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SOME SOLITONS ON LORENTZIAN PARA-KENMOTSU MANIFOLDS

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ABSTRACT. In this paper we study the nature of the Einstein soliton and η-Einstein soliton in the framework of Lorentzian para-Kenmotsu manifolds (briefly, LP-Kenmotsu manifolds). We find an expression for scalar curvature of LP-Kenmotsu manifolds admitting the Einstein soliton and η-Einstein soliton in various cases. We prove that if an LP-Kenmotsu manifold contains an η-Einstein soliton with a parallel Reeb vector field then the manifold is η-Einstein. We study the nature of the η-Einstein soliton on these manifolds with conformal, collinear and torse forming potential vector fields. We also study the η-Einstein soliton on these manifolds satisfying the curvature conditions: $(\xi.)_R.S = 0$, $(\xi.)_{W_2}.S = 0$ and $(\xi.)_S.W_2 = 0$, where R, S and W_2 are, respectively, the Riemannian curvature tensor, Ricci curvature tensor and W_2-curvature tensor.

1. INTRODUCTION

In 2018, the notion of Lorentzian para-Kenmotsu manifold (LP-Kenmotsu manifold) has been introduced by A. Haseeb and R. Prasad [6]. Later, N. V. C. Shukla and A. Dixit [17] studied ϕ-recurrent Lorentzian para-Kenmotsu manifolds and found that such type of manifolds are η-Einstein. Further, V. Chandra and S. Lal [3] studied some special results on 3-dimensional Lorentzian para-Kenmotsu manifolds. This manifold was also studied by many authors, namely, K. L. Sai Prasad, S. Sunitha Devi [14], T. Mert and M. Atceken ([9], [10], [11]).

The concept of Ricci flow was first introduced by R. S. Hamilton in the early 1980s. Hamilton [5] observed that the Ricci flow is an excellent tool for simplifying the structure of a manifold. It is the process which deforms the metric of a Riemannian manifold by smoothing out the irregularities. The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S, \tag{1.1}$$

where g is a Riemannian metric, S is Ricci tensor and t is time. The solitons for the Ricci flow are the solutions of the above equation, where the metrices at different

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times differ by a diffeomorphism of the manifold. A Ricci soliton is represented by a triple (g,V,λ), where V is a vector field and λ is a scalar, which satisfies the equation

$$L_V g+2S+2\lambda g=0, \tag{1.2}$$

where S is the Ricci curvature tensor and $L_V g$ denotes the Lie derivative of g along the vector field V. A Ricci soliton is said to be shrinking, steady, expanding accordingly $\lambda<0, \lambda=0, \lambda>0$, respectively. The vector field V is called potential vector field and if it is a gradient of a smooth function, then the Ricci soliton (g,V,λ) is called a gradient Ricci soliton and the associated function is called the potential function. The Ricci soliton was further studied by many researchers. For instance, we see ([7], [13], [16], [18]) and their references.

Catino and Mazzieri [2] in 2016 first introduced the notion of the Einstein soliton as a generalization of the Ricci soliton. An almost contact manifold M with structure (ϕ,ξ,η,g) is said to have an Einstein soliton (g,V,λ) if

$$L_V g+2S+(2\lambda-r)g=0, \tag{1.3}$$

holds, where r is the scalar curvature. The Einstein soliton (g,V,λ) is said to be shrinking, steady, expanding accordingly $\lambda<0, \lambda=0, \lambda>0$, respectively. The Einstein soliton creates some self-similar solutions of the Einstein flow given by

$$\frac{\partial g}{\partial t}=-2S+rg.$$

Again as a generalization of the Einstein soliton, the η-Einstein soliton on manifold $M(\phi,\xi,\eta,g)$ is introduced by A. M. Blaga [1] and it is given by

$$L_V g+2S+(2\lambda-r)g+2\beta\eta\otimes\eta=0, \tag{1.4}$$

where, β is some constant. When $\beta=0$ the notion of η-Einstein soliton simply reduces to the notion of the Einstein soliton. And when $\beta\neq 0$, the data (g,V,λ,β) is called a proper η-Einstein soliton on M. The η-Einstein soliton is called shrinking if $\lambda<0$, steady if $\lambda=0$, and expanding if $\lambda>0$.

