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GLOBAL DYNAMICS AND BIFURCATION OF A HIGHER ORDER DIFFERENCE EQUATION

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The work is dedicated to the 65th birthday of Prof. Mehmed Nurkanović

ABSTRACT. This study is devoted to the dynamical analysis of the following higher order difference equation

$$x_{n+1} = px_n + \frac{q}{rx_{n-k}^2},\ k \in \{1, 2, ...\},$$

where p, q, r and the initial conditions are positive real numbers. In particular, we discuss the existence of periodic solutions of the difference equation. We also handle the boundedness, local and global stability of solutions of the difference equation. Moreover, we study the existence of Neimark-Sacker bifurcation of solutions of the difference equation for $k = 1$ and also give an invariant curve of the difference equation. Finally, we provide some numerical examples to support our results and present some open problems for future works.

1. INTRODUCTION

Higher order difference equations and their systems have garnered increased interest from researchers over the past few decades for several reasons. One principal reason is that they provide a natural means of describing many discrete mathematical models utilized across a broad range of fields such as biology, physics, engineering, economics, and population dynamics. Consequently, these models have received considerable attention and analysis in these areas. We anticipate that the study of difference equations will surge in popularity as researchers uncover more intriguing and innovative applications. Despite their apparent simplicity, these equations still present a challenging task, and comprehending the behaviors of the solutions they offer requires rigorous investigation. However, it is these challenges that make difference equations an enticing subject for future research in the field, fueling interest among scholars. Researchers in this field have extensively studied

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the dynamical properties of various difference equations, including their boundedness, stability, periodicity, and oscillations. However, only a few researchers have investigated the bifurcation analysis of the equilibrium points, which can provide insights into the qualitative behavior of the solutions near these critical points. This gap in research presents an interesting opportunity to explore and further extend our understanding of the complex behavior of difference equations, as it is an area ripe for new insightful discoveries.

In [27], Ouyang et al. discussed a kind of Bobwhite quail population model

$$x_{n+1} = A + Bx_n + \frac{x_n}{x_{n-1}x_{n-2}},$$

where $n \geq 1$, the parameters and initial values are positive parabolic fuzzy numbers. They especially argued the conditional stability of this model and also the existence, boundedness and persistence of its unique positive fuzzy solution.

In [31], Taşdemir handled the dynamics of the following difference equations

$$x_{n+1} = A + B\frac{x_n}{x_{n-m}^2}, m \in \{2, 3, \cdots\}, \tag{1.1}$$

where A, B and the initial conditions are positive real numbers. The author studied the existence of bounded solutions, rate of convergence, global stability analysis and periodic solutions of the higher order difference equations.

In [33], Taşdemir et al. explored some dynamical properties of solutions of following higher order difference equations

$$x_{n+1} = A + B\frac{x_{n-m}}{x_n^2}, m = \{1, 2, ...\},$$

where A, B and initial conditions are positive real numbers. In particular, the authors dealt with the periodic solutions, bounded solutions, oscillation behaviours, stability and rate of convergence of the higher order difference equations.

In [21], Kulenovic et al. discussed the Neimark-Sacker bifurcation of the following quadratic fractional difference equation

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}^2 + \delta x_n}{Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n},$$

with the parameters $\beta, \gamma, \delta, B, C, D$ and the initial conditions non-negative numbers with $B + C + D > 0$ and the denominator is positive for all $n \geq 0$.

In [12], Kalabusic et al. investigated the dynamics of the following two difference equations

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1}}, x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2},$$

with the non-negative parameters $\alpha, \beta, \gamma, \delta, A, B$ and non-negative initial conditions.

In [3], Bešo et al. considered the dynamics of the second order difference equation

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2},$$

where γ, δ and the initial conditions are positive real numbers. The authors proved the boundedness, global attractivity and Neimark-Sacker bifurcation results of this difference equation.

In [10], Hassan discussed the dynamics of the following second order difference equation

$$x_{n+1} = px_n + \frac{q}{x_{n-1}^2}, \tag{1.2}$$

where $p, q \in (0,1)$. Hassan studied the periodic solutions, boundedness and stability of Equation (1.2).

There are also many papers related to difference equations (see, for example, [1,2,4,7,11,15,17–20,24–26,29,30,30–35] and references therein).

Considering the above studies, we extend Equation (1.2) to a higher order and also present many new results on boundedness, periodicity, global asymptotic stability and Neimark-Sacker bifurcation. Therefore, we handle the global dynamics of solutions of unique equilibrium point of the following higher order difference equation

$$x_{n+1} = px_n + \frac{q}{rx_{n-k}^2}, \tag{1.3}$$

where p, q, r and the initial conditions are positive real numbers and $k \in \{1, 2, ...\}$. In particular, we investigate the periodicity, boundedness, local and global stability of the solutions of the difference equation (1.3). Moreover, we study the existence of Neimark-Sacker bifurcation of solutions of the equilibrium point of the difference equation (1.3) for $k = 1$. We also give an invariant curve of the difference equation (1.3) for $k = 1$.

This paper is divided into seven sections. In the first section, we provide some brief information about the papers related to our study. We also give the important results and definitions related to the theory of difference equations. In section 2, we analyze the existence of periodic solutions of Equation (1.3) with period two. In section 3, we investigate the boundedness of solutions of Equation (1.3). In section 4, we study the local and global asymptotic stability of solutions of Equation (1.3). In section 5, we handle the existence of Neimark-Sacker bifurcation of solutions of Equation (1.3). In addition to this, we deal with the invariant curve of Equation (1.3). In section 6, we present some numerical examples to support to our results. In the last section, we summarize our results and offer some open problems for researchers.

We first transform Equation (1.3) using a change of variables as follows:

$$x_n = \sqrt[3]{\frac{q}{r}} y_n.$$

Hence, we obtain the difference equation

$$y_{n+1} = p y_n + \frac{1}{y_{n-k}^2} \tag{1.4}$$

where $p > 0$ and $k \in \{1, 2, ...\}$. Therefore, we handle the difference equation (1.4). Thus, we have the following solutions of Equation (1.4):

$$\bar{y}_1 = \frac{1}{\sqrt[3]{1-p}}, \bar{y}_2 = \frac{1}{\sqrt[3]{1-p}} e^{\frac{2\pi i}{3}}, \bar{y}_3 = \frac{1}{\sqrt[3]{1-p}} e^{\frac{4\pi i}{3}},$$

where $p \neq 1$. During this study, we consider the equilibrium point $\bar{y} = \bar{y}_1 > 0$ for $0 < p < 1$. The other equilibrium points can be handled in different studies.

We now provide a summary of the important results and definitions related to the theory of difference equations. For more information, see [5, 6, 13, 16] and the references contained therein.

Definition 1.1. *Let I be some interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Then, for every initial condition, the difference equation*

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1, \cdots, k = 1, 2, ... \tag{1.5}$$

has a unique solution $\{y_n\}_{n=-k}^{\infty}$.

Definition 1.2. *The equilibrium point $\bar{y}$ of the equation*

$$y_{n+1} = f(y_n, y_{n-k}), n = 0, 1, 2, \cdots, k = 1, 2, ...$$

is the point that satisfies the condition

$$\bar{y} = f(\bar{y}, \bar{y}).$$

Definition 1.3. *The equation*

$$y_{n+1} = q_0 y_n + q_k y_{n-k}, n = 0, 1, \cdots, \tag{1.6}$$

is called the linearized equation of Equation (1.5) about the equilibrium point $\overline{y}$ such that

$$q_0 = \frac{\partial f}{\partial y_n}, q_k = \frac{\partial f}{\partial y_{n-k}}.$$

