = (\sum_k |y_k|^p)^{1/p}$. It is also well known that the sequence spaces c and c_0 are Banach spaces accompanied by the supremum norm. For any sequence space μ and infinite matrix $\mathcal{A}$, the matrix domain $\mu_{\mathcal{A}}$ of $\mathcal{A}$ in a sequence space μ is defined as

$$\mu_{\mathcal{A}} = \{y \in \omega : \mathcal{A}y \in \mu\}. \tag{3.1}$$

Throughout the text, $\mathbb{N}$ is the set of natural numbers that include zero.
If a normed linear space $\mathcal{U}$ contains a sequence (b_n), then for every $y \in \mathcal{U}$ there is a unique sequence of scalars (α_n) such that

$$\|y - (\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n)\| \to 0 \ as \ n \to \infty,$$

and then (b_n) is known as the Schauder basis for $\mathcal{U}$. The series $\sum_{n=0}^{\infty} \alpha_n b_n$ has the sum y, known as the expansion of y about the basis (b_n), and we write $y = \sum_{n=0}^{\infty} \alpha_n b_n$, (see [18]).

Let $\mathcal{U}$ and $\mathcal{V}$ be any two sequence spaces. Then, the multiplier space $\mathcal{M}(\mathcal{U},\mathcal{V})$ is given as

$$\mathcal{M}(\mathcal{U},\mathcal{V}) = \left\{(a_n) \in \omega : ay = (a_n y_n) \in \mathcal{V}, \text{ for every } y \in \mathcal{U}\right\}.$$

Thus, the α-dual $\mathcal{U}^\alpha$, the β-dual $\mathcal{U}^\beta$ and the γ-dual $\mathcal{U}^\gamma$ of $\mathcal{U}$ are respectively defined as

$$\mathcal{U}^\alpha = \mathcal{M}(\mathcal{U},\ell_1), \ \mathcal{U}^\beta = \mathcal{M}(\mathcal{U},cs), \ \mathcal{U}^\gamma = \mathcal{M}(\mathcal{U},bs).$$

An infinite matrix can be observed as the linear operator from one sequence space to another sequence space. For this, let $\mathcal{U}$ and $\mathcal{V}$ be arbitrary subsets of ω. Let $\mathcal{A} = (a_{nt})$ be an infinite matrix with complex entries a_{nt}. By $\mathcal{A}y = (\mathcal{A}_n(y))$, we generally write the $\mathcal{A}-$transforms of a sequence $y = (y_t)$ provided the series $\mathcal{A}_n(y) = \sum_{t=0}^{\infty} a_{nt} y_t$ is convergent for each $n \geq 0$.

If $\mathcal{A}y \in \mathcal{V}$ with $y \in \mathcal{U}$, then $\mathcal{A}$ defines a matrix mapping from $\mathcal{U}$ to $\mathcal{V}$. In addition, $(\mathcal{U},\mathcal{V})$ indicates the family of all infinite matrices that map $\mathcal{U}$ into $\mathcal{V}$. Thus, $\mathcal{A}$ is in $(\mathcal{U},\mathcal{V})$ whenever $\mathcal{A}y = ((\mathcal{A}y)_n) \in \mathcal{V}$, $\forall\, y \in \mathcal{U}$, that is, $\mathcal{A} \in (\mathcal{U},\mathcal{V})$ if and only if $\mathcal{A}_n \in \mathcal{U}^\beta$, $\forall\, n$, [20].

4. The Spaces $c(\Delta_q^k)$ And $c_0(\Delta_q^k)$

In the present section, our aim is to introduce the generalized q-difference sequence spaces using the Δ_q^k operator and to study the domains of the newly defined matrix in the sequence spaces c and c_0. Now, let us define the sequence spaces

$c(\Delta_q^k)$ and $c_0(\Delta_q^k)$, as the set of all sequences such that their Δ_q^k-transforms are in the sequence spaces c and c_0, that is,

$$c(\Delta_q^k) = \left\{ y = (y_t) \in \omega : \lim_{t\to\infty} \left| q^{kt} \sum_{n=0}^{k} (-1)^n q^{\binom{n}{2}} \binom{k}{n}_q y_{n+t} \right| \text{ exists} \right\},$$

$$c_0(\Delta_q^k) = \left\{ y = (y_t) \in \omega : \lim_{t\to\infty} q^{kt} \sum_{n=0}^{k} (-1)^n q^{\binom{n}{2}} \binom{k}{n}_q y_{n+t} = 0 \right\}.$$

It is evident that spaces $c(\Delta_q^k)$ and $c_0(\Delta_q^k)$ reduce to ordinary difference spaces $c(\Delta^k)$ and $c_0(\Delta^k)$ when $q \to 1^-$ as studied and introduced by [1]. With the definition of the matrix domain (3.1), the spaces $c(\Delta_q^k)$ and $c_0(\Delta_q^k)$ can be redefined as $c(\Delta_q^k) = (c)_{\Delta_q^k}$ and $c_0(\Delta_q^k) = (c_0)_{\Delta_q^k}$.

