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ON THE GROWTH RATES OF THE COMPOSITION OF INTEGER TRANSLATED ENTIRE AND MEROMORPHIC FUNCTIONS

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ABSTRACT. The integer translation of a function $f(z)$ is denoted by $f(z+n)$ for each $n \in \mathbb{Z}$. It is possible to obtain a function with certain characteristics for each $n \in \mathbb{Z}$. This study examines the impact of integer translations on the growth and behavior of a meromorphic function $f(z)$. Specifically, we consider the family of meromorphic functions generated by integer shifts of $f(z)$, denoted as,

$$f_n(z) = \{f(z+n) : n \in \mathbb{Z}\}.$$

The primary focus is on understanding how these integer translations affect the Nevanlinna characteristic function $T(r,f)$, which is a key tool for assessing the growth of meromorphic functions. Our study also includes a comparative growth analysis between integer-translated versions of both entire and meromorphic functions. By examining a range of conditions, we provide insights into how translation influences the growth and value distribution of the functions. This investigation contributes to a deeper understanding of translation-invariant properties in complex analysis and offers new perspectives on the dynamic growth behavior of meromorphic and entire functions.

1. INTRODUCTION

Let f be a non-constant meromorphic function (i.e., regular except for poles) defined on the complex plane $\mathbb{C}$ and be an entire function in the absence of poles. To study how the values of f are distributed it is essential to explore the distribution of the solutions of the equation $f(z) = a$, where a is a complex number finite or infinite. This includes an estimate of the number of roots $n(r,a;f)$, counted with or without multiplicity in a disc $|z| \leq r$ for any non-negative real number r, estimates on the growth and the asymptotics of the number of such solutions in terms of r, the comparison of the various estimates when the constant a varies, etc.

An oldest such result is the Fundamental Theorem of Algebra, which states that a polynomial of degree n has n complex roots (counted with proper multiplicity). This

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theorem allows us to write any polynomial $f(z)$ in the form

$$f(z) = cz^p \prod_{r=p+1}^{n} \left(1 - \frac{z}{z_r}\right),$$

where z_r are the zeros of $f(z)$ other than those at the origin. Precisely, when $z \to \infty$ the growth of a polynomial $f(z)$ of degree $n \geq 1$ is equal to n. Then

$$\lim_{z\to\infty} \frac{f(z)}{z^n} = d,$$

where d is the coefficient of the monomial of highest degree of f. Thus, growth of the polymonial f can be read on its coefficients. In 1879 Picard [30, 31] obtained two generalized results:

(i) For any non-constant entire function f, the equation $f(z) = a$ has a solution for every value of a except possibly for one value. An example of an entire function with an exceptional value is the exponential map $f(z) = e^z$, with $z = 0$.

(ii) An arbitrary meromorphic function, in the neighborhood of an essential singularity, takes any complex value infinitely often, with at most one exception.

Borel [10] (1897) first gave the concept order of growth for an entire function f by the quantity $\rho_f = \{\inf \mu : \log M(r,f) = O(r^\mu), 0 \leq \mu \leq \infty\}$, where the maximum modulus $M(r,f) = \max_{|z|=r} |f(z)|$. The ρ_f is a measure of how fast the maximum modulus of the function f grows. With such a definition, the order of a polynomial is zero, the exponential function e^z has order one and the function e^{e^z} has infinite order. Later he reformulated the Picard's theorem using the notion of order and obtained the following sharper result, called the Picard-Borel theorem [11] : An entire function f of order ρ_f $(0 \leq \rho_f \leq \infty)$ satisfies

$$\limsup_{r\to\infty} \frac{\log n(r,a;f)}{\log r} \leq \rho_f$$

for every finite complex value 'a', with equality holding except possibly for one value a. Here, $n(r,a;f)$ is the number of roots with multiplicity of the equation $f(z) = a$ in the disc $|z| \leq r$. Borel's definition of order was used later by Nevanlinna [28, 29] who generalized it to the setting of meromorphic functions that are not necessarily entire.

For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $n(t,a;f)$ $(\overline{n}(t,a;f))$ the number of a-points (distinct a -points) of a non-constant meromorphic function f in $|z| \leq t$, where an ∞-point is a pole of f. We put

$$N(r,a;f) = \int_0^r \frac{n(t,a;f) - n(0,a;f)}{t} dt + n(0,a;f) \log r$$

and

$$\overline{N}(r,a;f) = \int_0^r \frac{\overline{n}(t,a;f) - \overline{n}(0,a;f)}{t} dt + \overline{n}(0,a;f)\log r.$$

The function $N(r,a;f)\left(\overline{N}(r,a;f)\right)$ is called the counting function for a-points (distinct a-points) of f . On many occassions $N(r,\infty;f)$ and $\overline{N}(r,\infty;f)$ are denoted by $N(r,f)$ and $\overline{N}(r,f)$ respectively . We also put

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta,$$

where $\log^+ x = \max\{\log x, 0\}$ for every real $x \geq 0$. For $a \in \mathbb{C}$, we denote by $m(r,a;f)$ the function $m(r,\frac{1}{f-a})$ and by $m(r,\infty;f)$ the function $m(r,f)$ which is called the proximity function of f. The function $T(r,f) = m(r,f) + N(r,f)$ is called the Nevanlinna's Characteristic function of f. Further, for entire $f(z)$

$$T(r,f) = m(r,f)$$

and in addition for $0 < r \leq R$,

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{R+r}{R-r} T(R,f) \text{ \{Hayman [17]\}}. \tag{1.1}$$

We assume that the reader has a basic understanding of Nevanlinna's value distribution theory. For a comprehensive introduction, please refer to [17, 37]. For all $r \in \mathbb{R}$, we define $\exp^{[1]} r = e^r$and $\exp^{[p+1]} r = \exp\left(\exp^{[p]} r\right), p \in \mathbb{Z}^+$. We also define for all r sufficiently large $\log^{[1]} r = \log r$, $\log^{[p+1]} r = \log\left(\log^{[p]} r\right), p \in \mathbb{Z}^+$. Moreover, we denote by $\exp^{[0]} r = r, \log^{[0]} r = r, \log^{[-1]} r = \exp^{[1]} r$ and $\exp^{[-1]} r = \log^{[1]} r$.