Its characteristic equation is

$$\lambda^{k+1} - q_0 \lambda^k - q_k = 0.$$

Theorem 1.1 (See [14])**.** *Linearized Stability. Consider the difference equation*

$$y_{n+1} = q_0 y_n + q_1 y_{n-1}, n = 0, 1, \cdots.$$

- **a:** *If both roots of the equation have absolute values less than one, then the equilibrium* $\bar{y}$ *of the equation is locally asymptotically stable.*
- **b:** *If at least one of the roots of the equation has an absolute value greater than one, then* $\bar{y}$ *is unstable.*
- **c:** *Both roots of the equation have absolute values less than one if and only if* $|q_0| < 1 - q_1 < 2$*, in this case,* $\bar{y}$ *is a locally asymptotically stable.*
- **d:** *Both roots of the equation have absolute values greater than one if and only if* $|q_1| > 1$ *and* $|q_0| < |1 - q_1|$*, in this case,* $\bar{y}$ *is a repeller.*
- **e:** *One root of the equation has an absolute value greater than one while the other root has an absolute value less than one if and only if* $q_0^2 + 4q_1 > 0$ *and* $|q_0| > |1 - q_1|$*, in this case,* $\bar{y}$ *is unstable and is called saddle point.*
- **f:** *A necessary and sufficient condition for a root of the equation to have absolute value equal to one is* $|q_0| = |1 - q_1|$ *or* $q_1 = -1$ *and* $|q_0| \leq 2$*, in this case,* $\bar{y}$ *is called a nonhyperbolic point.*

Theorem 1.2 (See [22]). *Let a be a nonnegative real, b an arbitrary real and k a positive integer. The difference equation*

$$x_{n+1} - ax_n + bx_{n-k} = 0, n = 0, 1, 2, ..., \tag{1.7}$$

is asymptotically stable if and only if $|a| < \frac{k+1}{k}$*, and*

- **a:** $|a| - 1 < b < \left(a^2 + 1 - 2|a|\cos\phi\right)^{\frac{1}{2}}$*, for k odd,*
- **b:** $|b - a| < 1$ *and* $|b| < \left(a^2 + 1 - 2|a|\cos\phi\right)^{\frac{1}{2}}$*, for k even,*

where ϕ *is the solution in* $\left(0, \frac{\pi}{k+1}\right)$ *of* $\sin(k\theta)/\sin[(k+1)\theta] = 1/|a|$.

Theorem 1.3 (See [8]). *Let* $f : [a,b]^{k+1} \to [a,b]$ *be a continuous function, where k is a positive integer, and where* $[a,b]$ *is an interval of real numbers and consider the following difference equation*

$$x_{n+1} = f\left(x_n, \cdots, x_{n-k}\right), n = 0, 1, \cdots. \tag{1.8}$$

Suppose that f satisfies the following conditions:

- **i:** *For each integer i with* $1 \leq i \leq k+1$*, the function* $f(z_1, z_2, \cdots, z_{k+1})$ *is weakly monotonic in* z_i *for fixed* $z_1, z_2, \cdots, z_{i-1}, z_{i+1}, \cdots, z_{k+1}$.
- **ii:** *If* (m, M) *is a solution of the system* $m = f(m_1, m_2, \cdots, m_{k+1})$ *and* $M = f(M_1, M_2, \cdots, M_{k+1})$*, then,* $m = M$*, where for each* $i = 1, 2, \cdots, k+1$*, we set*

$$m_i = \left\{ \begin{array}{c} m \text{ if } f \text{ nondecreasing in } z_i, \\ M \text{ if } f \text{ nonincreasing in } z_i, \end{array} \right\}$$

and

$$M_i = \left\{ \begin{array}{c} M \text{ if } f \text{ nondecreasing in } z_i, \\ m \text{ if } f \text{ nonincreasing in } z_i \end{array} \right\}.$$

Then, there exists exactly one equilibrium point $\bar{x}$ of the difference equation (1.8), and every solution of (1.8) converges to $\bar{x}$.

Theorem 1.4 (See [4])**.** *Let $n \in N_{n_0}^{+}$ and $g(n,u,v)$ be a nondecreasing function in u and v for any fixed n. Suppose that, for $n \geq n_0$, the inequalities*

$$\begin{aligned} y_{n+1} &\leq g(n, y_n, y_{n-1}), \\ u_{n+1} &\geq g(n, u_n, u_{n-1}) \end{aligned}$$

hold. Then

$$\begin{aligned} y_{n_0-1} &\leq u_{n_0-1}, \\ y_{n_0} &\leq u_{n_0} \end{aligned}$$

implies that

$$y_n \leq u_n, n \geq n_0.$$

2. Existence of Periodic Solutions of Equation (1.4)

This section is devoted to whether Equation (1.4) has two periodic solutions.

Theorem 2.1. *Let $p > 0$ and $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of Equation (1.4). Then, the following are true:*

- **a:** *If k is an odd number, then Equation (1.4) has no two periodic solutions.*
- **b:** *If k is an even number and if $p \in \left(0, \frac{1}{3}\right)$, then Equation (1.4) has two periodic solutions. But, if $p \geq \frac{1}{3}$, then Equation (1.4) has no two periodic solutions.*

Proof. **a:** Suppose that Equation (1.4) has a periodic solution with period two such that

$$\cdots, a, b, a, b, a, \cdots$$

where a, $b \in \mathbb{R}^{+}$ and different from the other. Let k be an odd number. Hence, from Equation (1.4) we get

$$\begin{aligned} a &= pb + \frac{1}{a^2}, \\ b &= pa + \frac{1}{b^2}. \end{aligned}$$

Therefore, we obtain that

$$(a-b)\left(1 + p + \frac{a+b}{a^2 b^2}\right) = 0.$$

From $p > 0$, we have

$$1 + p + \frac{a+b}{a^2 b^2} > 0,$$

and thus, we get $a = b$ a contradiction.

b: Assume that Equation (1.4) has a periodic solution with period two such that

$$\cdots, a, b, a, b, a, \cdots$$

where a, $b \in \mathbb{R}^+$ and different from the other. Let k be an even number. Then, we have from Equation (1.4) that

$$a = pb + \frac{1}{b^2}, \tag{2.1}$$

$$b = pa + \frac{1}{a^2}. \tag{2.2}$$

Hence, we get the following

$$ab^2 - pb^3 - 1 = 0, \tag{2.3}$$

$$ba^2 - pa^3 - 1 = 0. \tag{2.4}$$

Subtracting (2.4) from (2.3), we obtain that

$$(a-b)\left(pa^2 + (p-1)\,ab + pb^2\right) = 0.$$

Here, if the following equation has two roots different from each other

$$pa^2 + (p-1)\,ab + pb^2 = 0,$$

then we get two periodic solutions as $a \neq b$. On the other hand, we have the equilibrium solutions of Equation (1.4) as $a = b$. Therefore, we have $p \in \left(-1, \frac{1}{3}\right)$. Via our assumption, we complete the proof. Moreover, when (2.2) is substituted into (2.1), we obtain that

$$a = p\left(pa + \frac{1}{a^2}\right) + \frac{1}{\left(pa + \frac{1}{a^2}\right)^2}. \tag{2.5}$$

Therefore, we get with $p \in \left(0, \frac{1}{3}\right)$

$$a = \frac{1}{\sqrt[3]{1-p}},$$

$$a = \sqrt[3]{\frac{(1+p)(1-2p) - \sqrt{(1-3p)(1+p)}}{2\,(p^2+p^3)}},$$

$$a = \sqrt[3]{\frac{(1+p)(1-2p) + \sqrt{(1-3p)(1+p)}}{2\,(p^2+p^3)}}.$$

Note that the other roots of Equation (2.5) are not taken into account because they are not real numbers. We also know that $a = \frac{1}{\sqrt[3]{1-p}}$ is an equilibrium solution so it is not a periodic solution. Therefore, we have two cases with periodic solutions with two periods as follows:

$$a = \sqrt[3]{\frac{(1+p)(1-2p)-\sqrt{(1-3p)(1+p)}}{2(p^2+p^3)}},$$
$$b = \sqrt[3]{\frac{(1+p)(1-2p)+\sqrt{(1-3p)(1+p)}}{2(p^2+p^3)}},$$

and

$$a = \sqrt[3]{\frac{(1+p)(1-2p)+\sqrt{(1-3p)(1+p)}}{2(p^2+p^3)}},$$
$$b = \sqrt[3]{\frac{(1+p)(1-2p)-\sqrt{(1-3p)(1+p)}}{2(p^2+p^3)}}.$$

□

3. EXISTENCE OF BOUNDED SOLUTIONS OF EQUATION (1.4)

In this section, we investigate the boundedness of solutions of Equation (1.4). Here, we reveal under what conditions the solutions of Equation (1.4) are bounded or unbounded. We now discuss the existence of bounded solutions of Equation (1.4).