In [12], Pokhariyal and Mishra have defined the W_2-curvature tensor given by

$$W_2(X,Y)Z=R(X,Y)Z-\frac{1}{n-1}\left[g(Y,Z)QX-g(X,Z)QY\right], \tag{1.5}$$

for all $X,Y,Z\in\chi(M)$, where R is the Riemannian curvature tensor and $\chi(M)$ being the set of all vector fields on M.

As a generalization of concircular, concurrent and parallel vector fields, Yano [20] introduced the notion of a torse-forming vector field. A non-zero vector field V is said to be torse forming on a Riemannian manifold M if

$$\nabla_X V=fX+\pi(X)V, \tag{1.6}$$

where f is a smooth function defined on M and π is a 1-form. If the 1-form π becomes zero, then the vector field V is concircular [4,19]. If $f=1$ and $\pi=0$, then V is concurrent [21], also if $f=\pi=0$, then the vector field V is parallel.

Definition 1.1. *A Lorentzian para-Kenmotsu manifold M is called an η-Einstein manifold if its Ricci tensor is of the form*

$$S(Y,Z)=k_1 g(Y,Z)+k_2\eta(Y)\eta(Z),$$

for all $Y,Z\in\chi(M)$, where k_1,k_2 are scalars.

This paper is structured as follows:

The first two sections of the paper have been kept for introduction and preliminaries. In Section 3, we introduce the Einstein soliton on the LP-Kenmotsu manifold. In Section 4, we study the η-Einstein soliton on the LP-Kenmotsu manifold. Section 5 is concerned with the η-Einstein soliton on LP-Kenmotsu manifold satisfying $(\xi.)_R.S=0$. Section 6 deals with the η-Einstein soliton on the LP-Kenmotsu manifolds satisfying $(\xi.)_{W_2}.S=0$. In Section 7, we discuss the η-Einstein soliton on the LP-Kenmotsu manifolds satisfying $(\xi.)_S.W_2=0$. Finally, Section 8 contains an example of an LP-Kenmotsu manifold admitting the Einstein and η-Einstein soliton.

2. Preliminaries

Let M be an n-dimensional Lorentzian almost para-contact manifold with structure (ϕ,ξ,η,g), where η is a 1-form, ξ is the structure vector field, ϕ is a $(1,1)$-tensor field and g is a Lorentzian metric satisfying

$$\phi^2(X) = X+\eta(X)\xi, \eta(\xi)=-1, \tag{2.1}$$

$$g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\phi X,\phi Y) = g(X,Y)+\eta(X)\eta(Y), \tag{2.3}$$

for all vector fields X,Y on M. A Lorentzian almost para-contact manifold is said to be a Lorentzian para-contact manifold if η becomes a contact form. In a Lorentzian para-contact manifold the following relations also hold [8,15]:

$$\phi(\xi) = 0, \eta\circ\phi=0, \tag{2.4}$$

$$g(X,\phi Y) = g(\phi X,Y). \tag{2.5}$$

The manifold M is called a Lorentzian para-Kenmotsu manifold if

$$(\nabla_X\varphi)Y=-g(\phi X,Y)\xi-\eta(Y)\phi X, \tag{2.6}$$

for any smooth vector fields X,Y on M.