The order ρ_f and lower order λ_f of a meromorphic function f are defined by

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}\ ,\ \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

respectively. Also for a positive integer $p > 1$, the p-order ρ_f^p and lower p-order λ_f^p of a meromorphic function are respectively defined by

$$\rho_f^p = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log r} \text{ and } \lambda_f^p = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log r} \text{ \{Sato [32]\}}$$

and if f is entire, then $T(r,f)$ can be replaced with $\log M(r,f)$.

Juneja, Kapoor and Bajpai [21] defined the (p,q)-order and lower (p,q)-order of an entire function respectively as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M(r,f)}{\log^{[q]} r},$$

where p, q are positive integers with $p > q$. When f is meromorphic, one can easily verify that

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r} \text{ and } \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p-1]} T(r,f)}{\log^{[q]} r},$$

see [4, 7, 33]. Obviously, $\rho_f(2,1) = \rho_f^2 = \rho_f$ and $\lambda_f(2,1) = \lambda_f^2 = \lambda_f$.

The translation of a meromorphic function $f(z)$ is denoted by $f(z+n)$, where $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, one may obtain a function with some properties. Let us denote this family by $f_n(z)$, i.e.,

$$f_n(z) = \{f(z+n) : n \in \mathbb{Z}\}.$$

It is clear that the number of poles of f may be changed in a finite region after translation but it remains unaltered in the open complex plane $\mathbb{C}$, i.e.,

$$N(r, f(z+n)) = N(r,f) + e_n, \tag{1.2}$$

where e_n is a residue term such that $e_n \to 0$ as $r \to \infty$. Also

$$\begin{aligned} m(r, f(z+n)) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta} + n\right)\right| d\theta \\ &= m(r,f) + e'_n, \end{aligned} \tag{1.3}$$

where e'_n (may be distinct from e_n) is such that $e'_n \to 0$ as $r \to \infty$. Therefore, from (1.2) and (1.3) we get

$$N(r, f(z+n)) + m(r, f(z+n)) = N(r,f) + e_n + m(r,f) + e'_n$$
$$\text{i.e., } T(r, f(z+n)) = T(r,f) + e_n + e'_n.$$

Now if n varies, then the Nevanlinna's Characteristic function for the family $f_n(z)$ is

$$T(r, f_n) = nT(r,f) + \sum_n \left(e_n + e'_n\right). \tag{1.4}$$

Since $T(r,f)$ is an increasing function of r,

$$\lim_{r\to\infty} \frac{T(r,f_n)}{T(r,f)} = n, \text{ see [8].}$$

Again it is obvious that the maximum modulus $M(r,f)$ of an entire function $f(z)$ may be changed in a finite region after translation but it remains unaltered in the open complex plane $\mathbb{C}$, and for the family $f_n(z)$ we obtain

$$M(r, f_n) = M(r,f) + \sum_n e''_n,$$

where e''_n (may be distinct from e_n, e'_n) is such that $e''_n \to 0$ as $r \to \infty$. This implies,

$$\log M(r, f_n) = \log M(r,f) + O(1). \tag{1.5}$$

Now from (1.4) we obtain

$$\log^{[p-1]} T(r, f_n) = \log^{[p-1]} n + \log^{[p-1]} T(r, f) + \log^{[p-1]} \left(1 + \frac{\sum_n (e_n + e'_n)}{nT(r, f)} \right),$$

where $e_n \to 0$, $e'_n \to 0$ as $r \to \infty$. Since $T(r, f)$ is an increasing function of r,

$$\log^{[p-1]} T(r, f_n) = \log^{[p-1]} T(r, f) + O(1). \tag{1.6}$$

This implies,

$$\rho_{f_n}(p, q) = \rho_f(p, q).$$

Similary, it can be proved that

$$\lambda_{f_n}(p, q) = \lambda_f(p, q).$$

Biswas and Datta [6], Biswas et al. [7], and Biswas [8] etc. investigated the changes in Nevanlinna's characteristic function for integer-translated entire and meromorphic functions. Their research focused on how these changes affect the comparative growth of the composition of entire and meromorphic functions. Building on their contributions, we aim to explore this topic in greater depth.

In this paper, we present some new results regarding the growth rates of the composition of integer-translated entire and meromorphic functions with respect to one of the factors of the composition. We utilise concepts of (p, q)-order and (p, q)-lower order, where p and q are positive integers with $p > q$.

This naturally raises the question: Why focus on integer translations? Translations of meromorphic functions by integers introduce subtle yet significant changes in their growth behavior and characteristics, which may not be evident in the unshifted case. These shifts present a rich area for further study. Unresolved issues include understanding the exact effects of these translations on more complex function compositions and determining how these changes influence order and type within Nevanlinna's theory.

2. Preparatory Lemmas

We now recall a well-known definition : The Nevanlinna's deficiency of a finite or infinite complex number 'a' with respect to a meromorphic function g is defined as

$$\delta(a; g) = 1 - \limsup_{r \to \infty} \frac{N(r, a; g)}{T(r, g)} = \liminf_{r \to \infty} \frac{m(r, a; g)}{T(r, g)}.$$

The following lemmas are necessary.

Lemma 2.1. [14] *If f and g are two entire functions, then for all sufficiently large values of r,*

$$M\left(\frac{1}{8}M\left(\frac{r}{2},g\right)-|g(0)|,f\right)\leq M(r,f\circ g)\leq M(M(r,g),f).$$

Lemma 2.2. [5] *If f is a meromorphic function and g is an entire fucntion, then for all sufficiently large values of r,*

$$T(r,f\circ g)\leq\{1+o(1)\}\frac{T(r,g)}{\log M(r,g)}T(M(r,g),f).$$

Lemma 2.3. [7, 15, 21] *Let f be an entire function with non zero finite l-order ρ_f^l $\left(\text{non zero finite lower } l\text{-order } \lambda_f^l\right)$. If $p-q=l-1$, then the (p,q)-order $\rho_f(p,q)$ (lower (p,q)-order $\lambda_f(p,q)$) of f will be equal to 1. If $p-q\neq l-1$, then $\rho_f(p,q)$ ($\lambda_f(p,q)$) is either zero or infinity.*

3. Main results

In this section we present the main results of the paper.