Theorem 3.1. *Let $p \in (0,1)$. Then, every solution of Equation (1.4) is bounded.*

Proof. Let $p \in (0,1)$ and $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of Equation (1.4). From Equation (1.4), there exists a c_0 such that

$$y_{n+1} = py_n + \frac{1}{y_{n-k}^2} = p\left(y_n + \frac{1}{py_{n-k}^2}\right) \geq pc_0,$$

where c_0 is a positive real number and $n \geq -k$. Hence, we obtain that

$$y_{n+1} = py_n + \frac{1}{y_{n-k}^2} \leq py_n + \frac{1}{p^2c_0^2}.$$

Now we consider Theorem 1.4. Then, we handle the $\{u_n\}_{n=0}^{\infty}$, and $y_n \leq u_n, n = 0,1,2,\ldots$, and

$$u_{n+1} = pu_n + \frac{1}{p^2c_0^2}, \tag{3.1}$$

for $n \geq 1$ such that

$$u_{s+i} = y_{s+i}, i \in \{0,1\}, s \in \{0,1,2,\ldots\}, n \geq s.$$

Therefore, we obtain the solution of first order difference equation (3.1) as follows

$$u_n = p^n c_1 + \frac{p^n - 1}{(p-1)\, p^2 c_0^2},$$

where $c_1 = y_1$. Thus, we get that

$$y_{n+1} - u_{n+1} \le p\,(y_n - u_n)\,,$$

where $n > s$ and $p \in (0,1)$. Hence, we obtain that $y_n \le u_n, n > s$. Moreover, we have the following

$$pc_0 \le y_n \le p^n c_1 + \frac{p^n - 1}{(p-1)\, p^2 c_0^2},$$

where c_0 is a positive real number, $c_1 = y_1$, $n \ge 1$ and $p \in (0,1)$. So, the proof has been completed as desired. □

Here, we handle the unbounded solutions of Equation (1.4).

Theorem 3.2. *Let $p \ge 1$. Then, every solution of Equation (1.4) is unbounded from above.*

Proof. Let $p \ge 1$ and $\{y_n\}_{n=-k}^{\infty}$ be a positive solution of Equation (1.4). We have from Equation (1.4)

$$y_{n+1} = py_n + \frac{1}{y_{n-k}^2} \ge py_n \ge y_n.$$

From this, the result follows. □

4. STABILITY ANALYSIS OF EQUATION (1.4)

In this section, we study the local stability of Equation (1.4) about the equilibrium point $\bar{y}$. We also handle the global asymptotic stability of Equation (1.4) about the equilibrium point $\bar{y}$. We first examine the linearized equation and characteristic equation of Equation (1.4) about the equilibrium point $\bar{y}$ as follows:

The linearized equation of Equation (1.4) about the equilibrium point $\bar{y} = \frac{1}{\sqrt[3]{1-p}}$ is

$$z_{n+1} = pz_n + (2p-2)\, z_{n-k}. \tag{4.1}$$

Hence, we have the following characteristic equation about the equilibrium point $\bar{y} = \frac{1}{\sqrt[3]{1-p}}$,

$$\lambda^{k+1} - p\lambda^k + 2 - 2p = 0.$$

First, we consider Equation (1.4) for $k = 1$.

Theorem 4.1. *The followings are true:*

- **a:** *If $p \in \left(\frac{1}{2}, 1\right)$, then the equilibrium point $\bar{y}$ of Equation (1.4) is locally asymptotically stable.*

b: *If* $p \in \left(0, \frac{1}{2}\right)$, *then the equilibrium point* $\bar{y}$ *of Equation (1.4) is a repeller.*

c: *If* $p = \frac{1}{2}$, *then the equilibrium point* $\bar{y}$ *of Equation (1.4) is a nonhyperbolic point.*

Proof. Since the proof of the Theorem 4.1 can be easily completed via Theorem 1.1, we left it to the readers. □

Now, we consider Equation (1.4) for $k = \{2, 3, ...\}$. Here we apply Theorem 1.2 for difference equation (4.1) as follows:

Theorem 4.2. *Let* $p > 0$*. The difference equation (4.1) is asymptotically stable if and only if* $|p| < \frac{k+1}{k}$, *and*

a: $p < 1$ *and* $2 - 2p < \left(p^2 + 1 - 2p\cos\phi\right)^{\frac{1}{2}}$, *for k odd,*

b: $\frac{1}{3} < p < 1$ *and* $|2 - 2p| < \left(p^2 + 1 - 2p\cos\phi\right)^{\frac{1}{2}}$, *for k even,*

where ϕ *is the solution in* $\left(0, \frac{\pi}{k+1}\right)$ *of* $\sin(k\theta)/\sin[(k+1)\theta] = 1/p$.

Proof. From Theorem 1.2, we can deduce the proof of the theorem for k is odd or even. □

Remark 4.1. Consider Theorem 4.2 with $k = 2$. Then, we find ϕ that is the solution in $\left(0, \frac{\pi}{3}\right)$ of

$$p\sin 2\theta = \sin 3\theta.$$

Hence, we obtain that

$$\sin\theta\left(4\cos^2\theta - 2p\cos\theta - 1\right) = 0.$$

Thus, we have

$$\begin{aligned} \sin\theta &= 0, \\ \cos\theta &= \frac{p - \sqrt{p^2+4}}{4}, \\ \cos\theta &= \frac{p + \sqrt{p^2+4}}{4}. \end{aligned}$$

Since ϕ is the solution in $\left(0, \frac{\pi}{3}\right)$, we get

$$\cos\theta = \cos\phi = \frac{p + \sqrt{p^2+4}}{4}.$$

Therefore, we consider the following inequilities

$$\begin{aligned} |2 - 2p| &< \sqrt{p^2 + 1 - 2p\cos\phi}, \\ |2 - 2p| &< \sqrt{p^2 + 1 - p\left(\frac{p + \sqrt{p^2+4}}{2}\right)}. \end{aligned} \tag{4.2}$$

From the solution of the inequality (4.2) with $\frac{1}{3} < p < 1$, we obtain that

$$p > \frac{3-\sqrt{3}}{2}.$$

So, if $\frac{3-\sqrt{3}}{2} < p < 1$ and $k = 2$, then the equilibrium point $\bar{y}$ of Equation (1.4) is locally asymptotically stable.

Theorem 4.3. *Let $p \in (0,1)$, suppose p satisfies the conditions in Theorems 4.1 and 4.2. Then, the equilibrium point $\bar{y}$ of Equation (1.4) is globally asymptotically stable.*

Proof. For the proof of this theorem, we consider Theorem 1.3. According to this Theorem, we observe the function

$$f(y_n, y_{n-k}) = f(u,v) = pu + \frac{1}{v^2}. \tag{4.3}$$

The function f (4.3) is non-decreasing in u and non-increasing in v. Let (m,M) be a solution of the following system: $m = f(m,M)$ and $M = f(M,m)$. Thus, we have, via the function f (4.3)

$$\begin{aligned} m &= pm + \frac{1}{M^2}, \\ M &= pM + \frac{1}{m^2}. \end{aligned}$$

From these, we obtain that

$$mM(m-M)(1-p) = 0.$$

Therefore, we get, from $p < 1$,

$$m = M.$$

Thus, every solution of Equation (1.4) converges to the equilibrium point $\bar{y}$. So, this completes the proof. □

5. Analysis of Neimark-Sacker Bifurcation

During this section, we consider the Equation (1.3) for $k = 1$ and $r = 1$. Here, we consider Equation (1.3) for $k = 1$. Hence, we obtain the following second order difference equation

$$x_{n+1} = px_n + \frac{q}{y_{n-1}^2}, \tag{5.1}$$

where $p,q > 0$ and the initial conditions $x_{-1}, x_0 > 0$. Hence, we obtain the following solutions of Equation (5.1) as follows:

$$\bar{x}_1 = \sqrt[3]{\frac{q}{1-p}}, \bar{x}_2 = -\sqrt[3]{\frac{q}{1-p}}e^{\frac{\pi i}{3}}, \bar{x}_3 = \sqrt[3]{\frac{q}{1-p}}e^{\frac{2\pi i}{3}},$$

where $p,q>0$ and $p\neq 1$. In this section, we consider the equilibrium point $\bar{x}=\bar{x}_1$ since it is a real number. From this, we have the linearized equation and characteristic equation about its equilibrium point $\bar{x}$ respectively

$$\begin{aligned} z_{n+1}-pz_n-(2p-2)\,z_{n-1} &= 0, \\ \lambda^2-p\lambda+2-2p &= 0. \end{aligned}$$

From the Linearized Stability Theorem (1.1), we obtain the following:

a: If $p\in\left(\frac{1}{2},1\right)$, then the equilibrium point $\bar{y}$ of Equation (5.1) is locally asymptotically stable.

b: If $p\in\left(0,\frac{1}{2}\right)$, then the equilibrium point $\bar{y}$ of Equation (5.1) is a repeller.

c: If $p=\frac{1}{2}$, then the equilibrium point $\bar{y}$ of Equation (5.1) is a nonhyperbolic point.