Define the sequence $z = (z_t)$ as the Δ_q^k-transforms of a sequence $y = (y_t)$, that is,

$$z_t = q^{kt} \sum_{n=0}^{t} (-1)^n q^{\binom{n}{2}} \binom{k}{n}_q y_{n+t}, \ (t \in \mathbb{N}). \tag{4.1}$$

In addition, $y = (y_t)$ is defined as

$$y_t = \sum_{n=0}^{t} \binom{k+t-n-1}{t-n}_q z_n, \ (t \in \mathbb{N}).$$

Theorem 4.1. *$c(\Delta_q^k)$ and $c_0(\Delta_q^k)$ are BK-spaces accompanied with the norm defined as*

$$\|y\|_{c(\Delta_q^k)} = \|y\|_{c_0(\Delta_q^k)} = \left\|\Delta_q^k y\right\|_c = \sup_{t\in\mathbb{N}} \left| q^{kt} \sum_{n=0}^{k} (-1)^n q^{\binom{n}{2}} \binom{k}{n}_q y_{n+t} \right|.$$

Proof. The proof is obvious, so we omit the details. □

Theorem 4.2. *$c(\Delta_q^k) \cong c$ and $c_0(\Delta_q^k) \cong c_0$.*

Proof. Let $\lambda \in \{c, c_0\}$. Define the mapping $\phi : \lambda(\Delta_q^k) \to \lambda$ by $\phi y = \Delta_q^k y$ for all $y \in \lambda(\Delta_q^k)$. It is trivial that ϕ is a linear bijection preserving the norm. Hence, $\lambda(\Delta_q^k) \cong \lambda$. □

Theorem 4.3. *The space $\lambda(\Delta_q^k)$ for $\lambda \in \{c, c_0\}$ is a non-absolute type.*

Proof. Let $z = (z_t) = \{(-1)^t\}$. Then, it is evident from equality (2.1) that

$$\begin{aligned}
&\Delta_q z = \{2, -2q, 2q^2, -2q^3, \cdots\} = 2\{1, -q, q^2, -q^3, \cdots\},\\
&\Delta_q^2 z = 2(1+q)\{1, -q^2, q^4, \cdots\},\\
&\Delta_q^3 z = 2(1+q)(1+q^2)\{1, -q^3, q^6, \ldots\},\\
&\quad\vdots
\end{aligned}$$

$$\Delta_q^k z = 2(1+q)(1+q^2)(1+q^3)\cdots(1+q^{k-1})\{1, q^k, q^{2k}, \cdots\}.$$

Thus, $(\Delta_q^k z)_t = 2(1+q)_q^{t-1}(-1)^t q^{kt}$. However, $\Delta_q |z| = \{0, 0, 0, \cdots\}$.

This shows that $\|y\|_{\lambda(\Delta_q^k)} \neq \||z|\|_{\lambda(\Delta_q^k)}$, for $z = |z_t|$. □

Theorem 4.4. *The inclusion $c \subset c(\Delta_q^k)$ strictly holds for $q \in \mathbb{R}^+$.*

Proof. Let $z \in c$. Since Δ_q^k is conservative for each $q \in \mathbb{R}^+$, we observe that $\Delta_q^k z \in c$, which gives us that $z \in c(\Delta_q^k)$. Hence $c \subset c(\Delta_q^k)$. To prove strictness, take the sequence $u_t = (-t)$ for all $t \in \mathbb{N}$. Then $u \notin c$. By using the equality (2.1), it is deduced that

$$\begin{aligned}\Delta_q^k u &= ((1-q)(1-q^2)(1-q^3)\cdots(1-q^{k-1})q^{kt})\\ &= ((1-q)_q^{k-1} q^{kt}).\end{aligned}$$

This shows that $\lim_{t\to\infty} \Delta_q^k u$ exists and so $\Delta_q^k u \in c$ which gives us $u \in c(\Delta_q^k)$. Hence, the relation $c \subset c(\Delta_q^k)$ is strict, as claimed. □

Theorem 4.5. *The inclusion $c_0 \subset c_0(\Delta_q^m)$ strictly holds for $q \geq 1$.*

Proof. To establish the proof, one can proceed in a similar fashion using Theorem 4.4. □

To end this segment, let us obtain the Schauder basis for the spaces $c(\Delta_q^k)$ and $c_0(\Delta_q^k)$. We know that the matrix domain $\mu_{\mathcal{A}}$ of the triangle $\mathcal{A}$ in a sequence space μ has a basis whenever μ has a basis. Consequently, Theorem 4.2 gives the following result.