Theorem 3.1. *Let f be meromorphic and g be entire such that $\rho_g(s,t)<\lambda_f(p,q)\leq\rho_f(p,q)<\infty$, where p,q,s,t are positive integers with $p>q$ and $s>t$. If $f_m(z)=\{f(z+m):m\in\mathbb{Z}\}$ and $g_n(z)=\{g(z+n):n\in\mathbb{Z}\}$, then*

$$(i)\lim_{r\to\infty}\frac{\log^{[p-1]}T\left(\exp^{[t-1]}r,f_m\circ g_n\right)}{\log^{[p-2]}T\left(\exp^{[q-1]}r,f_m\right)}=0,\ \text{if } q\geq s$$

and

$$(ii)\lim_{r\to\infty}\frac{\log^{[p+s-q-2]}T\left(\exp^{[t-1]}r,f_m\circ g_n\right)}{\log^{[p-2]}T\left(\exp^{[q-1]}r,f_m\right)}=0,\ \text{if } q<s.$$

Proof. From the definition of the (p,q)-lower order of f_m, we have for any positive ε and for all sufficiently large values of r that

$$\log^{[p-1]}T\left(\exp^{[q-1]}r,f_m\right)\geq(\lambda_{f_m}(p,q)-\varepsilon)\log^{[q]}\left(\exp^{[q-1]}r\right)$$

$$or,\log^{[p-1]}T\left(\exp^{[q-1]}r,f_m\right)\geq(\lambda_{f_m}(p,q)-\varepsilon)\log r$$

$$or,\log^{[p-2]}T\left(\exp^{[q-1]}r,f_m\right)\geq r^{\left(\lambda_{f_m}(p,q)-\varepsilon\right)}$$

$$or,\log^{[p-2]}T\left(\exp^{[q-1]}r,f_m\right)\geq r^{\left(\lambda_f(p,q)-\varepsilon\right)},\ \text{by } \lambda_{f_m}=\lambda_f. \tag{3.1}$$

□

Let $f_m \circ g_n = h_t$, where h is a meromorphic function and $t \in \mathbb{Z}$. So, h_t can be expressed as $h_t(z) = \{h(z+t) : t \in \mathbb{Z}\}$ .Then,

$$T(r, h_t) = tT(r, h) + \sum_t \left(e_t + e_t'\right),$$

where $e_t \to 0$, $e_t' \to 0$ as $r \to \infty$.

$$i.e, T(r, f_m \circ g_n) = tT(r, f \circ g) + \sum_t \left(e_t + e_t'\right). \tag{3.2}$$

Using Lemma 2.2 and the inequality $T(r,g) \le \log^+ M(r,g)$, we get from (3.2) for all sufficiently large values of r that

$$\begin{aligned}
\log^{[p-1]} T(r, f_m \circ g_n) &\le \log^{[p-1]} T(M(r,g), f) + O(1) \\
or, \log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) &\le \log^{[p-1]} T\left(M\left(\exp^{[t-1]} r, g\right), f\right) + O(1)
\end{aligned}$$

$$\begin{aligned}
& or, \log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) \\
\le\ & (\rho_f(p,q) + \varepsilon) \log^{[q]} M\left(\exp^{[t-1]} r, g\right) + O(1).
\end{aligned} \tag{3.3}$$

Now the following two cases may arise.

Case − I : Let $q \ge s$.

Then, from (3.3) we get for all sufficiently large values of r that

$$\begin{aligned}
& \log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) \\
\le\ & (\rho_f(p,q) + \varepsilon) \log^{[s-1]} M\left(\exp^{[t-1]} r, g\right) + O(1).
\end{aligned} \tag{3.4}$$

Again for all sufficiently large values of r,

$$\begin{aligned}
\log^{[s]} M\left(\exp^{[t-1]} r, g\right) &\le (\rho_g(s,t) + \varepsilon) \log^{[t]} \left(\exp^{[t-1]} r\right) \\
or, \log^{[s]} M\left(\exp^{[t-1]} r, g\right) &\le (\rho_g(s,t) + \varepsilon) \log r
\end{aligned}$$

$$or, \log^{[s-1]} M\left(\exp^{[t-1]} r, g\right) \le r^{(\rho_g(s,t)+\varepsilon)}. \tag{3.5}$$

From (3.4) and (3.5) it follows for all sufficiently large values of r,

$$\log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) \le (\rho_f(p,q) + \varepsilon) r^{(\rho_g(s,t)+\varepsilon)} + O(1). \tag{3.6}$$

Now, from (3.1) and (3.6) we have for all sufficiently large values of r that

$$\begin{aligned}
\frac{\log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T\left(\exp^{[q-1]} r, f_m\right)} &\le \frac{(\rho_f(p,q) + \varepsilon) r^{(\rho_g(s,t)+\varepsilon)} + O(1)}{r^{\left(\lambda_f(p,q) - \varepsilon\right)}} \\
\le (\rho_f(p,q) + \varepsilon) \cdot \frac{1}{r^{2\left(\frac{\lambda_f(p,q) - \rho_g(s,t)}{2} - \varepsilon\right)}} &+ \frac{O(1)}{r^{\left(\lambda_f(p,q) - \varepsilon\right)}}.
\end{aligned} \tag{3.7}$$

Since $\rho_g(s,t) < \lambda_f(p,q)$, we can choose $\varepsilon(>0)$ in such a way that

$$\begin{aligned} 0 &< \varepsilon < \frac{\lambda_f(p,q) - \rho_g(s,t)}{2} \\ i.e, 0 &< (\rho_g(s,t) + \varepsilon) < \lambda_f(p,q) - \varepsilon. \end{aligned} \tag{3.8}$$

In view of (3.8), we get from (3.7) that

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T\left(\exp^{[q-1]} r, f_m\right)} &= 0 \\ or, \lim_{r\to\infty} \frac{\log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T\left(\exp^{[q-1]} r, f_m\right)} &= 0. \end{aligned}$$

This proves the first part of the theorem.
Case – II : Let $q < s$.
Then, from (3.3) we have for all sufficiently large values of r that

$$\begin{aligned} &\log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) \\ \leq\ & (\rho_f(p,q) + \varepsilon) \exp^{[s-q]} \log^{[s]} M\left(\exp^{[t-1]} r, g\right) + O(1). \end{aligned} \tag{3.9}$$

