

AN EQUIVALENT FORM OF THE PRIME NUMBER THEOREM

G. SUDHAAMSH MOHAN REDDY, S. SRINIVAS RAU, AND B. UMA

ABSTRACT. A simple proof is given that $\sum_n \frac{\mu(n)d(n)}{n} = 0$ using the Prime Number Theorem. It is shown that this is equivalent to the estimate $\sum_{n \leq x} \mu(n)d(n) = o(x)$ and to the Prime Number Theorem.

1. INTRODUCTION

Let $\mu(n)$ denote the Moebius function and $d(n)$ the divisor function. The question " $\sum_n \frac{\mu(n)d(n)}{n} = 0$?" was posed in ([4], p 1599). It can indeed be settled positively by applying the estimate in Ram Murty ([3], 4.4.6) $\sum_{n \leq x} \mu(n)d(n) = o(x)$ (see Remark 1.6 below) W.Narkiewicz in a letter indicated that it can be proved by contour integrals. In this note we give one more proof (Prop 1.7). We also show (Prop 1.11) that this convergence is equivalent to the Prime Number Theorem.

Lemma 1.1.

$$\prod_p \left(1 - \frac{2}{p^s}\right) = \sum_n \frac{\mu(n)d(n)}{n^s} \quad (\operatorname{Re} s > 1)$$

Proof. For any multiplicative function $f(n)$ we have ([1], Theorem 11.7)

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \prod_p \left(1 + \frac{f(p)}{p^s} + \dots + \frac{f(p^k)}{p^{ks}} + \dots\right)$$

Take $f(n) = \mu(n)d(n)$, $f(p) = \mu(p)d(p) = -2$ and $f(p^k) = \mu(p^k)d(p^k) = 0$ for $k > 2$. Hence the equality. $\square$

Lemma 1.2.

$$\prod_p \left(1 - \frac{2}{p^s}\right) = \prod_p \left\{\left(1 - \frac{1}{p^s}\right)^2\right\} \prod_p \left(1 - \frac{1}{p^{2s}\left(1 - \frac{1}{p^s}\right)^2}\right) \quad (\operatorname{Re} s > 1)$$

Proof. ([3], 4.4.6) We verify the factorization

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$$(1 - \frac{2}{x}) = (1 - \frac{1}{x})^2 (1 - \frac{1}{x^2(1 - \frac{1}{x})^2})$$

and put $x = p^s$, take product over all primes p , and rearrange terms using absolute convergence for $\text{Re } s > 1$.

$$\begin{aligned} RHS &= (1 - \frac{1}{x})^2 (\frac{x^2(1 - \frac{1}{x})^2 - 1}{x^2(1 - \frac{1}{x})^2}) \\ &= (1 - \frac{1}{x})^2 (\frac{(x-1)^2 - 1}{(x-1)^2}) \\ &= (1 - \frac{1}{x})^2 (\frac{x^2 - 2x + 1 - 1}{x^2(1 - \frac{1}{x})^2}) \\ &= (1 - \frac{1}{x})^2 (\frac{x^2 - 2x}{x^2(1 - \frac{1}{x})^2}) \\ &= \frac{x-2}{x} \\ &= 1 - \frac{2}{x} \\ &= LHS. \end{aligned}$$

□

Corollary 1.1. For $\text{Re } s > 1$

$$\sum_{n=1}^{\infty} \frac{\mu(n)d(n)}{n^s} = \prod_p (1 - \frac{2}{p^s}) = (\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s})^2 \prod_p (1 - \frac{1}{p^{2s}(1 - \frac{1}{p^s})^2})$$

where the last product converges uniformly for $\text{Re } s > \frac{1}{2}$.

Proof. We use the product formula ([1], p 231) for $\text{Re } s > 1$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_p (1 - \frac{1}{p^s})$$

and the convergence condition for infinite products:

$\prod_n (1 - \alpha_n)$, $0 \leq \alpha_n < 1$ converges iff $\sum_n \alpha_n$ converges.

□

Remark 1.1. Recall([1], p 97) that the Prime Number Theorem is equivalent to

$$\sum_n \frac{\mu(n)}{n} = 0$$

Lemma 1.3. The Cauchy product

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = (\sum_{l=1}^{\infty} \frac{\mu(l)}{l}) (\sum_{m=1}^{\infty} \frac{\mu(m)}{m})$$

converges to 0.

Proof. We note that the formal Cauchy product holds at $s = 1$ and that $a_n = \sum_{lm=n} \mu(l)\mu(m)$ as in ([3], 4.4.5). We use the estimate there:

$$(i) \sum_{n \leq x} a_n = t_x = O(x \exp(-c(\log x)^{\frac{1}{10}}))$$

so that $t_x = o(x)$ and $|\frac{t_x}{x}| \leq K e^{-c(\log x)^{\frac{1}{10}}}$ for suitable positive constants c, K .

(ii) By theorem 9.63(c), $\sum \frac{a_n}{n}$ is Cesaro summable if and only if $\sum \frac{t_n}{n(n+1)}$ converges. We check absolute convergence by the integral test, applying the above estimate.

$$\int_1^\infty \frac{t_x}{x(x+1)} dx \leq K \int_1^\infty \frac{1}{(x+1)} \frac{1}{e^{c(\log x)^{1/10}}} dx \leq K \int_1^\infty \frac{1}{x} \frac{1}{e^{c(\log x)^{1/10}}} dx$$

changing variables to $y = \log x$ we have the convergence of the integral.