Theorem 4.6. *For each fixed $t \in \mathbb{N}$, define the sequence $g^{(t)}(q) = (g_n^{(t)}(q))$ of the elements of the space $c_0(\Delta_q^k)$ by*

$$g^{(t)}(q) = \begin{cases} \binom{k+t-n-1}{t-n}_q, & 0 \leq t < n, \\ 0, & t \geq n. \end{cases}$$

(i) The set $\{g^{(0)}(q), g^{(1)}(q), g^{(2)}(q), \cdots\}$ forms the basis for $c_0(\Delta_q^k)$, and every $y \in c_0(\Delta_q^k)$ is uniquely expressed as $y = \sum_{t=0}^{\infty} v_t g^{(t)}(q)$.
(ii) The set $\{e, g^{(0)}(q), g^{(1)}(q), g^{(2)}(q), \cdots\}$ forms the basis for $c(\Delta_q^k)$, and every $y \in c(\Delta_q^k)$ is uniquely expressed as $y = he + \sum_{t=0}^{\infty}(v_t - h)g^t(q)$, where $h = \lim_{t\to\infty} v_t = \lim_{t\to\infty}(\Delta_q^k y)_t$ and e is the unit sequence.

5. THE ALPHA-DUAL, THE BETA-DUAL AND THE GAMMA-DUAL

The current segment involves $\mathcal{U}^\alpha$, $\mathcal{U}^\beta$ and $\mathcal{U}^\gamma$ for $\mathcal{U} \in \{c(\Delta_q^k), c_0(\Delta_q^k)$. Since the calculation of duals is quite similar for both spaces, we skip the proof for the space $c(\Delta_q^k)$. Firstly, we list the well-known lemmas due to Stielglitz and Tietz [23], which are needed to obtain the alpha-dual, the beta-dual and the gamma-dual. In the remainder of the paper, $\mathcal{F}$ will denote the collection of all finite subsets of $\mathbb{N}$.

Lemma 5.1. *$\mathcal{A} = (a_{nt}) \in (c_0, \ell_1)$ if and only if*

$$\sup_{K\in\mathcal{F}} \left(\sum_{t=0}^{\infty} \left| \sum_{n\in K} a_{nt} \right| \right) < \infty. \tag{5.1}$$

Lemma 5.2. *$\mathcal{A} = (a_{nt}) \in (c_0, c)$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_{t=0}^{n} |a_{nt}| < \infty, \tag{5.2}$$

$$\lim_{n\to\infty} a_{nt} = \beta_t, \text{ for each } t \in \mathbb{N}. \tag{5.3}$$

Lemma 5.3. *$\mathcal{A} = (a_{nt}) \in (c_0, \ell_\infty)$ if and only if (5.2) holds.*

Theorem 5.1. *Let us define the set $k_1(q)$ as*

$$k_1(q) = \left\{ a = (a_n) \in \omega : \sup_{K\in\mathcal{F}} \sum_{n=0}^{\infty} \left| \sum_{t\in K} \binom{k+t-n-1}{t-n}_q a_t \right| < \infty \right\}.$$

Then, $\{c(\Delta_q^k)\}^\alpha = \{c_0(\Delta_q^k)\}^\alpha = k_1(q)$.

Proof. Consider the below equality

$$a_t y_t = \sum_{n=0}^{t} \binom{k+t-n-1}{t-n}_q a_t z_n = (F(q)z)_t \tag{5.4}$$

for all $t \in \mathbb{N}$, and let $z = (z_t)$ be the (Δ_q^k)-transform of $y = (y_t)$ and the matrix $F(q) = (f_{nt}^q)$ be given as

$$f_{nt}^q = \begin{cases} \binom{k+t-n-1}{t-n}_q, & 0 \le t \le n, \\ 0, & t > n. \end{cases}$$