In a Lorentzian para-Kenmotsu manifold the following relations also hold [6]:

$$
\begin{aligned}
\nabla_X \xi &= -X - \eta(X)\xi, && (2.7)\\
(\nabla_X \eta) Y &= -g(X,Y) - \eta(X)\eta(Y), && (2.8)\\
\eta(R(X,Y)Z) &= g(Y,Z)\eta(X) - g(X,Z)\eta(Y), && (2.9)\\
R(X,Y)\xi &= \eta(Y)X - \eta(X)Y, && (2.10)\\
R(\xi,X)Y &= g(X,Y)\xi - \eta(Y)X, && (2.11)\\
R(\xi,X)\xi &= X + \eta(X)\xi, && (2.12)\\
S(X,\xi) &= (n-1)\eta(X) && (2.13)\\
S(\xi,\xi) &= -(n-1), && (2.14)\\
Q\xi &= (n-1)\xi, && (2.15)\\
S(\phi X,\phi Y) &= S(X,Y) + (n-1)\eta(X)\eta(Y), && (2.16)
\end{aligned}
$$

for any smooth vector fields X,Y,Z on M.

3. EINSTEIN SOLITON ON THE LP-KENMOTSU MANIFOLD

In this section we examine the nature of the Einstein soliton on the LP-Kenmotsu manifold with a structure vector field and a torse forming vector field as a potential vector field.

Theorem 3.1. *Let M be an LP-Kenmotsu manifold admitting an Einstein soliton (g,V,λ). If the non-zero potential vector field V is collinear with the structure vector field, then the soliton is*

1. *expanding if $r > 2(n-1)$,*
2. *steady if $r = 2(n-1)$,*
3. *shrinking if $r < 2(n-1)$.*

Proof. After expanding (1.3), we have

$$g(\nabla_X V,Y) + g(X,\nabla_Y V) + 2S(X,Y) + (2\lambda - r)g(X,Y) = 0. \tag{3.1}$$

Setting $V = \xi$ in (3.1) we get

$$0 = -2g(X,Y) - 2\eta(X)\eta(Y) + 2S(X,Y) + (2\lambda - r)g(X,Y). \tag{3.2}$$

Putting $X = \xi$ and using (2.1), (2.13) we get

$$0 = -2(n-1)\eta(Y) - (2\lambda - r)\eta(Y). \tag{3.3}$$

Since $\eta(Y) \neq 0$, we have

$$\lambda = \frac{r}{2} - (n-1),$$

which implies the theorem. □

Theorem 3.2. *Let M be an LP-Kenmotsu manifold admitting an Einstein soliton (g, V, λ). If the potential vector field is torse-forming, then the soliton is*
1. *expanding if* $\frac{1}{n}(\pi(V)+r)-\frac{r}{2}+f<0$,
2. *steady if* $\frac{1}{n}(\pi(V)+r)-\frac{r}{2}+f=0$,
3. *shrinking if* $\frac{1}{n}(\pi(V)+r)-\frac{r}{2}+f>0$.

Proof. Using (1.6) in (3.1) we get

$$\begin{aligned} 0 &= (2f+2\lambda-r)g(X,Y)+\pi(X)g(V,Y) \\ &\quad +\pi(Y)g(X,V)+2S(X,Y). \end{aligned} \tag{3.4}$$

Contracting (3.4) over X and Y we obtain

$$\lambda=-\left[\frac{1}{n}(\pi(V)+r)\right]+\frac{r}{2}-f,$$

which implies the theorem. □

4. η-EINSTEIN SOLITON ON LP-KENMOTSU MANIFOLDS

Theorem 4.1. *If an n-dimensional LP-Kenmotsu manifold admits an η-Einstein soliton (g, ξ, λ, β), then the soliton scalars are given by the following equations*

$$\begin{aligned} \lambda &= -\frac{div\xi}{n-1}+\frac{r}{2}(\frac{n-3}{n-1})+1, \\ \beta &= -\frac{div\xi}{n-1}-\frac{r}{n-1}+n, \end{aligned}$$

where divξ*=Divergence of* ξ.