(iii) Now " $t_n = o(n)$ " and Cesaro summability together imply the convergence of the series $\sum_n \frac{a_n}{n}$ (Theorem 9.63(b), [2]). The value of the sum is 0 since all three series above are convergent and $\sum_n \frac{\mu_n}{n} = 0$ (Prime Number Theorem) so that the sum on the left is $(0)(0) = 0$.

Remark 1.2. We may derive our main claim " $\sum \frac{\mu(n)d(n)}{n} = 0$ " by the same argument as in Lemma 1.5 using [3], 4.4.6, in particular using $\sum_{n \leq x} \mu(n)d(n) = o(x)$. But our way shows a link with the Prime Number Theorem. We show that our convergence statement is equivalent to the Prime Number Theorem (Prop 1.11 below). $\square$

Proposition 1.1.

$$\sum_n \frac{\mu(n)d(n)}{n} = 0$$

Proof. By Cor 1.3 with $s = 1$ we have the formal Cauchy product equality since the partial sums are limits as $s \rightarrow 1^+$

$$\sum_n \frac{\mu(n)d(n)}{n} = \left(\sum_n \frac{\mu(n)}{n} \right)^2 \left(\sum_n \frac{b_n}{n} \right)$$

$\sum_n \frac{b_n}{n}$ converges absolutely to a positive limit L and $\left(\sum_n \frac{\mu(n)}{n} \right)^2 = 0$ by Lemma 1.5. So by Mertens' Criterion (if one of the series is absolutely convergent and the other is convergent then the Cauchy product converges to the product of the limits) the Cauchy product converges to $(0)(L) = 0$ as claimed. $\square$

Corollary 1.2.

$$\sum_{n \leq x} \mu(n)d(n) = o(x)$$

Proof. We apply ([2], Theorem 9.6.3) let $s_n = \sum_{i=1}^n u_i$, $\sigma_n = \frac{1}{n} \sum_{i=1}^n s_i$

$$\begin{aligned} t_n &= u_1 + 2u_2 + \cdots + nu_n \\ &= (n+1)s_n - n\sigma_n \end{aligned}$$

Hence

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{t_n}{n} &= \lim_{n \rightarrow \infty} \frac{(n+1)}{n} s_n - \lim_{n \rightarrow \infty} \sigma_n \\ &= (1)(L) - L \\ &= 0 \end{aligned}$$

Take $u_n = \frac{\mu(n)d(n)}{n}$ to obtain $\sum_{n \leq x} \mu(n)d(n) = o(x)$. $\square$

Lemma 1.4. $\sum_n \frac{\mu(n)d(n)}{n} = 0$ implies $\sum_n \frac{a_n}{n} = (\sum \frac{\mu(m)}{m})(\sum \frac{\mu(l)}{l})$ converges to 0.

Proof. As above we use Mertens on

$$(\sum_n \frac{\mu(n)d(n)}{n})(\sum_n \frac{c_n}{n}) = \sum_n \frac{a_n}{n}$$

where $\sum \frac{c_n}{n^s} = \prod_p (1 - \frac{1}{p^{2s}(1-\frac{1}{p^s})^2})^{-1}$, inverse of the earlier product. $\square$

Lemma 1.5. $\sum_n \frac{a_n}{n} = 0$ implies $\sum_n \frac{\mu(n)}{n} = 0$ (the Prime Number Theorem).

Proof. Consider the Power series $\sum_n \frac{a_n}{n} x^n$ and apply Abel's Convergence Theorem to obtain uniform convergence on an interval $[x_0, 1]$. Write the power series as the sum of partial sum and remainder

$$\sum_n \frac{a_n}{n} x^n = s_k(x) + R_k(x)$$

with

$$s_k(x) = \sum_{i+j \leq k} p_i(x)p_j(x)$$

$$R_k(x) = \sum_{i+j > k} p_i(x)p_j(x)$$

$$p_m(x) = \sum_{n=1}^m \frac{\mu(n)}{n} x^n = \text{partial sum of } \sum_n \frac{\mu(n)}{n} x^n$$

Now uniform convergence on $[x_0, 1]$ implies $R_k(x) \rightarrow 0$ uniformly as $k \rightarrow \infty$. Hence the " n^{th} term" in the convergent series for $R_k(x)$ tends to 0 uniformly. Choosing $x = 1$ and the subsequence with $i = j$, we have $p_i^2(1) \rightarrow 0$ and $p_i(1) \rightarrow 0$ as $i \rightarrow \infty$ which is the Prime Number Theorem (The use of power series may be avoided by considering $x = 1$ and only $s_k(1) + R_k(1)$ etc). $\square$

Proposition 1.2. " $\sum_n \frac{\mu(n)d(n)}{n} = 0$ " is equivalent to the Prime Number Theorem.

Proof. Combine Remark 1.4, Prop 1.7 and Lemmas 1.9, 1.10. $\square$

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G.Sudhaamsh Mohan Reddy
 Faculty of Science and Technology
 ICFAI Foundation for Higher Education
 Department of Mathematics
 Hyderabad-501203, INDIA
 e-mail: dr.sudhamshreddy@gmail.com

and

S. Srinivas Rau
 Faculty of Science and Technology
 ICFAI Foundation for Higher Education
 Department of Mathematics
 Hyderabad-501203, INDIA
 e-mail: rauindia@yahoo.co.in

and

B. Uma
 CTW, Military College
 Secunderabad, 500015, INDIA
ubidarahalli699@gmail.com