We observe that using the equality (5.4), $ay = (a_n y_n) \in \ell_1$ whenever $y \in c_0(\Delta_q^k)$ if and only if $F(q)z \in \ell_1$ for $z \in c_0$. Thus, we conclude that $a = (a_n) \in \{k_1(q)\}^\alpha$ if and only if the matrix $(\Delta_q^k) \in (c_0, \ell_1)$. Thus, we conclude by Lemma 5.1 that $\{c_0(\Delta_q^k)\}^\alpha = k_1(q)$. □

Theorem 5.2. *Let us define the sets $k_2(q)$, $k_3(q)$ and $k_4(q)$ by*

$$k_2(q) = \left\{ a = (a_n) \in \omega : \sum_{n=t}^{\infty} \binom{k+n-t-1}{n-t}_q a_n \text{ exists for each } t \in \mathbb{N} \right\},$$

$$k_3(q) = \left\{ a = (a_n) \in \omega : \sup_{n\in\mathbb{N}} \sum_{t=0}^{n} \left| \sum_{m=t}^{n} \binom{k+n-t-1}{n-t}_q a_m \right| < \infty \right\},$$

$$k_4(q) = \left\{ a = (a_n) \in \omega : \lim_{n\to\infty} \sum_{m=t}^{n} \binom{k+n-t-1}{n-t}_q a_m \text{ exists} \right\}.$$

Then, $\{c_0(\Delta_q^k)\}^\beta = k_2(q) \cap k_3(q)$ *and* $\{c(\Delta_q^k)\}^\beta = k_2(q) \cap k_3(q) \cap k_4(q)$.

Proof. Consider the equality

$$\begin{aligned} \sum_{t=0}^{n} a_t y_t &= \sum_{t=0}^{n} \left[\sum_{m=0}^{t} \binom{k+t-n-1}{t-n}_q z_m \right] a_t \\ &= \sum_{t=0}^{n} \left[\sum_{m=t}^{n} \binom{k+n-t-1}{n-t}_q a_m \right] z_t \\ &= (G(q)z)_n \end{aligned}$$

for each $n \in \mathbb{N}$. Then, the sequence $z = (z_t)$ is the Δ_q^k-transform of sequence $y = (y_t)$, where the matrix $G(q) = (g_{nt}^q)$ is given by

$$g_{nt}^q = \begin{cases} \sum_{m=t}^{n} \binom{k+n-t-1}{n-t}_q a_m, & 0 \le t \le n \\ 0, & t > n \end{cases}$$

for every $t, n \in \mathbb{N}$. Thus, we see that $ay = (a_t x_t) \in cs$ for $y \in c_0(\Delta_q^k)$ if and only if $G(q)z \in c$ whenever $y \in c_0$. This implies that $a = (a_n)$ is a sequence in the β-dual of $c_0(\Delta_q^k)$ if and only if the matrix $G(q) \in (c_0, c)$. Hence, by Lemma 5.2, it is deduced that $\sup_{n\in\mathbb{N}} \sum_{t=0}^{n} \left| g_{nt}^q \right| < \infty$ and therefore $\lim_{n\to\infty} g_{nt}^q$ exists for each $t \in \mathbb{N}$. Therefore, $\{c_0(\Delta_q^k)\}^\beta = k_2(q) \cap k_3(q)$. □

Theorem 5.3. *The gamma dual of* $c(\Delta_q^k)$ *and* $c_0(\Delta_q^k)$ *is* $k_3(q)$.

Proof. The proof for the γ-dual is similar to Theorem 5.2 using Lemma 5.3. □

6. Matrix Mappings

The present section examines the necessary and sufficient conditions for a matrix mapping from spaces $c(\Delta_q^k)$ and $c_0(\Delta_q^k)$ to space ν, where $\nu \in \{\ell_\infty, \ell_1, c, c_0\}$. The Theorem below is useful in our study.

Theorem 6.1. *Let* $\nu \subset \omega$ *and* T *be any one of the spaces* c_0 *or* c. *Then* $\mathcal{A} = (a_{nt}) \in (T(\Delta_q^k), \nu)$ *if and only if* $E^{(n)} = (e_{mt}^{(n)}) \in (T, c)$ *for each* $n \in \mathbb{N}$ *and* $E = (e_{nt}) \in (T, \nu)$, *where*

$$e_{mt}^{(n)} = \begin{cases} 0, & t > m, \\ \sum_{j=t}^{m} \binom{k+n-t-1}{n-t}_q a_{nj}, & 0 \le t \le m, \end{cases}$$

and

$$e_{nt} = \sum_{j=t}^{m} \binom{k+n-t-1}{n-t}_q a_{nj}, \ \forall\, t, n \in \mathbb{N}.$$