Proof. Applying $V=\xi$ in (1.4), we get

$$0=\left(L_{\xi}g\right)(X,Y)+2S(X,Y)+(2\lambda-r)g(X,Y)+2\beta\eta(X)\eta(Y). \tag{4.1}$$

Using (2.7), we get

$$\begin{aligned} 0 &= g(\nabla_X\xi,Y)+g(X,\nabla_Y\xi)+2S(X,Y) \\ &\quad +(2\lambda-r)g(X,Y)+2\beta\eta(X)\eta(Y), \\ &= (2\lambda-r-2)g(X,Y)+2S(X,Y)+2(\beta-1)\eta(X)\eta(Y). \end{aligned} \tag{4.2}$$

Setting $X=Y=\xi$ in (4.2), we get

$$\beta=\lambda-\frac{1}{2}\left[r-2(n-1)\right]. \tag{4.3}$$

Taking an orthonormal frame field and contracting (4.1) over X and Y, we obtain

$$\beta=\mathrm{div}\xi+\lambda n-\frac{r}{2}(n-2). \tag{4.4}$$

Comparing the value of β from (4.3) and (4.4), we get

$$\lambda = -\frac{\mathrm{div}\xi}{n-1} + \frac{r}{2}\left(\frac{n-3}{n-1}\right) + 1. \tag{4.5}$$

Putting the value of λ from (4.3) in (4.5), we get

$$\beta = -\frac{\mathrm{div}\xi}{n-1} - \frac{r}{n-1} + n.$$

□

Corollary 4.1. *If an n-dimensional LP-Kenmotsu manifold M contains an η-Einstein soliton (g,ξ,λ,β), then M is an η-Einstein manifold*

Proof. From equation (4.2) we have

$$S(X,Y) = \left(\frac{r}{2} - \lambda + 1\right) g(X,Y) - (\beta - 1)\eta(X)\eta(Y), \tag{4.6}$$

which shows that M is η-Einstein manifold. □

Theorem 4.2. *If an LP-Kenmotsu manifold M contains an η-Einstein soliton (g,ξ,λ,β) with the structure vector field ξ being parallel i.e., $\nabla_X\xi = 0$, then M is an η-Einstein manifold.*

Proof. After expanding the Lie derivative we get from (1.4)

$$\begin{aligned} 0 &= g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X,Y) \\ &\quad + (2\lambda - r)g(X,Y) + 2\beta\eta(X)\eta(Y). \end{aligned} \tag{4.7}$$

Setting $V = \xi$ and $\nabla_X\xi = 0$ in (4.7), we get

$$S(X,Y) = -\left(\lambda - \frac{r}{2}\right) g(X,Y) - \beta\eta(X)\eta(Y), \tag{4.8}$$

which shows that M is η-Einstein. □

Corollary 4.2. *If an n-dimensional LP-Kenmotsu manifold admits an η-Einstein soliton (g,ξ,λ,β), then the scalar curvature of M is constant.*

Proof. Tracing (4.8), we have

$$r = \frac{2(n\lambda - \beta)}{n-2},$$

which implies

$$\delta r = 0.$$

□

Theorem 4.3. *If M is an LP-Kenmotsu manifold admitting an η-Einstein soliton (g,V,λ,β) such that $V \in D = \ker\eta$, then the scalar curvature of M is given by*

$$r = 2(n-1+\lambda) - 2\beta.$$

Proof. Consider the distribution D on the LP-Kenmotsu manifold M as $D = \ker\eta$. If $V \in D$, then

$$\eta(V) = 0.$$

Taking the covariant derivative with respect to ξ and using $\left(\nabla_\xi \eta\right) V = 0$, we get

$$\eta\left(\nabla_\xi V\right) = 0. \tag{4.9}$$

Setting $X = Y = \xi$ in (4.7) and using (2.14), (4.9), we obtain

$$\begin{aligned} 0 &= 2g(\nabla_\xi V, \xi) + 2S(\xi, \xi) + (2\lambda - r)\eta(\xi) + 2\beta\eta(\xi)\eta(\xi) \\ &= 2\eta\left(\nabla_\xi V\right) - 2(n-1) - (2\lambda - r) + 2\beta \\ &= -2(n-1+\lambda) + r + 2\beta. \end{aligned}$$

