SARAJEVO JOURNAL OF MATHEMATICS DOI: 10.5644/SJM.21.01.02-1
Vol.21 (34), No.1 (2025), 5–19

(WEAK) PRE-L-IDEALS AND ATOMS IN L-ALGEBRAS

DANIEL ABRAHAM ROMANO

ABSTRACT. The notion of L-algebras was introduced in 2008 by W. Rump. He also introduced the concept of L-ideals in this class of logical algebras. This paper introduces the concept of (weak) pre-L-ideals and atoms in L-algebras and discusses some of their important properties. A connection has been established between pre-L-ideals and atoms in L-algebras.

1. INTRODUCTION

L-algebras, which are closely related to non-classical logical algebras and quantum Yang-Baxter equation solutions, were first introduced 2008 by Rump in [7]. Since then it was shown that many non-classical logical algebras can be unified into L-algebras. In [8], Rump showed that three classes of algebras (Heyting algebras, MV-algebras, and orthomodular lattices) were associated to specializations of a bounded L-algebra. Since then, this class of logical algebras has been the focus of interest of many researchers (see, for example [1–3, 5, 6, 10]). Thus, in the articles [1, 8, 10], the relation of L-algebra to other types of logical algebras was analyzed. Categories of L-algebras are discussed in articles [3, 9]. Substructures of ideals and corresponding congruence relations in these algebras are treated in [5].

In this paper, by using the notion of L-algebras, we introduce the notions of (weak) pre-L-ideals and atoms in L-algebras. We have found examples of (weak) pre-L-ideals that are not L-ideals. The concept of pre-L-ideals designs an equivalence on A left compatible with the operation in A but the reverse does not have to be true. However, if the left congruence ρ on an L-algebra A is right-cancellative, then the set $[1]_\rho$ is a pre-L-ideal in A. The connection between the concepts of weak pre-L-ideals and atoms is established. In addition, it is shown that the family of all (weak) pre-L-ideals in an L-algebra forms a complete lattice. In Subsection 3.2 of the main part of this paper, the concept of atoms in (K)L-algebras is discussed and some of its basic properties are shown. While, for example, the set L(A) of all atoms of an L-algebra A is an anti-chain, the set L(A) of a KL-algebra A is an subalgebra in A.

2010 *Mathematics Subject Classification.* 03B50, 03G25.

Key words and phrases. L-algebra, (weak) pre-L-ideal, atoms in L-algebras, extension of L-algebras.

In addition to the previous one, two types of extensions of the (K)L-algebra A to the (K)L-algebra $A\cup\{a\}$ were designed so that the element a is an atom in $A\cup\{a\}$.

2. Preliminaries: Concept of L-algebras

Let us emphasize here that the formulas in this text are written according to the standard rules for writing formulas in Mathematical Logic. Likewise, the logical functions in these formulas have literal meanings. If a formula is not closed by a quantifier, it means that the formula is open and all variables in it are free variables. The notation $:=$, and notation $:\Longleftrightarrow$, when they appear in formulas, means that the symbol on the left side in relation to $:=$, in relation to $:\Longleftrightarrow$, is an abbreviation for the formula on the right side according to the same label respectively.

Recall that an L-algebra [7] is a set A with a binary operation $A\times A\ni(x,y)\longmapsto x\cdot y\in A$ and a 0-ary operation $1\in A$ such that

(L0) $(\forall x\in A)(1\cdot x=x)$,

(L1) $(\forall x\in A)(x\cdot x=1)$,

(L2) $(\forall x\in A)(x\cdot 1=1)$,

(L3) $(\forall x,y,z\in A)((x\cdot y)\cdot(x\cdot z)=(y\cdot x)\cdot(y\cdot z))$, and

(L4) $\forall x,y\in A)((x\cdot y=1\wedge y\cdot x=1)\Longrightarrow x=y)$.

Similarly, we define a pre-L-algebra assuming that only properties (L0)-(L3) holds [5]. These pre-L-algebras are called unital cycloids in [7]. It is easily seen that in a pre-L-algebra the element 1 with properties (L0)-(L2) is unique. It is called the logical unit of the pre-L-algebra. An example of a pre-L-algebra that is not an L-algebra can be found in [5], Example 3.1. Equation (L3) holds in most generalizations of classical logic, including intuitionistic, many-valued, and quantum logic.

On any pre-L-algebra A there is a natural quasi-order $\leqslant$, i.e., a reflexive and transitive relation, defined by $(\forall x,y\in A)(x\leqslant y:\Longleftrightarrow x\cdot y=1)$. A pre-L-algebra is an L-algebra if and only if this natural quasi-order is a partial order, that is, if and only if it is also an antisymmetric relation.

According to (L2), it is obvious that any L-algebra A satisfies

(L5) $(\forall x\in A)(x\leqslant 1)$.