Now, for all sufficiently large values of r,

$$\log^{[s]} M\left(\exp^{[t-1]} r, g\right) \leq (\rho_g(s,t) + \varepsilon) \log^{[t]} \exp^{[t-1]} r$$

$$or, \log^{[s]} M\left(\exp^{[t-1]} r, g\right) \leq (\rho_g(s,t) + \varepsilon) \log r$$

$$or, \exp^{[s-q]} \log^{[s]} M\left(\exp^{[t-1]} r, g\right) \leq \exp^{[s-q]} \log r^{(\rho_g(s,t)+\varepsilon)}$$

$$or, \exp^{[s-q]} \log^{[s]} M\left(\exp^{[t-1]} r, g\right) \leq \exp^{[s-q-1]} r^{(\rho_g(s,t)+\varepsilon)}. \tag{3.10}$$

Now, from (3.9) and (3.10) we get for all sufficiently large values of r that

$$\log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) \leq (\rho_f(p,q) + \varepsilon) \exp^{[s-q-1]} r^{(\rho_g(s,t)+\varepsilon)} + O(1)$$

$$or, \log^{[p]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) \leq \exp^{[s-q-2]} r^{(\rho_g(s,t)+\varepsilon)} + O(1)$$

$$or, \log^{[p+s-q-2]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) \leq r^{(\rho_g(s,t)+\varepsilon)} + O(1). \tag{3.11}$$

Using (3.1) and (3.11), we get for all sufficiently large values of r that

$$\begin{aligned}\frac{\log^{[p+s-q-2]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T\left(\exp^{[q-1]} r, f_m\right)} &\leq \frac{r^{(\rho_g(s,t)+\varepsilon)} + O(1)}{r^{\lambda_f(p,q)-\varepsilon}} \\ &\leq \frac{1}{r^{2\left(\frac{\lambda_f(p,q)-\rho_g(s,t)}{2}-\varepsilon\right)}} + \frac{O(1)}{r^{\lambda_f(p,q)-\varepsilon}}.\end{aligned}$$

In view of (3.8), we get from above that

$$\begin{aligned}\limsup_{r\to\infty} \frac{\log^{[p+s-q-2]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T\left(\exp^{[q-1]} r, f_m\right)} &= 0 \\ or, \lim_{r\to\infty} \frac{\log^{[p+s-q-2]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T\left(\exp^{[q-1]} r, f_m\right)} &= 0.\end{aligned}$$

This completes the proof of the second part.

Remark 3.1. The condition $\rho_g(s,t) < \lambda_f(p,q)$ in Theorem 3.1 is essential as we see in the following example.

Example 3.1. *Let $f = g = \exp z$ and $p = s = 2$, $q = t = 1$. Then, $\rho_g(s,t) = \lambda_f(p,q) = \rho_f(p,q) = 1$. Now, $f_m(z) = \exp(z+m)$, $g_n(z) = \exp(z+n)$ and $(f_m \circ g_n)(z) = \exp(\exp(z+n)+m)$. Then,*

$$T(r, f_m \circ g_n) \geq T\left(\frac{r}{2}, f_m \circ g_n\right) \sim \frac{\exp\left(\frac{r+n}{2}\right)}{\left(2\pi^3 \frac{r+n}{2}\right)^{\frac{1}{2}}} + O(1)\,(r \to \infty)$$

and $T(r, f_m) \leq \log M(r, f_m) = r + m$.

Therefore,

$$\begin{aligned}\lim_{r\to\infty} \frac{\log^{[p+s-q-2]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T\left(\exp^{[q-1]} r, f_m\right)} &= \lim_{r\to\infty} \frac{\log T(r, f_m \circ g_n)}{T(r, f_m)} \\ &\geq \lim_{r\to\infty} \frac{\frac{r+n}{2} - \frac{1}{2}\log(r+n) + O(1)}{r+m} \\ &= \frac{1}{2} \neq 0, \text{ which is contrary to Theorem 3.1.}\end{aligned}$$

Theorem 3.2. *Let f be meromorphic and g be entire such that $\lambda_g(s,t) < \lambda_f(p,q) \leq \rho_f(p,q) < \infty$, where p,q,s,t are positive integers with $p > q, s > t$. If $f_m(z) = \{f(z+m) : m \in \mathbb{Z}\}$ and $g_n(z) = \{g(z+n) : n \in \mathbb{Z}\}$, then*

$$(i) \liminf_{r\to\infty} \frac{\log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T(\exp^{[q-1]} r, f_m)} = 0, \ \mathit{if}\, q \geq s$$

and

$$(ii)\ \liminf_{r\to\infty}\frac{\log^{[p+s-q-2]}T\left(\exp^{[t-1]}r,f_m\circ g_n\right)}{\log^{[p-2]}T(\exp^{[q-1]}r,f_m)}=0,\ \textit{if}\ q<s.$$

Proof. For any positive ε and for a sequence of values of r tending to infinity we have that

$$\log^{[s]}M\left(\exp^{[t-1]}r,g_n\right)\leqslant(\lambda_g(s,t)+\varepsilon)\log^{[t]}\exp^{[t-1]}r$$

$$\textit{or, }\log^{[s]}M\left(\exp^{[t-1]}r,g_n\right)\leqslant(\lambda_g(s,t)+\varepsilon)\log r$$

$$\textit{or, }\log^{[s]}M\left(\exp^{[t-1]}r,g_n\right)\leqslant\log r^{\lambda_g(s,t)+\varepsilon}$$

$$\textit{or, }\log^{[s-1]}M\left(\exp^{[t-1]}r,g_n\right)\leq r^{\lambda_g(s,t)+\varepsilon}. \tag{3.12}$$

□

Case I. If $q\geqslant s$, then from (3.4) and (3.12) we obtain, for a sequence of values of r tending to infinity, that

$$\log^{[p-1]}T\left(\exp^{[t-1]}r,f_m\circ g_n\right)\leqslant(\rho_f(p,q)+\varepsilon)\,r^{\lambda_g(s,t)+\varepsilon}+O(1). \tag{3.13}$$

Combining (3.1) and (3.13) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]}T\left(\exp^{[t-1]}r,f_m\circ g_n\right)}{\log^{[p-2]}T\left(\exp^{[q-1]}r,f_m\right)}\leq\frac{(\rho_f(p,q)+\varepsilon)\,r^{\lambda_g(s,t)+\varepsilon}+O(1)}{r^{(\lambda_f(p,q)-\varepsilon)}}$$

$$\leq(\rho_f(p,q)+\varepsilon)\,.\,\frac{1}{r^{2\left(\frac{\lambda_f(p,q)\,-\,\lambda_g(s,t)}{2}-\varepsilon\right)}}+\frac{O(1)}{r^{\lambda_f(p,q)-\varepsilon}}. \tag{3.14}$$