This implies the theorem. □

5. η-EINSTEIN SOLITON ON THE LP-KENMOTSU MANIFOLDS SATISFYING $(\xi.)_R .S = 0$

Theorem 5.1. *Let $M(\phi, \xi, \eta, g)$ be an n-dimensional LP-Kenmotsu manifold admitting η-Einstein soliton (g, ξ, λ, β) and satisfying $(\xi.)_R .S = 0$, then the soliton constants are given by*

$$\lambda = \frac{1}{2}\left[r - 2(n-2)\right], \beta = 1.$$

Proof. The condition that must be satisfied by S is

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0, \tag{5.1}$$

for all $X, Y, Z \in \chi(M)$.

Using (2.11) and replacing the expression of S from (4.6) in (5.1), we get

$$(\beta - 1)\left[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)\right] = 0, \tag{5.2}$$

For $Z = \xi$ in (5.2) we have

$$(\beta - 1)g(\phi X, \phi Y) = 0,$$

for all $X, Y \in \chi(M)$, which gives

$$\beta = 1. \tag{5.3}$$

From (4.3) and (5.3) we obtain

$$\lambda = \frac{1}{2}\left[r - 2(n-2)\right].$$

□

Corollary 5.1. *The η-Einstein soliton (g, ξ, λ, β) on an n-dimensional LP-Kenmotsu manifold M satisfying $(\xi.)_R .S = 0$ is shrinking, steady and expanding accordingly*

$$r < 2(n-2),\ r = 2(n-2),\ r > 2(n-2).$$

Corollary 5.2. *On an LP-Kenmotsu manifold M satisfying* $(\xi.)_R.S = 0$*, there is no Einstein soliton with the potential vector field* ξ*.*

6. η-EINSTEIN SOLITON ON LP-KENMOTSU MANIFOLDS SATISFYING $(\xi.)_{W_2}.S = 0$

Theorem 6.1. *Let* $M(\phi,\xi,\eta,g)$ *be an n-dimensional LP-Kenmotsu manifold admitting the* η*-Einstein soliton* (g,ξ,λ,β) *and satisfying* $(\xi.)_{W_2}.S = 0$*, then the soliton constants are given by*

$$\lambda = \frac{r}{2}+1, \beta = n.$$

Proof. The condition that must be satisfied by S is

$$0 = S(W_2(\xi,X)Y,Z)+S(Y,W_2(\xi,X)Z), \tag{6.1}$$

for all $X,Y,Z \in \chi(M)$.

Replacing the expression of S from (4.6) in (6.1), we get

$$0 = \left[\frac{r}{2}-\lambda+1\right] g(Y,W_2(\xi,X)Z)+g(W_2(\xi,X)Y,Z). \tag{6.2}$$

By the help of (1.5), (2.9), (2.11) and (6.2), we get

$$\begin{aligned} 0 \quad = \quad & \left[\frac{r}{2}-\lambda+1\right]\left[\frac{r}{2}-\lambda-n+2\right][g(X,Z)\eta(Y)+g(X,Y)\eta(Z)] \\ & -(\beta-1)\left[\frac{r}{2}-\lambda+1\right]\eta(X)\eta(Y)\eta(Z), \end{aligned} \tag{6.3}$$

which gives

$$\lambda = \frac{r}{2}+1.$$

Putting the value of λ in (4.3), we get

$$\beta = n.$$

□

Corollary 6.1. *On an LP-Kenmotsu manifold M satisfying* $(\xi.)_{W_2}.S = 0$*, there is no Einstein soliton with the potential vector field* ξ*.*

7. η-EINSTEIN SOLITON ON LP-KENMOTSU MANIFOLDS SATISFYING $(\xi.)_S.W_2 = 0$

Theorem 7.1. *Let* $M(\phi,\xi,\eta,g)$ *be an n-dimensional LP-Kenmotsu manifold admitting the* η*-Einstein soliton* (g,ξ,λ,β) *and satisfying* $(\xi.)_S.W_2 = 0$*, then the soliton constants are given by*

$$\lambda = \left(\frac{r}{2}+1\right), \beta = n, \ or\ \lambda = \left(\frac{r}{2}-n+2\right), \beta = 1.$$