Now, we investigate the Neimark-Sacker bifurcation of the equilibrium point $\bar{x}$ of Equation (5.1). In this section, we need the following theorem which is also known as Poincare-Andronov-Hopf bifurcation theorem for maps, see [9, 28, 36].

Theorem 5.1. *Let $f:\mathbb{R}\times\mathbb{R}^2\to\mathbb{R}^2;(\lambda,x)\to f(\lambda,x)$ be a C^4 map depending on the real parameter λ satisfying the following conditions:*

i: *$f(\lambda,0)=0$ for λ near some fixed λ_0;*

ii: *$Df(\lambda,0)=0$ has two non-real eigenvalues $\mu(\lambda)$ and $\overline{\mu(\lambda)}$ for λ near λ_0, $|\mu(\lambda_0)|=1$;*

iii: *$\frac{d}{d\lambda}|\mu(\lambda)|=d(\lambda_0)\neq 0$ at $\lambda=\lambda_0$;*

iv: *$\mu^k(\lambda_0)\neq 1$ for $k=1,2,3,4$.*

Then, there is a smooth λ-dependent change for coordinate bringing f into the form

$$f(\lambda,x)=G(\lambda,x)+O(\|x\|^5).$$

and there are smooth functions $a(\lambda)$, $b(\lambda)$ and $w(\lambda)$ so that in polar coordinates the function $G(\lambda,x)$ is given by

$$\begin{pmatrix} r \\ \theta \end{pmatrix}=\begin{pmatrix} |\mu(\lambda)|\,r-a(\lambda)r^3 \\ \theta+w(\lambda)+b(\lambda)r^2 \end{pmatrix}. \tag{5.2}$$

If $a(\lambda_0)>0$ and $d(\lambda_0)>0$ ($d(\lambda_0)<0$), then there is a neighborhood U of the origin and a, $\delta>0$ such that for $|\lambda-\lambda_0|<\delta$ and $x_0\in U$, then the w-limit set of x_0 is the origin if $\lambda<\lambda_0$ ($\lambda>\lambda_0$) and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda<\lambda_0$ ($\lambda>\lambda_0$). Furthermore, $\Gamma(\lambda_0)=0$.

If $a(\lambda_0)<0$ and $d(\lambda_0)>0$ ($d(\lambda_0)<0$), then there is a neighborhood U of the origin and a, $\delta>0$ such that for $|\lambda-\lambda_0|<\delta$ and $x_0\in U$, then w-limit set of x_0 is the origin if $\lambda>\lambda_0$ ($\lambda<\lambda_0$) and belongs to a closed invariant C^1 curve $\Gamma(\lambda)$ encircling the origin if $\lambda>\lambda_0$ ($\lambda<\lambda_0$). Furthermore, $\Gamma(\lambda_0)=0$.

Considering a general map $f(\lambda,x)$ *that has a fixed point at the origin with complex eigenvalues* $\mu(\lambda)=\alpha(\lambda)+i\beta(\lambda)$ *and* $\overline{\mu(\lambda)}=\alpha(\lambda)-i\beta(\lambda)$ *satisfying* $\alpha(\lambda)^2+\beta(\lambda)^2=1$ *and* $\beta(\lambda)\neq 0$.

By putting the linear part of such a map into Jordan canonical form, we may assume f *to have the following form near the origin*

$$f(\lambda,x)=\begin{pmatrix}\alpha(\lambda) & -\beta(\lambda)\\ \beta(\lambda) & \alpha(\lambda)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}g_1(\lambda,x_1,x_2)\\ g_2(\lambda,x_1,x_2)\end{pmatrix}.$$

Moreover, for all sufficiently small positive (negative) λ, f *has an attracting (repelling) invariant circle if* $a(\lambda_0)<0$ $(a(\lambda_0)>0)$ *respectively; and* $a(\lambda_0)$ *is given by the following formula:*

$$a(\lambda_0) = \boldsymbol{Re}\left[\frac{(1-2\mu(\lambda_0))^{-2}\mu(\lambda_0)}{1-\mu(\lambda_0)}\gamma_{11}\gamma_{20}\right]+\frac{1}{2}|\gamma_{11}|^2+|\gamma_{20}|^2 \tag{5.3}$$

$$-\boldsymbol{Re}\left(\overline{\mu(\lambda_0)}\gamma_{21}\right), \tag{5.4}$$

where

$$\begin{aligned}
\gamma_{20} &= \frac{1}{8}\left\{\begin{array}{c}(g_1)_{x_1x_1}-(g_1)_{x_2x_2}+2(g_2)_{x_1x_2}\\ +i\left[(g_2)_{x_1x_1}-(g_2)_{x_2x_2}-2(g_1)_{x_1x_2}\right]\end{array}\right\},\\
\gamma_{11} &= \frac{1}{4}\left\{(g_1)_{x_1x_1}+(g_1)_{x_2x_2}+i\left[(g_2)_{x_1x_1}+(g_2)_{x_2x_2}\right]\right\},\\
\gamma_{02} &= \frac{1}{8}\left\{\begin{array}{c}(g_1)_{x_1x_1}-(g_1)_{x_2x_2}-2(g_2)_{x_1x_2}\\ +i\left[(g_2)_{x_1x_1}-(g_2)_{x_2x_2}+2(g_1)_{x_1x_2}\right]\end{array}\right\},\\
\gamma_{21} &= \frac{1}{8}\left\{\begin{array}{c}(g_1)_{x_1x_1x_1}+(g_1)_{x_1x_2x_2}+(g_2)_{x_1x_1x_2}+(g_2)_{x_2x_2x_2}\\ +i\left[(g_2)_{x_1x_1x_1}+(g_2)_{x_1x_2x_2}-(g_1)_{x_1x_1x_2}-(g_1)_{x_2x_2x_2}\right]\end{array}\right\}.
\end{aligned}$$

Now, we focus on the Neimark-Sacker bifurcation of the unique equilibrium point of Equation (5.1).

First, we take a change of variable such that

$$y_n=x_n-\bar{x}.$$

Thus, from Equation (5.1), we have that

$$y_{n+1}=(p-1)\bar{x}+py_n+\frac{q}{(y_{n-1}+\bar{x})^2}. \tag{5.5}$$

By using the substitution $u_n=y_{n-1}, v_n=y_n$, we write Equation (5.5) in the equivalent form

$$\left\{\begin{array}{c}u_{n+1}=v_n\\ v_{n+1}=(p-1)\bar{x}+pv_n+\frac{q}{(u_n+\bar{x})^2}\end{array}\right. . \tag{5.6}$$

We define a corresponding map, denoted as F and defined by:

$$F\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ (p-1)\overline{x} + pv + \frac{q}{(u+\overline{x})^2} \end{pmatrix}.$$

Therefore, we get

$$F\begin{pmatrix} u \\ v \end{pmatrix} = J_F(0,0)\begin{pmatrix} u \\ v \end{pmatrix} + G\begin{pmatrix} u \\ v \end{pmatrix} \tag{5.7}$$

where

$$J_F(u,v) = \begin{pmatrix} 0 & 1 \\ -\frac{2q}{(u+\overline{x})^3} & p \end{pmatrix},$$

and

$$G\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ (p-1)\overline{x} + \frac{q}{(u+\overline{x})^2} + \frac{2qu}{\overline{x}^3} \end{pmatrix}.$$

Also, we obtain that

$$J_F(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{2q}{\overline{x}^3} & p \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2(p-1) & p \end{pmatrix}.$$