Since $\lambda_g(s,t)<\lambda_f(p,q)$, we can choose $\varepsilon\,(>0)$ in such a way that

$$\begin{aligned}0\quad&<\quad\varepsilon<\frac{\lambda_f(p,q)-\lambda_g(s,t)}{2}\\ i.e,0\quad&<\quad(\lambda_g(s,t)+\varepsilon)<\lambda_f(p,q)-\varepsilon.\end{aligned} \tag{3.15}$$

In view of (3.15), it follows from (3.14) that

$$\liminf_{r\to\infty}\frac{\log^{[p-1]}T\left(\exp^{[t-1]}r,f_m\circ g_n\right)}{\log^{[p-2]}T\left(\exp^{[q-1]}r,f_m\right)}=0.$$

This proves the first part of the theorem.

Case II. If $q < s$, then from (3.9) we have for a sequence of values of r tending to infinity that

$$\begin{aligned} & \log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) \\ \leq \quad & (\rho_f(p,q)+\varepsilon) \exp^{[s-q]} \log^{[s]} M\left(\exp^{[t-1]} r, g\right) + O(1). \end{aligned} \tag{3.16}$$

Again for a sequence of values of r tending to infinity,

$$\begin{aligned} \log^{[s]} M\left(\exp^{[t-1]} r, g\right) &\leqslant (\lambda_g(s,t)+\varepsilon) \log^{[t]} \exp^{[t-1]} r \\ or, \log^{[s]} M\left(\exp^{[t-1]} r, g\right) &\leqslant (\lambda_g(s,t)+\varepsilon) \log r \\ or, \log^{[s]} M\left(\exp^{[t-1]} r, g\right) &\leqslant \log r^{(\lambda_g(s,t)+\varepsilon)} \\ or, \exp^{[s-q]} \log^{[s]} M\left(\exp^{[t-1]} r, g\right) &\leqslant \exp^{[s-q]} \log r^{(\lambda_g(s,t)+\varepsilon)} \\ or, \exp^{[s-q]} \log^{[s]} M\left(\exp^{[t-1]} r, g\right) &\leqslant \exp^{[s-q-1]} r^{(\lambda_g(s,t)+\varepsilon)}. \end{aligned} \tag{3.17}$$

Now from (3.16) and (3.17) we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[p-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) &\leqslant (\rho_f(p,q)+\varepsilon) \exp^{[s-q-1]} r^{(\lambda_g(s,t)+\varepsilon)} + O(1) \\ or, \ \log^{[p]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) &\leqslant \exp^{[s-q-2]} r^{(\lambda_g(s,t)+\varepsilon)} + O(1) \end{aligned}$$

$$or, \log^{[p+s-q-2]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right) \leq r^{(\lambda_g(s,t)+\varepsilon)} + O(1). \tag{3.18}$$

Combining (3.1) and (3.18) we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \frac{\log^{[p+s-q-2]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T(\exp^{[q-1]} r, f_m)} &\leq \frac{r^{(\lambda_g(s,t)+\varepsilon)} + O(1)}{r^{\lambda_f(p,q)-\varepsilon}} \\ &\leq \frac{1}{r^{2\left(\frac{\lambda_f(p,q)\,-\,\lambda_g(s,t)}{2} - \varepsilon\right)}} + \frac{O(1)}{r^{\lambda_f(p,q)-\varepsilon}}. \end{aligned} \tag{3.19}$$

Now, in view of (3.15), it follows from (3.19) that

$$\liminf_{r\to\infty} \frac{\log^{[p+s-q-2]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T(\exp^{[q-1]} r, f_m)} = 0.$$

This establishes the second part of the theorem.

Remark 3.2. The condition $\lambda_g(s,t) < \lambda_f(p,q)$ in Theorem 3.2 is necessary which is evident from the following example:

Example 3.2. *Let* $f = g = \exp z$ *and* $p = s = 2$, $q = t = 1$. *Then,* $\lambda_g(s,t) = \lambda_f(p,q) = \rho_f(p,q) = 1$. *Now,* $f_m(z) = \exp(z+m)$, $g_n(z) = \exp(z+n)$ *and* $(f_m o g_n)(z) = \exp(\exp(z+n)+m)$. *Then,*

$$T(r, f_m \circ g_n) \geq T\left(\frac{r}{2}, f_m \circ g_n\right) \sim \frac{\exp\left(\frac{r+n}{2}\right)}{\left(2\pi^3 \frac{r+n}{2}\right)^{\frac{1}{2}}} + O(1)\,(r \to \infty)$$

and $T(r, f_m) \leq \log M(r, f_m) = r + m$.

Therefore,

$$\begin{aligned}
\liminf_{r\to\infty} \frac{\log^{[p+s-q-2]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-2]} T(\exp^{[q-1]} r, f_m)} &= \liminf_{r\to\infty} \frac{\log T(r, f_m \circ g_n)}{T(r, f_m)} \\
&\geq \liminf_{r\to\infty} \frac{\frac{r+n}{2} - \frac{1}{2}\log(r+n) + O(1)}{r+m} \\
&= \frac{1}{2} \neq 0, \text{ which is contrary to Theorem 3.2.}
\end{aligned}$$

Theorem 3.3. *Let* f *and* g *are any two entire functions with*
(i) $\lambda_{f\circ g}(a,b) > 0$
(ii) $0 < \rho_g < \infty$ *and*
(iii) $\lambda_f^l > 0$,
where l, a, b *are all positive integers with* $l > 1$, *and* $a > b$. *Also, let* $0 < A < \rho_g$. *If* $f_m(z) = \{f(z+m) : m \in \mathbb{Z}\}$ *and* $g_n(z) = \{g(z+n) : n \in \mathbb{Z}\}$, *then for any two positive integers* s, t *such that* $s - t = 1$ *and* $s > 2$,

$$\limsup_{r\to\infty} \frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_m \circ g_n) \log^{[l-1]} T(r, f_m \circ g_n)}{\left\{\log^{[s-1]} T(\exp^{[t]} r^A, g_n)\right\}\left\{\log^{[s-1]} T(\exp^{[t-1]} r, g_n)\right\}} = \infty.$$