Proof. The condition that must be satisfied by S is

$$\begin{aligned} 0 &= S(X,W_2(Y,Z)V)\xi - S(\xi,W_2(Y,Z)V)X + S(X,Y)W_2(\xi,Z)V \\ &\quad - S(\xi,Y)W_2(X,Z)V + S(X,Z)W_2(Y,\xi)V - S(\xi,Z)W_2(Y,X)V \\ &\quad + S(X,V)W_2(Y,Z)\xi - S(\xi,V)W_2(Y,Z)X, \end{aligned} \tag{7.1}$$

for all $X,Y,Z,V \in \chi(M)$. Taking the inner product with ξ the relation (7.1) becomes

$$\begin{aligned} 0 &= -S(X,W_2(Y,Z)V) - S(\xi,W_2(Y,Z)V)\eta(X) \\ &\quad + S(X,Y)\eta(W_2(\xi,Z)V) - S(\xi,Y)\eta(W_2(X,Z)V) \\ &\quad + S(X,Z)\eta(W_2(Y,\xi)V) - S(\xi,Z)\eta(W_2(Y,X)V) \\ &\quad + S(X,V)\eta(W_2(Y,Z)\xi) - S(\xi,V)\eta(W_2(Y,Z)X). \end{aligned} \tag{7.2}$$

Replacing the expression of S from (4.6), we get

$$\begin{aligned} 0 &= \left(\frac{r}{2} - \lambda + 1\right)[g(X,W_2(Y,Z)V) + \eta(W_2(Y,Z)V)] \\ &\quad - \left(\frac{r}{2} - \lambda + 1\right)[g(X,Y)\eta(W_2(\xi,Z)V + g(X,V)\eta(W_2(Y,Z)\xi] \\ &\quad - (\beta-1)\eta(X)[\eta(Y)\eta(W_2(\xi,Z)V) + \eta(V)\eta(W_2(Y,Z)\xi] \\ &\quad - (n-1)[\eta(Y)\eta(W_2(X,Z)V) + \eta(Z)\eta(W_2(Y,X)V)] \\ &\quad - (\beta-1)\eta(X)\eta(Z)\eta(W_2(Y,\xi)V) - (n-1)\eta(V)\eta(W_2(Y,Z)X) \\ &\quad - \left(\frac{r}{2} - \lambda + 1\right)g(X,Z)\eta(W_2(Y,\xi)V). \end{aligned} \tag{7.3}$$

Setting $V = \xi$ in (7.3) and using (1.5), (2.1), (2.9), (2.10), (2.13) and (2.14), we get

$$0 = \left(\frac{r}{2} - \lambda + 1\right)\left[\frac{r}{2} - \lambda - n + 2\right][g(X,Z)\eta(Y) - g(X,Y)\eta(Z)]. \tag{7.4}$$

Setting $Z = \xi$ and using (2.3) in (7.4), we obtain

$$\left(\frac{r}{2} - \lambda + 1\right)\left[\frac{r}{2} - \lambda - n + 2\right]g(\phi X,\phi Y) = 0,$$

which gives

$$\lambda = \left(\frac{r}{2} + 1\right), \text{ or } \left(\frac{r}{2} - n + 2\right). \tag{7.5}$$

In view of (4.3) and (7.5), we have

$$\lambda = \left(\frac{r}{2} + 1\right), \beta = n, \text{ or } \lambda = \left(\frac{r}{2} - n + 2\right), \beta = 1.$$

□

Corollary 7.1. *On an LP-Kenmotsu manifold M satisfying $(\xi.)_S.W_2 = 0$ there is no Einstein soliton with the potential vector field ξ.*

8. Example of 5-dimensional Lorentzian para-Kenmotsu manifold admitting η-Einstein soliton

We consider a 5-dimensional manifold

$$M = \left\{(x,y,z,u,v) \in R^5\right\},$$

where (x,y,z,u,v) are the standard co-ordinates in R^5.