Hence, we get the eigenvalues of $J_F(0,0)$ as follows

$$\mu_{\pm}(p) = \frac{p \pm \sqrt{p^2+8p-8}}{2}.$$

Thus, we have

$$|\mu_{\pm}(p)|^2 = \mu(p)\overline{\mu(p)} = 2(1-p). \tag{5.8}$$

Here, we consider the non-hyperbolic equilibrium point. Then, let $p = p_0 = \frac{1}{2}$ and so $\overline{x}_{p_0} = \sqrt[3]{2q}$. Therefore, we have the Jacobian matrix of F at $(0,0)$ and $p = p_0 = \frac{1}{2}$

$$J_F(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{2} \end{pmatrix} = A.$$

Thus, we get the eigenvalues of Jacobian matrix A such that

$$\mu(p_0) = \frac{1+i\sqrt{15}}{4},$$

and

$$|\mu(p_0)| = 1.$$

μ also satisfies

$$\begin{aligned}\mu^2(p_0) &= \frac{-7+i\sqrt{15}}{8},\\ \mu^3(p_0) &= \frac{-11-3i\sqrt{15}}{16},\\ \mu^4(p_0) &= \frac{17-7i\sqrt{15}}{32},\end{aligned}$$

and

$$\mu^k(p_0) \neq 1,$$

for $k = 1,2,3,4$. Additionally, we have the eigenvectors corresponding to $\mu(p_0)$ as follows

$$q = \left(\frac{1-i\sqrt{15}}{4}, 1\right)^T,$$

and

$$\gamma = \left(\frac{2\sqrt{15}i}{15}, \frac{1}{2} - \frac{\sqrt{15}i}{30}\right),$$

and $Aq = \mu q$, $\gamma A = \mu\gamma$ and $\gamma q = 1$ hold. Furthermore, we obtain from (5.8)

$$\frac{d}{dp}|\mu(p)|\bigg|_{p_0=\frac{1}{2}} = -1 < 0. \tag{5.9}$$

Theorem 5.2. *We consider Equation (5.1). There exists a neighbourhood U of the equilibrium point $(\bar{x},\bar{x})$ and a $\delta > 0$ such that for $\left|p - \frac{1}{2}\right| < \delta$ and $x_{-1}, x_0 \in U$, then the w-limit set of solutions of Equation (5.1), with the initial condition x_{-1}, x_0 is the equilibrium point $\bar{x}$ if $p > \frac{1}{2}$ and belongs to a closed invariant C^1 curve $\Gamma(\alpha)$ encircling the equilibrium point if $p < \frac{1}{2}$. Moreover, $\Gamma(p_0) = 0$ and invariant curve $\Gamma(p)$ can be approximated by*

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix} + \begin{pmatrix} \sqrt{\frac{\sqrt[3]{4q^2}}{3}\left(\frac{1}{2}-p\right)}\left(\cos\theta + \sqrt{15}\sin\theta\right) \\ +\frac{4\sqrt[3]{4q^2}}{3}\left(\frac{1}{2}-p\right)\left(\frac{-17}{24\sqrt[3]{2q}}\cos 2\theta + \frac{7\sqrt{15}}{24\sqrt[3]{2q}}\sin 2\theta + \frac{2}{\sqrt[3]{2q}}\right) \\ 4\sqrt{\frac{\sqrt[3]{4q^2}}{3}\left(\frac{1}{2}-p\right)}\cos\theta \\ +\frac{4\sqrt[3]{4q^2}}{3}\left(\frac{1}{2}-p\right)\left(\frac{7}{6\sqrt[3]{2q}}\cos 2\theta - \frac{\sqrt{15}}{6\sqrt[3]{2q}}\sin 2\theta + \frac{2}{\sqrt[3]{2q}}\right) \end{pmatrix}.$$

Proof. Assume that $p = p_0 + \delta$ with δ a sufficiently small parameter. From above results, we can transform (5.6) into the normal form as follows

$$z_{n+1} = \mu(\delta) z_n + c(\delta) z_n^2 \bar{z}_n + O\left(|z_n|^4\right). \tag{5.10}$$

Equation (5.10) can be obtained in the polar coordinates as follows

$$\begin{pmatrix} r_{n+1} \\ \theta_{n+1} \end{pmatrix} = \begin{pmatrix} |\mu(\delta)| r_n + a(\delta) r_n^3 + O\left(r_n^4\right) \\ \theta_n + \arg\mu(\delta) + b(\delta) r_n^2 + O\left(r_n^3\right) \end{pmatrix}, \tag{5.11}$$

for $a(\delta) = \mathbf{Re}\left(\frac{c(\delta)}{\mu(\delta)}\right)$ and $b(\delta) = \mathbf{Im}\left(\frac{c(\delta)}{\mu(\delta)}\right)$. By performing the Taylor expansion of the coefficients of the first equation of (5.11), we have

$$r_{n+1} = (1 + d\delta) r_n + a(0) r_n^3 + O\left(r_n^4\right).$$

Substituting $p = p_0 = \frac{1}{2}$ and $\bar{x}$ into (5.7), we get

$$F_{p_0}\begin{pmatrix} u \\ v \end{pmatrix} = A\begin{pmatrix} u \\ v \end{pmatrix} + G_{p_0}\begin{pmatrix} u \\ v \end{pmatrix},$$

where

$$G_{p_0}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2}\sqrt[3]{2q} + \frac{q}{\left(u+\sqrt[3]{2q}\right)^2} + u \end{pmatrix}.$$

Hence, (5.6) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = A\begin{pmatrix} u_n \\ v_n \end{pmatrix} + G_{p_0}\begin{pmatrix} u_n \\ v_n \end{pmatrix}.$$

Define the basis of $\mathbb{R}^2$ by $\phi = (q, \bar{q})$, then we can substitute for (u, v)

$$\begin{pmatrix} u \\ v \end{pmatrix} = \phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (qz + \bar{q}\bar{z}) = \begin{pmatrix} \frac{1-i\sqrt{15}}{4}z + \frac{1+i\sqrt{15}}{4}\bar{z} \\ z + \bar{z} \end{pmatrix}.$$

Therefore, we have that

$$G\left(\phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ -\frac{1}{2}\sqrt[3]{2q} + \frac{q}{\left(\frac{1-i\sqrt{15}}{4}z + \frac{1+i\sqrt{15}}{4}\bar{z} + \sqrt[3]{2q}\right)^2} + \frac{1-i\sqrt{15}}{4}z + \frac{1+i\sqrt{15}}{4}\bar{z} \end{pmatrix}.$$

Let

$$G\left(\phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right) = \frac{1}{2}\left(g_{20}z^2 + 2g_{11}z\bar{z} + g_{02}\bar{z}^2\right) + O\left(|z|^3\right).$$

Thus, we obtain

$$g_{20} = \frac{\partial^2}{\partial z^2} G\left(\phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ \frac{-21-3\sqrt{15}i}{8\sqrt[3]{2q}} \end{pmatrix},$$

$$g_{11} = \frac{\partial^2}{\partial z \partial \bar{z}} G\left(\phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ \frac{3}{\sqrt[3]{2q}} \end{pmatrix},$$

$$g_{02} = \frac{\partial^2}{\partial \bar{z}^2} G\left(\phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix}\right)\Big|_{z=0} = \begin{pmatrix} 0 \\ \frac{-21+3\sqrt{15}i}{8\sqrt[3]{2q}} \end{pmatrix},$$

and

$$K_{20} = \left(\mu^2 I - A\right)^{-1} g_{20} = \begin{pmatrix} \frac{-17-7i\sqrt{15}}{24\sqrt[3]{2q}} \\ \frac{7+i\sqrt{15}}{6\sqrt[3]{2q}} \end{pmatrix},$$

$$K_{11} = (I-A)^{-1} g_{11} = \begin{pmatrix} \frac{2}{\sqrt[3]{2q}} \\ \frac{2}{\sqrt[3]{2q}} \end{pmatrix},$$

$$K_{02} = \left(\bar{\mu}^2 I - A\right)^{-1} g_{02} = \begin{pmatrix} \frac{-17+7i\sqrt{15}}{24\sqrt[3]{2q}} \\ \frac{7-i\sqrt{15}}{6\sqrt[3]{2q}} \end{pmatrix} = \bar{K}_{20}.$$