Proof. Let $\mathcal{A} \in (T(\Delta_q^k), \nu)$ and $y \in T(\Delta_q^k)$. Then, we have the following equality

$$\sum_{t=0}^{m} a_{nt} y_t = \sum_{t=0}^{n}\sum_{j=0}^{t} \binom{k+n-t-1}{n-t} z_j a_{nt} = \sum_{t=0}^{m}\sum_{j=t}^{m} \binom{k+n-t-1}{n-t}_q a_{nj} z_t = \sum_{t=0}^{m} e_{mt}^{(n)} z_t \tag{6.1}$$

for every $n, m \in \mathbb{N}$. Also, $\mathcal{A}y$ exists and so $E^{(n)} \in (T, c)$. Again, as $m \to \infty$ in the equality (6.1), it yields that $\mathcal{A}y = Ez$. Since $\mathcal{A}y \in \nu$ and so $Ez \in \nu$ which leads to the fact that $E \in (T, \nu)$.
Conversely, suppose that $E^{(n)} = (e_{mt}^{(n)}) \in (T, c)$ for every $n \in \mathbb{N}$ and $E = (e_{nt}) \in (T, \nu)$. Take $y \in T(\Delta_q^k)$. Then $(a_{nt})_{t\in\mathbb{N}} \in T^{\beta}$ for each $n \in \mathbb{N}$ which gives us the fact that $(a_{nt})_{t\in\mathbb{N}} \in \{T(\Delta_q^k)\}^{\beta}$ for each $n \in \mathbb{N}$. Also, from the equality (6.1), $\mathcal{A}y = Ez$ as $m \to \infty$. This gives us $\mathcal{A} \in (T(\Delta_q^k), \nu)$, as claimed. □

Following [23], let us define the following conditions:

$$\lim_{m\to\infty} e_{mt}^{(n)} \text{ exists for every } n, t \in \mathbb{N}, \tag{6.2}$$

$$\sup_{m\in\mathbb{N}} \sum_{t=0}^{\infty} \left| e_{mt}^{(n)} \right| < \infty \text{ for each } n \in \mathbb{N}, \tag{6.3}$$

$$\lim_{m\to\infty} \sum_{t=0}^{\infty} e_{mt}^{(n)} \text{ exists for each } n \in \mathbb{N}, \tag{6.4}$$

$$\lim_{n\to\infty} \sum_{t=0}^{n} |e_{nt}| = \alpha_t \text{ for each } t \in \mathbb{N}. \tag{6.5}$$

The following results can be derived using the above conditions together with Theorem 6.1 as follows:

Corollary 6.1. *The following statements hold:*
(i) $\mathcal{A} = (a_{nt}) \in (c_0(\Delta_q^k), \ell_\infty)$ if and only if (6.2), (6.3) hold, and also (5.2) holds with e_{nt} instead of a_{nt}.
(ii) $\mathcal{A} = (a_{nt}) \in (c_0(\Delta_q^k), c)$ if and only if (6.2), (6.3) hold, and also (5.2) and (5.3) hold with e_{nt} instead of a_{nt}.
(iii) $\mathcal{A} = (a_{nt}) \in (c_0(\Delta_q^k), c_0)$ if and only if (6.2), (6.3) hold and also (5.2) and (5.3) hold with $\alpha_t = 0$, with e_{nt} instead of a_{nt}.
(iv) $\mathcal{A} = (a_{nt}) \in (c_0(\Delta_q^k), \ell_1)$ if and only if (6.2), (6.3) hold, and also (5.1) holds with e_{nt} instead of a_{nt}.

Corollary 6.2. *The following statements hold:*
(i) $\mathcal{A} = (a_{nt}) \in (c(\Delta_q^k), \ell_\infty)$ if and only if (6.2), (6.3) and (6.4) hold, and also (5.2) holds with e_{nt} instead of a_{nt}.
(ii) $\mathcal{A} = (a_{nt}) \in (c(\Delta_q^k), c)$ if and only if (6.2), (6.3), (6.4) and (6.5) hold, and also (5.2) and (5.3) hold with e_{nt} instead of a_{nt}.
(iii) $\mathcal{A} = (a_{nt}) \in (c(\Delta_q^k), c_0)$ if and only if (6.2), (6.3), (6.4) and (6.5) hold with $\alpha_t = 0$,

and (5.2) and (5.3) also hold with e_{nt} instead of a_{nt}.
(iv) $\mathcal{A} = (a_{nt}) \in (c(\Delta_q^k), \ell_1)$ if and only if (6.2), (6.3) and (6.4) hold, and also (5.1) holds with e_{nt} instead of a_{nt}.

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