Proof. Let us choose $0 < \varepsilon < \min\left\{\lambda_f^l, \lambda_{f\circ g}(a,b), \rho_g\right\}$. □

Using the inequality $T\left(\frac{r}{2}, fog\right) \leq \log^+ M\left(\frac{r}{2}, fog\right) \leq 3T(r, fog)$ and Lemma 2.1 we get for all sufficiently large values of r that

$$T(r, fog) \geq \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4}, g\right) - |g(0)|, f\right). \tag{3.20}$$

From (3.2) and (3.20) for all sufficiently large values of r, we get

$$\begin{aligned}
\log^{[l-1]} T(r, f_m \circ g_n) &\geq \log^{[l]} M\left(\frac{1}{8}M\left(\frac{r}{4}, g\right) - |g(0)|, f\right) + O(1) \\
or, \log^{[l-1]} T(r, f_m \circ g_n) &\geq (\lambda_f^l - \varepsilon)\log M\left(\frac{r}{4}, g\right) + O(1) \\
or,\ \log^{[l-1]} T(r, f_m \circ g_n) &\geq (\lambda_f^l - \varepsilon)\left(\frac{r}{4}\right)^{\rho_g - \varepsilon} + O(1). \qquad (3.21)
\end{aligned}$$

Now, in view of (1.6), for the family g_n we may write for all sufficiently large values of r that

$$\log^{[s-1]} T(\exp^{[t]} r^A, g_n) = \log^{[s-1]} T(\exp^{[t]} r^A, g) + O(1).$$

Since $\rho_g^2 = \rho_g < \infty$ and $s - t = 1$, by Lemma 2.3 we get $\rho_g(s,t) = 1$ and so, it follows from above for all sufficiently large values of r that

$$\begin{aligned} &\log^{[s-1]} T(\exp^{[t]} r^A, g_n) \le (1+\varepsilon)\log^{[t]} \exp^{[t]} r^A + O(1) \\ &or, \log^{[s-1]} T(\exp^{[t]} r^A, g_n) \le (1+\varepsilon) r^A + O(1). \end{aligned} \tag{3.22}$$

Now, from (3.21) and (3.22) we get for all sufficiently large values of r that

$$\frac{\log^{[l-1]} T(r, f_m \circ g_n)}{\log^{[s-1]} T(\exp^{[t]} r^A, g_n)} \ge \frac{(\lambda_f^l - \varepsilon)\left(\frac{r}{4}\right)^{\rho_g - \varepsilon} + O(1)}{(1+\varepsilon) r^A + O(1)}. \tag{3.23}$$

From (3.2) for all sufficiently large values of r,

$$\begin{aligned} \log^{[a-1]} T(r, f_m \circ g_n) &= \log^{[a-1]} T(r, f \circ g) + O(1) \\ or, \log^{[a-1]} T(r, f_m \circ g_n) &\ge (\lambda_{f\circ g}(a,b) - \varepsilon)\log^{[b]} r + O(1) \\ or, \log^{[a-1]} T(\exp^{[b-1]} r, f_m \circ g_n) &\ge (\lambda_{f\circ g}(a,b) - \varepsilon)\log r + O(1). \end{aligned} \tag{3.24}$$

Again in view of (1.6) for all sufficiently large values of r we obtain

$$\begin{aligned} \log^{[s-1]} T(r, g_n) &= \log^{[s-1]} T(r, g) + O(1) \\ or, \log^{[s-1]} T(\exp^{[t-1]} r, g_n) &= \log^{[s-1]} T(\exp^{[t-1]} r, g) + O(1). \end{aligned}$$

Since $\rho_g(s,t) = 1$, we get from above that

$$\log^{[s-1]} T(\exp^{[t-1]} r, g_n) \le (1+\varepsilon)\log r + O(1). \tag{3.25}$$

Therefore, from (3.24) and (3.25) we get for all sufficiently large values of r that

$$\frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_m \circ g_n)}{\log^{[s-1]} T(\exp^{[t-1]} r, g_n)} \ge \frac{(\lambda_{f\circ g}(a,b) - \varepsilon)\log r + O(1)}{(1+\varepsilon)\log r + O(1)}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[a-1]} T(\exp^{[b-1]} r, f_m \circ g_n)}{\log^{[s-1]} T(\exp^{[t-1]} r, g_n)} \ge \lambda_{f\circ g}(a,b). \tag{3.26}$$

As $0 < A < \rho_g$, we can choose $\varepsilon(>0)$ in such a way that

$$0 < \varepsilon < \rho_g - A. \tag{3.27}$$

Now, from (3.27) and (3.23) it follows that

$$\limsup_{r\to\infty} \frac{\log^{[l-1]} T(r, f_m \circ g_n)}{\log^{[s-1]} T(\exp^{[t]} r^A, g_n)} = \infty. \tag{3.28}$$

Thus, the theorem follows from (3.26) and (3.28).

Corollary 3.1. *Under the same conditions of Theorem 3.3 when $s=2$,*

$$\limsup_{r\to\infty}\frac{\log^{[a-1]}T(\exp^{[b-1]}r,f_m\circ g_n)\log^{[l-1]}T(r,f_m\circ g_n)}{\{\log T(\exp r^A,g_n)\}\{\log T(r,g_n)\}}=\infty.$$

Remark 3.3. The condition $\lambda_f^l>0$ in Corollary 3.1 is essential as we see in the following example:

Example 3.3. *Let $f=z$ and $g=\exp z$. Also, let $l=a=2$ and $b=1$. Then, $\lambda_f^l=\lambda_f^2=\lambda_f=0$ and $\rho_g=\lambda_{f\circ g}=1$. Now $f_m(z)=z+m$, $g_n(z)=\exp(z+n)$ and $(f_m o g_n)(z)=\exp(z+n)+m$.*