We choose the linearly independent vector fields

$$E_1 = x\frac{\partial}{\partial x}, E_2 = x\frac{\partial}{\partial y}, E_3 = x\frac{\partial}{\partial z}, E_4 = x\frac{\partial}{\partial u}, E_5 = x\frac{\partial}{\partial v}.$$

Let g be the Riemannian metric defined by $g(E_i,E_j) = 0,\ if\ i \neq j\ for\ i,j = 1,2,3,4,5,$

and $g(E_1,E_1) = -1, g(E_2,E_2) = 1, g(E_3,E_3) = 1, g(E_4,E_4) = 1, g(E_5,E_5) = 1.$

Let η be the 1-form defined by $\eta(X) = g(X,E_1)$ for any $X \in \chi\left(M^5\right)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi E_1 = 0, \phi E_2 = -E_3, \phi E_3 = -E_2, \phi E_4 = -E_5, \phi E_5 = -E_4. \tag{8.1}$$

Let $X,Y,Z \in \chi\left(M^5\right)$ be given by

$$\begin{aligned} X &= x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4 + x_5E_5,\\ Y &= y_1E_1 + y_2E_2 + y_3E_3 + y_4E_4 + y_5E_5,\\ Z &= z_1E_1 + z_2E_2 + z_3E_3 + z_4E_4 + z_5E_5. \end{aligned}$$

Then, we have

$$\begin{aligned} g(X,Y) &= x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5,\\ \eta(X) &= -x_1,\\ g(\phi X,\phi Y) &= x_2y_2 + x_3y_3 + x_4y_4 + x_5y_5. \end{aligned}$$

Using the linearity of g and ϕ, $\eta(E_1) = -1, \phi^2X = X + \eta(X)E_1$ and $g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y)$ for all $X,Y \in \chi(M)$.

We have

$$\begin{aligned} &[E_1,E_2] = E_2, [E_1,E_3] = E_3, [E_1,E_4] = E_4, [E_1,E_5] = E_5,\\ &[E_2,E_1] = -E_2, [E_3,E_1] = -E_3, [E_4,E_1] = -E_4, [E_5,E_1] = -E_5,\\ &[E_i,E_j] = 0 \text{ for all others } i \text{ and } j. \end{aligned}$$

Let the Levi-Civita connection with respect to g be ∇, then using Koszul formula we get the following

$$\begin{aligned}
&\nabla_{E_1}E_1 = 0, \nabla_{E_1}E_2 = 0, \nabla_{E_1}E_3 = 0, \nabla_{E_1}E_4 = 0, \nabla_{E_1}E_5 = 0,\\
&\nabla_{E_2}E_1 = -E_2, \nabla_{E_2}E_2 = -E_1, \nabla_{E_2}E_3 = 0,\ \nabla_{E_2}E_4 = 0, \nabla_{E_2}E_5 = 0,\\
&\nabla_{E_3}E_1 = -E_3,\ \nabla_{E_3}E_2 = 0\ , \nabla_{E_3}E_3 = -E_1\ , \nabla_{E_3}E_4 = 0\ , \nabla_{E_3}E_5 = 0,\\
&\nabla_{E_4}E_1 = -E_4, \nabla_{E_4}E_2 = 0, \nabla_{E_4}E_3 = 0, \nabla_{E_4}E_4 = -E_1, \nabla_{E_4}E_5 = 0,\\
&\nabla_{E_5}E_1 - E_5, \nabla_{E_5}E_2 = 0, \nabla_{E_5}E_3 = 0, \nabla_{E_5}E_4 = 0,\ \nabla_{E_5}E_5 = -E_1.
\end{aligned}$$

From the above results we see that the structure (ϕ, ξ, η, g) satisfies

$(\nabla_X\phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \forall X, Y \in \chi\left(M^5\right)$, where $\eta(\xi) = \eta(E_1) = -1$. Hence $M^5(\phi, \xi, \eta, g)$ is a LP-Kenmotsu manifold.