By using K_{20}, K_{11} and K_{02}, we get the following

$$\begin{aligned} g_{21} &= \frac{\partial^3}{\partial z^2 \partial \bar{z}} G\left(\phi\begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \frac{1}{2}K_{20}z^2 + K_{11}z\bar{z} + \frac{1}{2}K_{02}\bar{z}^2\right)\Bigg|_{z=0} \\ &= \begin{pmatrix} 0 \\ \frac{11-3i\sqrt{15}}{4\sqrt[3]{4q^2}} \end{pmatrix}. \end{aligned}$$

Consequently, we obtain that

$$a(0) = \mathbf{Re}\left(\frac{c(0)}{\mu}\right) = \frac{1}{2}\mathbf{Re}\left(\gamma g_{21}\bar{\mu}\right) = -\frac{3}{4\sqrt[3]{4q^2}} < 0.$$

Also we have from (5.9)

$$\frac{d}{dp}\left|\mu(p)\right|\Bigg|_{p_0=\frac{1}{2}} = -1 < 0.$$

In addition, we have an asymptotic approximation of the invariant curve as given in [23]:

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix} + 2\rho_0 \mathbf{Re}\left(qe^{i\theta}\right) + \rho_0^2\left(\mathbf{Re}\left(K_{20}e^{2i\theta}\right) + K_{11}\right)$$

where

$$d = \frac{d}{dp}\left|\mu(p)\right|\Bigg|_{p_0=\frac{1}{2}}, \rho_0 = \sqrt{-\frac{d}{a(0)}\delta}, \theta \in \mathbb{R}.$$

Since $\rho_0 = \sqrt{\frac{4\sqrt[3]{4q^2}}{3}\left(\frac{1}{2}-p\right)}$ for $0 < \frac{1}{2}-p < \delta$, where $\delta > 0$ is a sufficiently small parameter, from the above calculations we have that

$$\begin{pmatrix} u \\ v \end{pmatrix} \approx \begin{pmatrix} \bar{x} \\ \bar{x} \end{pmatrix} + \begin{pmatrix} \sqrt{\frac{\sqrt[3]{4q^2}}{3}\left(\frac{1}{2}-p\right)}\left(\cos\theta + \sqrt{15}\sin\theta\right) \\ +\frac{4\sqrt[3]{4q^2}}{3}\left(\frac{1}{2}-p\right)\left(\frac{-17}{24\sqrt[3]{2q}}\cos 2\theta + \frac{7\sqrt{15}}{24\sqrt[3]{2q}}\sin 2\theta + \frac{2}{\sqrt[3]{2q}}\right) \\ 4\sqrt{\frac{\sqrt[3]{4q^2}}{3}\left(\frac{1}{2}-p\right)}\cos\theta \\ +\frac{4\sqrt[3]{4q^2}}{3}\left(\frac{1}{2}-p\right)\left(\frac{7}{6\sqrt[3]{2q}}\cos 2\theta - \frac{\sqrt{15}}{6\sqrt[3]{2q}}\sin 2\theta + \frac{2}{\sqrt[3]{2q}}\right) \end{pmatrix}.$$

□

6. Numerical Examples

In order to verify our results, we consider four numerical examples with different choices of p and k in Equation (1.4). Here, each example exhibits the visualization of the different behaviours of the solutions of Equation (1.4).

Example 6.1. *Consider Equation (1.4) with $p = \frac{1}{6}$ and $k = 4$. Then, we obtain the fifth order difference equation*

$$y_{n+1} = \frac{1}{6}y_n + \frac{1}{y_{n-4}^2}. \tag{6.1}$$

According to Theorem 2.1, given the initial conditions

$$y_{-4} = y_{-2} = y_0 = \sqrt[3]{12 - 18\sqrt{\frac{3}{7}}} \approx 0.60022,$$

$$y_{-3} = y_{-1} = \sqrt[3]{12 + 18\sqrt{\frac{3}{7}}} \approx 2.8758,$$

Equation (6.1) has two periodic solution as shown in Figure 1. Figure 1 shows the first 40 terms of Equation (6.1).

Example 6.2. *Consider Equation (1.4) with $p = 0.64$ and $k = 2$. Then, we have the following difference equation of order three*

$$y_{n+1} = 0.64y_n + \frac{1}{y_{n-2}^2}, \tag{6.2}$$

and $\bar{y} = 1.40572$. According to Theorem 4.2, Remark 4.1 and Theorem 4.3, if $p > \frac{3-\sqrt{3}}{2} \approx 0.63397$, then Equation (6.2) is globally asymptotically stable. Now, we consider Equation (6.2) with the initial conditions $y_{-2} = 6$, $y_1 = 1$ and $y_0 = 4$. Figure 2 shows the first 1.500 terms of Equation (6.2).

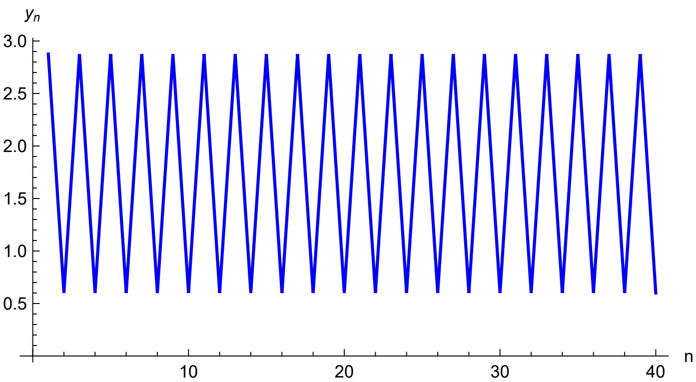

FIGURE 1. Two periodic solution of Equation (6.1).

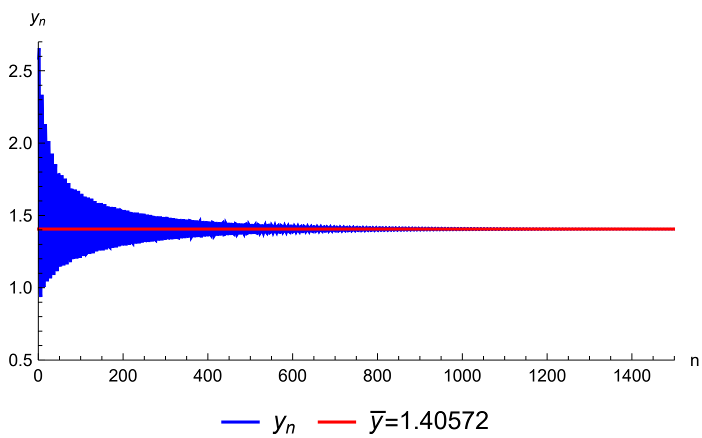

FIGURE 2. Globally asymptotic stability of Equation (6.2).

Example 6.3. *Consider Equation (1.4) with* $p = 0.51$ *and* $k = 1$*. Then, we get the following second order difference equation*

$$y_{n+1} = 0.51 y_n + \frac{1}{y_{n-1}^2}, \tag{6.3}$$

and $\bar{y} = 1.26843$*. According to Theorems 4.1 and 4.3, if* $\frac{1}{2} < p < 1$*, then Equation (6.3) is globally asymptotically stable. Now, we consider Equation (6.3) with the initial conditions* $y_1 = 7$ *and* $y_0 = 2$*. We now present two figures, Figure 3 shows the global asymptotic stability of the first 1.500 terms of Equation (6.3) and Figure 4 shows the phase portrait behaviour of the first 1.500 terms of Equation (6.3).*

Example 6.4. *Consider Equation (5.1) with* $x_{-1} = 0.1$ *and* $x_0 = 1.1$*. Then, we get the following Neimark-Sacker Bifurcation plots for the different* p *intervals. The*

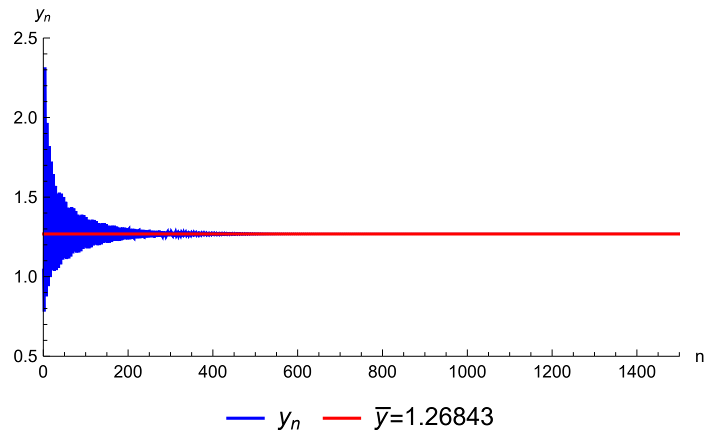

FIGURE 3. Global asymptotic stability of Equation (6.3).