Then,

$$\log T(r,f_m\circ g_n)\le\log^{[2]}M(r,f_m\circ g_n)=\log(r+n)+O(1)$$

and

$$\log T(r,g_n)\ge\log^{[2]}M(\frac{r}{2},g_n)+O(1)=\log(r+n)+O(1).$$

Hence,

$$\begin{aligned}\limsup_{r\to\infty}\frac{(\log T(r,f_m\circ g_n))^2}{\log T(\exp r^A,g_n)\log T(r,g_n)}&\le\limsup_{r\to\infty}\frac{\log(r+n)+O(1)}{\log(\exp r^A+n)+O(1)}\\&=\limsup_{r\to\infty}\frac{\left(1+\frac{n}{\exp r^A}\right)}{Ar^A\left(1+\frac{n}{r}\right).}\\&=0,\ \text{ which is contrary to Corollary 3.1.}\end{aligned}$$

Theorem 3.4. *Let f be an entire function such that $0<\lambda_f(p,q)\le\rho_f(p,q)<\infty$. Also, let g be an entire function with $\rho_g^l>0$. If $f_m(z)=\{f(z+m):m\in\mathbb{Z}\}$ and $g_n(z)=\{g(z+n):n\in\mathbb{Z}\}$, then*

$$(i)\ \limsup_{r\to\infty}\frac{\log^{[p-1]}T(r,f_m\circ g_n)}{\log^{[p-1]}T(\exp^{[l-1]}r^\mu,f_m)}=\infty,\ \text{if } q=l-1$$

$$(ii)\ \limsup_{r\to\infty}\frac{\log^{[p-1]}T(r,f_m\circ g_n)}{\log^{[p-1]}T(\exp^{[l-1]}r^\mu,f_m)}\ge\frac{\rho_g^l\lambda_f(p,q)}{\mu\rho_f(p,q)},\ \text{if } q=l$$

and

$$(iii)\ \limsup_{r\to\infty}\frac{\log^{[p-1]}T(r,f_m\circ g_n)}{\log^{[p-1]}T(\exp^{[l-1]}r^\mu,f_m)}\ge\frac{\lambda_f(p,q)}{\rho_f(p,q)},\ \text{if } q>l,$$

where $\mu<\rho_g^l$ and l,p,q are positive integers with $l>1$, $p>q$.

Proof. From (3.2) and (3.20) for all sufficiently large values of r,

$$\log^{[p-1]}T(r,f_m\circ g_n)\ge\log^{[p]}M\left(\frac{1}{8}M\left(\frac{r}{4},g\right)-|g(0)|,f\right)+O(1)$$

$$or, \log^{[p-1]} T(r, f_m \circ g_n) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q]} M\left(\frac{r}{4}, g\right) + O(1)$$

$$or, \log^{[p-1]} T(r, f_m \circ g_n) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q-l+1]} \log^{[l-1]} M\left(\frac{r}{4}, g\right) + O(1)$$

$$or, \log^{[p-1]} T(r, f_m \circ g_n) \geq (\lambda_f(p,q) - \varepsilon) \log^{[q-l+1]} \left(\frac{r}{4}\right)^{\left(\rho_g^l - \varepsilon\right)} + O(1). \quad (3.29)$$

□

Again from the definition of $\rho_{f_m}(p,q)$, it follows for all sufficiently large values of r that

$$\log^{[p-1]} T(\exp^{[l-1]} r^{\mu}, f_m) \leq (\rho_{f_m}(p,q) + \varepsilon) \log^{[q]} \exp^{[l-1]} r^{\mu}$$

$$\begin{aligned} & or, \log^{[p-1]} T(\exp^{[l-1]} r^{\mu}, f_m) \\ \leq \quad & (\rho_f(p,q) + \varepsilon) \log^{[q-l+1]} r^{\mu}, \text{by } \rho_{f_m} = \rho_f. \end{aligned} \quad (3.30)$$

Thus, from (3.29) and (3.30) we have for all sufficiently large values of r that

$$\frac{\log^{[p-1]} T(r, f_m \circ g_n)}{\log^{[p-1]} T(\exp^{[l-1]} r^{\mu}, f)} \geq \frac{(\lambda_f(p,q) - \varepsilon) \log^{[q-l+1]} \left(\frac{r}{4}\right)^{\left(\rho_g^l - \varepsilon\right)} + O(1)}{(\rho_f(p,q) + \varepsilon) \log^{[q-l+1]} r^{\mu}}. \quad (3.31)$$

Since $\mu < \rho_g^l$, the theorem follows from (3.31).

Remark 3.4. The condition $\mu < \rho_g^l$ in Theorem 3.4 is essential as we see in the following example:

Example 3.4. *Let $f = g = \exp z$ and $p = s = 2, q = t = 1, l = 2$. Also, let $\mu = 1$. Then, $\lambda_f(p,q) = \rho_f(p,q) = 1$ and $\rho_g^l = \rho_g^2 = \rho_g = 1$. Now $f_m(z) = \exp(z+m)$, $g_n(z) = \exp(z+n)$ and $(f_m \circ g_n)(z) = \exp(\exp(z+n)+m)$. Then, $T(r, f_m o g_n) \sim \frac{\exp(r+n)}{(2\pi^3(r+n))^{\frac{1}{2}}}$ $(r \to \infty)$ and $T(r, f_m) = \frac{r+m}{\pi} + O(1)$.*

Therefore,

$$\begin{aligned} & \limsup_{r \to \infty} \frac{\log^{[p-1]} T(r, f_m \circ g_n)}{\log^{[p-1]} T(\exp^{[l-1]} r^{\mu}, f_m)} \\ = \quad & \limsup_{r \to \infty} \frac{\log T(r, f_m \circ g_n)}{\log T(\exp r, f_m)} \\ \leq \quad & \limsup_{r \to \infty} \frac{(r+n) - \frac{1}{2}\log(r+n) + O(1)}{\frac{r+m}{\pi} + O(1)} \\ = \quad & 1 \neq \infty, \text{ which is contrary to Theorem 3.4.} \end{aligned}$$

Theorem 3.5. *Let f and g be any two entire functions such that $0 < \lambda_f(p,q) \leq \rho_f(p,q) < \infty$ and $\rho_g(s,t) < \infty$, where p,q,s,t are positive integers with $p > q, s > t$. If $f_m(z) = \{f(z+m) : m \in \mathbb{Z}\}$ and $g_n(z) = \{g(z+n) : n \in \mathbb{Z}\}$, then*

$$(i)\ \limsup_{r\to\infty} \frac{\log^{[p]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, f_m)} \leqslant \frac{\rho_g(s,t)}{\lambda_f(p,q)},\ \textit{if } q \geq s$$

and

$$(ii)\ \limsup_{r\to\infty} \frac{\log^{[p+s-q-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, f_m)} \leqslant \frac{\rho_g(s,t)}{\lambda_f(p,q)},\ \textit{if } q < s.$$