The non-zero components of Riemannian curvature with respect to the Levi-Civita connection ∇ are given by

$$\begin{aligned}
&R(E_1,E_2)E_1 = E_2, R(E_1,E_2)E_2 = -E_1, R(E_1,E_3)E_1 = E_3,\\
&R(E_1,E_3)E_3 = -E_1, R(E_1,E_4)E_1 = E_4, R(E_1,E_4)E_4 = -E_1,\\
&R(E_1,E_5)E_1 = E_5, R(E_1,E_5)E_5 = -E_1, R(E_2,E_1)E_2 = E_1,\\
&R(E_2,E_1)E_1 = -E_2, R(E_2,E_3)E_2 = E_3, R(E_2,E_3)E_3 = -E_2,\\
&R(E_2,E_4)E_2 = E_4, R(E_2,E_4)E_4 = -E_2, R(E_2,E_5)E_2 = E_5,\\
&R(E_2,E_5)E_5 = -E_2, R(E_3,E_1)E_3 = E_1, R(E_3,E_1)E_1 = -E_3,\\
&R(E_3,E_2)E_3 = E_2, R(E_3,E_2)E_2 = -E_3, R(E_3,E_4)E_3 = E_4,\\
&R(E_3,E_4)E_4 = -E_3, R(E_3,E_5)E_3 = E_5, R(E_3,E_5)E_5 = -E_3,\\
&R(E_4,E_1)E_4 = E_1, R(E_4,E_1)E_1 = -E_4, R(E_4,E_2)E_4 = E_2,\\
&R(E_4,E_2)E_2 = -E_4, R(E_4,E_3)E_4 = E_3, R(E_4,E_3)E_3 = -E_4,\\
&R(E_4,E_5)E_4 = E_5, R(E_4,E_5)E_5 = -E_4, R(E_5,E_1)E_5 = E_1,\\
&R(E_5,E_1)E_1 = -E_5, R(E_5,E_2)E_5 = E_2, R(E_5,E_2)E_2 = -E_5,\\
&R(E_5,E_3)E_5 = E_3, R(E_5,E_3)E_3 = -E_5, R(E_5,E_4)E_5 = E_4.
\end{aligned}$$

Using the above curvature tensors the Ricci curvature tensors with respect to ∇ are:

$$\begin{aligned}
&S(E_1,E_1) = -4, S(E_2,E_2) = S(E_3,E_3) = 2,\\
&S(E_4,E_4) = S(E_5,E_5) = 2.
\end{aligned}$$

Therefore, the scalar curvature tensor with respect to the Levi-Civita connection is $r = 4$.

If the manifold M contains an η-Einstein soliton (g, E_1, λ, β), then relation (4.3) becomes

$$\begin{aligned}
\beta &= \lambda - \frac{1}{2}[r - 2(n-1)]\\
&= \lambda - \frac{1}{2}[4 - 2(5-1)]\\
&= \lambda + 2. \qquad (8.2)
\end{aligned}$$

Now,

$$\begin{aligned}\mathrm{div}\xi = \mathrm{div}E_1 &= g(\nabla_{E_i}E_1, E_i), i = 1,2,3,4,5\\ &= -g(E_2,E_2) - g(E_3,E_3) - g(E_4,E_4) - g(E_5,E_5)\\ &= -4.\end{aligned}$$

Setting $\mathrm{div}\xi = -4, r = 4, n = 5$ in (4.4) and (4.5), we get

$$\lambda = 3, \beta = 5.$$

Obviously, these values of λ and β satisfy equation (8.2) and hence the Theorem 4.1 of Section 4 is verified.

Similarly, we can check all the results for the manifold under consideration.

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