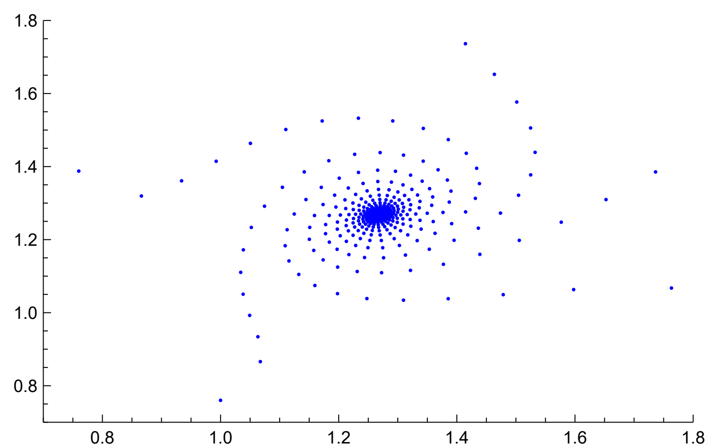

FIGURE 4. Phase portrait behaviour of Equation (6.3).

figures 5 and 6 show that if $p \in (0, 0.5)$, then the equilibrium point $\bar{x}$ of Equation (5.1) is unstable. Figure 7 shows that if $p \in (0.5, 1)$, then the equilibrium point $\bar{x}$ of Equation (5.1) is stable.

Example 6.5. *Consider Equation (5.1) with $p = 0.4999$ and $q = 3$. Then, we have the following second order difference equation*

$$x_{n+1} = 0.4999x_n + \frac{3}{x_{n-1}^2}, \tag{6.4}$$

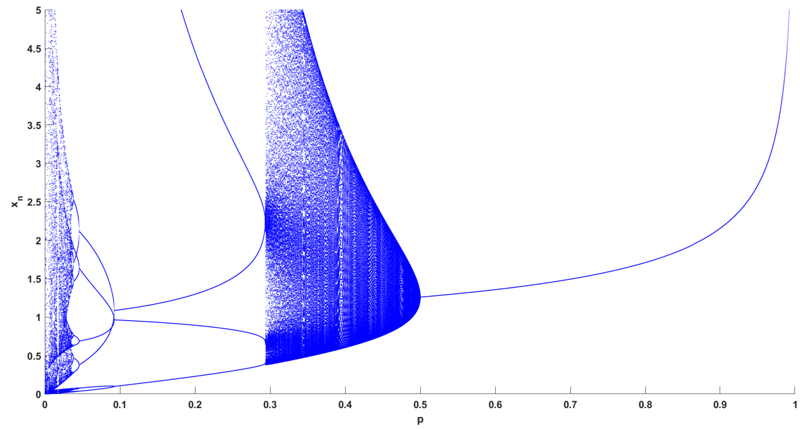

FIGURE 5. Bifurcation diagram for x_n with respect to $p \in (0,1)$

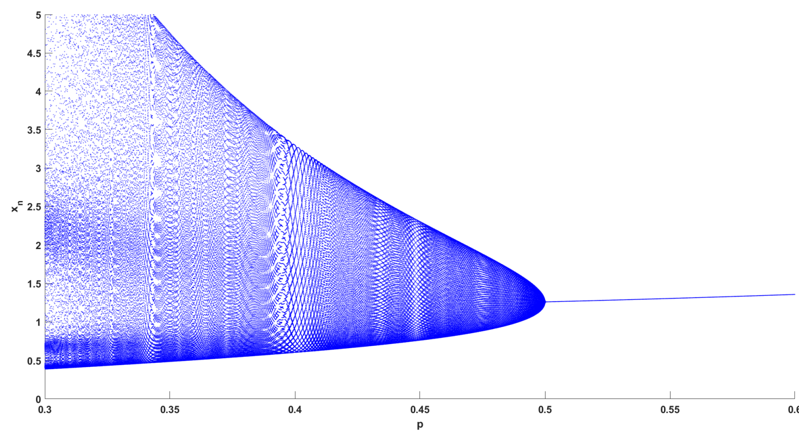

FIGURE 6. Bifurcation diagram for x_n with respect to $p \in (0.3,0.6)$

with the initial conditions $x_{-1} = 5$ and $x_0 = 8$. According to Theorem 5.2, Equation (6.4) has an invariant curve as shown in Figure 8. Figure 8 shows both first 100.000 terms of Equation (6.4) and the invariant curve of Equation (6.4).

7. CONCLUSION AND OPEN PROBLEMS

During this paper, we discuss the dynamical behavior of the solutions of Equation (1.3). We first give the existence of two periodic solutions of Equation (1.3).

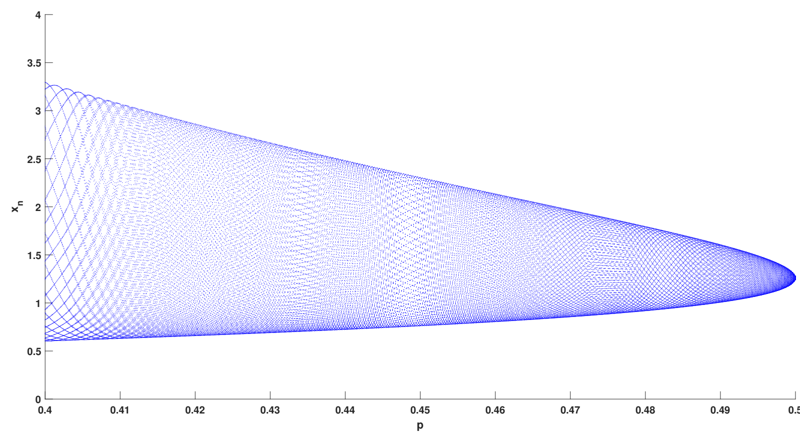

FIGURE 7. Bifurcation diagram for x_n with respect to $p \in (0.4, 0.5)$

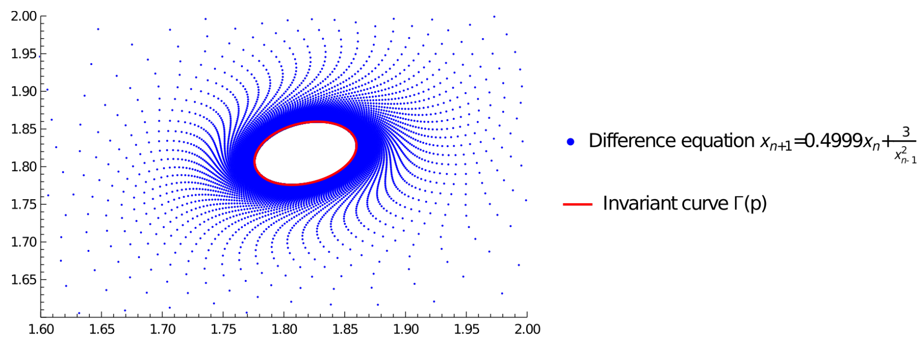

FIGURE 8. Phase portrait behavior of 100.000 terms of Equation (6.4) and the invariant curve of Equation (6.4).

We determine that if k is an odd number, then Equation (1.4) has no two periodic solutions and if k is an even number and if $p \in \left(0, \frac{1}{3}\right)$, then Equation (1.4) has two periodic solutions. But, if $p \geq \frac{1}{3}$, then Equation (1.4) has no two periodic solutions. We also reveal the bounded and unbounded solutions of Equation (1.3). We find that if $p \in (0,1)$, then all solutions of Equation (1.4) are bounded from below and above. Additionally, we present local and global stability of the solutions of Equation (1.3). We see that if $p \in (0,1)$, and p satisfies the conditions in Theorems 4.1 and 4.2, then the equilibrium point $\bar{y}$ of Equation (1.4) is globally asymptotically stable. Moreover, we discover the existence of Neimark-Sacker bifurcation of the solutions of Equation (5.1) and also give an invariant curve of Equation (5.1). Lastly, we consider some numerical simulations to support our results.