Proof. We have for all sufficiently large values of r,

$$\log^{[p-1]} T(\exp^{[q-1]} r, f_m) \geq (\lambda_{f_m}(p,q) - \varepsilon) \log^{[q]} \exp^{[q-1]} r$$

$$or, \log^{[p-1]} T(\exp^{[q-1]} r, f_m) \geq (\lambda_f(p,q) - \varepsilon) \log r, \text{by } \lambda_{f_m} = \lambda_f. \tag{3.32}$$

□

Case I. If $q \geqslant s$, then from (3.6) and (3.32) we get for all sufficiently large values of r that

$$\frac{\log^{[p]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, f_m)} \leqslant \frac{(\rho_g(s,t) + \varepsilon) \log r + O(1)}{(\lambda_f(p,q) - \varepsilon) \log r}$$

$$or, \limsup_{r\to\infty} \frac{\log^{[p]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, f_m)} \leqslant \frac{\rho_g(s,t) + \varepsilon}{\lambda_f(p,q) - \varepsilon}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from the above that

$$\limsup_{r\to\infty} \frac{\log^{[p]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, f_m)} \leqslant \frac{\rho_g(s,t)}{\lambda_f(p,q)}.$$

This proves the first part of the theorem.

Case II. If $q < s$, then from (3.11) and (3.32) we obtain for all sufficiently large values of r that

$$\frac{\log^{[p+s-q-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, f_m)} \leqslant \frac{(\rho_g(s,t) + \varepsilon) \log r + O(1)}{(\lambda_f(p,q) - \varepsilon) \log r}$$

$$or, \limsup_{r\to\infty} \frac{\log^{[p+s-q-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, f_m)} \leqslant \frac{\rho_g(s,t) + \varepsilon}{\lambda_f(p,q) - \varepsilon}.$$

As $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[p+s-q-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, f_m)} \leqslant \frac{\rho_g(s,t)}{\lambda_f(p,q)}.$$

This completes proof of the second part.

Remark 3.5. The condition $\rho_g(s,t) < \infty$ in Theorem 3.5 is necessary which is evident from the following example:

Example 3.5. *Let* $f = \exp z$, $g = \exp^{[2]} z$ *and* $p = s = 2$, $q = t = 1$. *Then* $\lambda_f(p,q) = \rho_f(p,q) = 1$ *and* $\rho_g(s,t) = \infty$. *Now* $f_m(z) = \exp(z+m)$, $g_n(z) = \exp^{[2]}(z+n)$ *and* $(f_m \circ g_n)(z) = \exp\left(\exp^{[2]}(z+n)+m\right)$.

Therefore,

$$\begin{aligned}
&\log^{[2]} T(r, f_m \circ g_n) \geq \log^{[3]} M(\frac{r}{2}, f_m \circ g_n) + O(1)\\
or,\ &\log^{[2]} T(r, f_m \circ g_n) \geq \log^{[3]} \exp\left(\exp^{[2]}\left(\frac{r}{2}+n\right)+m\right) + O(1)\\
or,\ &\log^{[2]} T(r, f_m \circ g_n) \geq \left(\frac{r}{2}+n\right) + O(1)
\end{aligned}$$

and

$$\log T(r, f_m) \leq \log^{[2]} M(r, f_m) = \log(r+m) + O(1).$$

So,

$$\begin{aligned}
&\limsup_{r\to\infty} \frac{\log^{[p+s-q-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, f_m)}\\
= \quad &\limsup_{r\to\infty} \frac{\log^{[2]} T(r, f_m \circ g_n)}{\log T(r, f_m)}\\
\geq \quad &\limsup_{r\to\infty} \frac{\left(\frac{r}{2}+n\right)+O(1)}{\log(r+m)+O(1)}\\
or, &\limsup_{r\to\infty} \frac{\log^{[p+s-q-1]} T\left(\exp^{[t-1]} r, f_m \circ g_n\right)}{\log^{[p-1]} T(\exp^{[q-1]} r, f_m)} = \infty,
\end{aligned}$$

which is contrary to Theorem 3.5.

4. Conclusions

The traditional growth indicators, such as the iterated p-order [25, 32], and the (p,q)-order [21], have the disadvantage of not adequately covering the arbitrary growth of entire and meromorphic functions, see [12, 13] for details. To address this limitation, researchers such as Shen et al. [35], Bouabdelli and Belaïdi [9], and Long et al. [26] introduced the $[p,q]_{,\varphi}$-order to measure arbitrary growth. Also, Chyzhykov et al. [12, 13] as well as Filevych et al. [16, 19, 20] developed a more general scale known as the φ-order, which provides new insights into classifying entire and meromorphic functions based on their growth behaviours.

Recently, Heittokangas et al. [18] refined the concept of φ-order by adding a subadditivity condition on φ, creating a more versatile tool. For more details, refer to [18]. This concept has since been applied to examine the growth properties of solutions to complex differential equations, as explored by Khedim et al. [24], and Belaïdi [1] as well as Kara et al. [22, 23].

Parallely, Mulyava et al. [27], and Sheremeta [34, 36] researched the generalised (α,β)-order, highlighting its role in characterising growth. Building on this, Belaïdi and Biswas [2, 3] introduced the (α,β,γ)-order, showing its relevance in various fields. These developments provide important tools for analysing the growth of entire and meromorphic functions.

These innovative growth indicators can also be applied within a broader framework to analyze the growth behavior of integer-translated entire and meromorphic functions. By advancing our understanding of function growth, these approaches have the potential to significantly enrich the theoretical foundation of this field, opening up new avenues for future research and practical applications.

Understanding integer translations in this context is essential, as these translations often uncover more subtle changes in function behavior that might otherwise go unnoticed. Exploring this area has wide-ranging applications, including improving the comparative growth of entire and meromorphic functions. It also provides insights relevant to fields such as differential equations and dynamic systems. This knowledge can be particularly useful in mathematical physics, where function growth is critical for modeling systems with complex-valued solutions.

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6. Conflicts of interest

The authors declare no conflict of interest.

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