We now offer two open problems to researchers.

Open problem 1: Investigate the dynamics of the following higher order difference equation

$$x_{n+1} = Ax_n + \frac{B}{Cx_{n-k}^r},$$

where A, B, C and the initial conditions are positive real numbers.

Open problem 2: Investigate the dynamics of the following higher order difference equation

$$x_{n+1} = a_n x_n + \frac{b_n}{c_n x_{n-k}^2},$$

where the initial conditions are positive real numbers and a_n, b_n, c_n can be bounded, convergent or periodic sequences.

REFERENCES

[1] M.A. Abbasi, Q. Din, O. Albalawi, R. Niaz, M.A. Alomair, A.M. Alomair, *Analysis of the Stability and Chaotic Dynamics of an Ecological Model*, Complexity, **2024** (2024), 1681557, 30 pages. https://doi.org/10.1155/2024/1681557

[2] Y. Akrour, M. Kara, N. Touafek, Y. Yazlık, *On The Three Dimensional Higher Order System of Difference Equations*, Discrete Contin. Dyn. Syst. Ser. B., **29** (2022), 77–108.

[3] E. Bešo, S. Kalabušić, N. Mujić, E. Pilav, *Boundedness of solutions and stability of certain second-order difference equation with quadratic term*, Adv. Differ. Equ., **2020 (19)** (2020), 1–22 . https://doi.org/10.1186/s13662-019-2490-9

[4] A. Bilgin, M.R.S. Kulenović, *Global asymptotic stability for discrete single species population models*, Discrete Dyn. Nat. Soc., **2017** (2017), 1–15. https://doi.org/10.1155/2017/5963594

[5] E. Camouzis, G. Ladas, *Dynamics of third order rational difference equations with open problems and conjectures, volume 5 of Advances in Discrete Mathematics and Applications*, Chapman & Hall/CRC, Boca Raton, 2008.

[6] S. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag, New York, 1996.

[7] M. Göcen, *The exact solutions of some difference equations associated with adjusted Jacobsthal-Padovan numbers*, Kırklareli University Journal of Engineering and Science, **8(1)** (2022), 1–14. https://doi.org/10.34186/klujes.1078836

[8] E.A. Grove, G. Ladas, *Periodicities in Nonlinear Difference Equations, Vol. 4*, Chapman & Hall / CRC, 2005.

[9] J.K. Hale, H. Kocak, *Dynamics and Bifurcations*, Springer-Verlag, New York, (1991).

[10] S.S. Hassan, *Dynamics of the Rational Difference Equation* $x_{n+1} = px_n + \frac{q}{x_{n-1}^2}$, Preprints 2020, https://doi.org/10.20944/preprints202004.0113.v1

[11] M. Kara, Y. Yazlık, N. Touafek, *On a difference equation whose solution is related to Fibonacci numbers*, Filomat, **38(20)** (2024), 7199–7207. https://doi.org/10.2298/FIL2420199K

[12] S. Kalabušić, M.R.S. Kulenović, M. Mehuljić, *Global dynamics and bifurcations of two quadratic fractional second order difference equations*, J. Comput. Anal. Appl., **21(1)** (2016), 132–143.

[13] V.L. Kocic, G. Ladas, *Global Asymptotic Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[14] W. Kosmala, M.R.S. Kulenović, G. Ladas, C.T. Teixeira, *On the recursive sequence* $y_{n+1} = \frac{p+y_{n-1}}{qy_n+y_{n-1}}$, J. Math. Anal. Appl., **251** (2000), 571–86.
[15] M.R.S. Kulenović, M. Nurkanović, Z. Nurkanović, *The Rate of Convergence of a Certain Mixed Monotone Difference Equation*, Sarajevo Journal of Mathematics, **20(1)** (2024), 121–135. https://doi.org/10.5644/SJM.20.01.11
[16] M.R.S. Kulenović, G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures*, Chapman & Hall/CRC, Boca Raton, 2002
[17] M.R.S. Kulenović, O. Merino, *A global attractivity result for maps with invariant boxes*, Discrete Contin. Dyn. Syst. Ser. B., **6(1)** (2006), 97–110.
[18] M.R.S. Kulenović, M. Nurkanović, *Asymptotic behavior of a two dimensional fractional system of difference equations*, Radovi matematički, **11** (2002), 11–19.
[19] M.R.S. Kulenović, M. Nurkanović, *Asymptotic behavior of a competitive system of linear fractional difference equation*, Adv. Differ. Equ., **3** (2006), 1–13.
[20] M.R.S. Kulenović, S. Moranjkić, M. Nurkanović, Z. Nurkanovic, *Global Asymptotic Stability and Naimark-Sacker Bifurcation of Certain Mix Monotone Difference Equation*, Discrete Dyn. Nat. Soc., **2018** (2018), 1–22. https://doi.org/10.1155/2018/7052935
[21] M.R.S. Kulenović, E. Pilav and E. Silić, *Naimark–Sacker bifurcation of a certain second order quadratic fractional difference equation*, J. Math. Comput. Sci., **4(6)** (2014), 1025–1043.
[22] S. A. Kuruklis, *The asymptotic stability of* $x_{n+1} - ax_n + bx_{n-k} = 0$, J. Math. Anal. Appl., **188** (1994), 719–731.
[23] K. Murakami, *The invariant curve caused by Naimark–Sacker bifurcation*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **9(1)** (2002), 121–132.
[24] M. Nurkanović, Z. Nurkanović, *Birkhoff Normal Forms, Kam Theory, Periodicity and Symmetries for Certain Rational Difference Equation With Cubic Terms*, Sarajevo Journal of Mathematics, **12(2)** (2024), 217–231. https://doi.org/10.5644/SJM.12.2.08
[25] B. Oğul, D. Şimşek, *Dynamical analysis and solutions of nonlinear difference equations of thirty order*, Universal Journal of Mathematics and Applications, **7(3)** (2024), 111–120. https://doi.org/10.32323/ujma.1483974
[26] B. Oğul, *Dynamical behavior of solution of twenty-fourth order rational difference equation*, Filomat, **39(5)** (2025), 1595–1607.
[27] M. Ouyang, Q. Zhang, M. Cai, Z. Zeng, *Dynamic analysis of a fuzzy Bobwhite quail population model under g-division law*, Scientific Reports, **14** (2024), Article number: 9682. https://doi.org/10.1038/s41598-024-60178-4
[28] C. Robinson, *Stability, Symbolic Dynamics and Chaos*, CRC Press, Boca Raton, (1995).
[29] S. Sadiq, N.F. Sher, A. Khaliq, M. Shabbir, *Stability and qualitative dynamics of a class of three-dimensional difference equation system of exponential type*, Math. Meth. Appl. Sci. **2024** (2024), 1–21. https://doi.org/10.1002/mma.10554.
[30] D. Şimşek, B. Oğul, F. Abdullayev, *Dynamical behavior of solution of fifteenth-order rational difference equation*. Filomat, **38(3)** (2024), 997–1008.
[31] E. Taşdemir, *Global dynamics of a higher order difference equation with a quadratic term*, J. Appl. Math. Comput. **67(1-2)** (2021), 423–437.
[32] E. Taşdemir, *Asymptotically Periodic Solutions of A Fifth Order Difference Equation*, Journal of Mathematical Analysis, **10(3)** (2019), 100–111.
[33] E. Taşdemir, M. Göcen, Y. Soykan, *Global Dynamical Behaviours and Periodicity of a Certain Quadratic-Rational Difference Equation with Delay*, Miskolc Math. Notes, **23(1)** (2022), 471–484.
[34] E. Taşdemir, *Global Dynamical Properties of a System of Quadratic-Rational Difference Equations with Arbitrary Delay*, Sarajevo Journal of Mathematics, **18(1)** (2024), 161–175. https://doi.org/10.5644/SJM.18.01.10

[35] N. Touafek, M. Boulouh, M. Kara, Y. Yazlık, Explicit form for the solutions of a system of difference equations. In *Modeling of discrete and continuous systems: Ordinary, partial and fractional derivatives* (2025). (pp. 55–71). Springer Nature Singapore.

[36] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Second Edition, Springer-Verlag, New York, (2003).

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