Example 2.1. *Let $A=\{x,y,z,1\}$ be a set and let the operation in A be given by the following table*

$\cdot$	1	x	y	z
1	1	x	y	z
x	1	1	y	x
y	1	1	1	x
z	1	1	x	1

Then $(A,\cdot,1)$ is an L-algebra. $\square$

Example 2.2. *Let $(A,\leqslant)$ be a partially ordered set with the greatest element 1 and $A\setminus\{1\}$ be discrete. Let us define the binary operations $\cdot$ and as following:*

$$x\cdot y=\begin{cases} y & \text{if } x,y\in A\setminus\{1\},\\ y & \text{if } x=1,\\ 1 & \text{if } y=1.\end{cases}$$

Then we can easily check that $(A,\cdot,1)$ is an L-algebra. $\square$

Example 2.3. *Set $A=\langle-\infty,1]\subseteq\mathbb{R}$. Define*

$$x\cdot y=\begin{cases} 1 & \text{if } x\leqslant y,\\ y & \text{if } y<x.\end{cases}$$

Then $(A,\cdot,1)$ is an L-algebra. $\square$

Regarding examples of L-algebra, the reader can refer to the article [4].

Although the following two results are known, we include them in this paper for consistency of presentation.

Lemma 2.1 ([7]). *Let $(A,\cdot,1)$ be an L-algebra. The induced partial order is left compatible with the operation in A in the following sense*

$$(\forall x,y,z\in A)(y\leqslant z\Longrightarrow x\cdot y\leqslant x\cdot z).$$

Lemma 2.2 ([7]). *Let $(A,\cdot,1)$ be an L-algebra. The induced partial order is reverse right compatible with the operation in A in the following sense*

$$(\forall x,y,z\in A)(x\leqslant y\Longrightarrow y\cdot z\leqslant x\cdot z)$$

if and only if the following formula

(K) $(\forall x,y\in A)(x\cdot(y\cdot x)=1)$

is valid in A.

According to [3], if the L-algebra A satisfies the condition (K), then it is called a KL-algebra. A CL-algebra is an L-algebra $(A,\cdot,1)$ such that

(C) $(\forall x,y,z\in A)((x\cdot(y\cdot z))\cdot(y\cdot(x\cdot z))=1)$.

It follows that in any L-algebra A satisfying the condition (C) we have

$$(\forall x,y,z\in A)(x\cdot(y\cdot z)=y\cdot(x\cdot z)).$$

Clearly, every CL-algebra is a KL-algebra, since for any $x,y\in A$, we have $x\cdot(y\cdot x)=y\cdot(x\cdot x)=y\cdot 1=1$.

Example 2.4. *Let the set $A = \{1, x, y, z\}$ be ordered by the order relation $\leqslant$ in the following way $x \leqslant 1$, $y \leqslant 1$ and $z \leqslant 1$. If an operation $\cdot$ is determined in L as follows*

$\cdot$	1	x	y	z
1	1	x	y	z
x	1	1	y	y
y	1	z	1	z
z	1	z	z	1

Then $(A, \cdot, 1)$ is an L-algebra, but it is neither a KL-algebra nor a CL-algebra. □

Example 2.5 ([7], Example 1)**.** *Let $(X, \leqslant)$ be a partially ordered set, and $A := X \cup \{1\}$, where $x < 1$ for all $x \in X$. Let us define an internal operation in L as follows*

$$x \cdot y = \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{if } x \nleqslant y \end{cases}$$

for all $x, y \in A$. Then $(A, \cdot, 1)$ is an KL-algebra. □

3. The main results

This section is the central part of this article. It consists of three subsections. In the first, the concept of (weak) pre-L-ideals in L-algebras is introduced and its basic properties are recorded. In the second, the concept of atoms in L-algebras A is determined and the behavior of the set L(A) of all atoms in A is determined. In the last subsection, an extension of the L-algebra $(A, \cdot, 1)$ to the L-algebra $(A \cup \{a\}, *, 1)$ is created so that the element a is an atom in $A \cup \{a\}$.

3.1. **(Weak) pre-L-ideals**

A subset J of a pre-L-algebra A is an L-ideal in A ([7]) if

(J0) $1 \in J$,

(J1) $(\forall x, y \in A)((x \in J \wedge x \cdot y \in J) \Longrightarrow y \in J)$,

(J2$_R$) $(\forall x, y \in A)(x \in J \Longrightarrow (x \cdot y) \cdot y \in J)$,

(J2$_L$) $(\forall x, y \in A)(x \in J \Longrightarrow y \cdot (x \cdot y) \in J)$,

(J3) $(\forall x, y \in A)(x \in J \Longrightarrow y \cdot x \in J)$.

For the purposes of this paper, we introduce the following concepts:

Definition 3.1. *If the conditions (J0) and (J1) hold for a subset J of A, it is said to be a weak pre-L-ideal in pre-L-algebra A. If J additionally satisfies the condition (J3), then we say that it is a pre-L-ideal in A.*

Although the following statements are obvious, we still write they in the form of lemmas due to the consistency.

Lemma 3.1. *Any pre-L-ideal in a L-algebra A is a weak pre-L-ideal in A.*

Let A be an L-algebra. For a nonempty subset S in A we say that it is a subalgebra in A if the following holds

(S1) $(\forall x,y \in A)((x \in S \wedge y \in S) \implies x \cdot y \in S)$.

It should be added here that the subalgebra S of the L-algebra A obviously also satisfies the condition

(S0) $1 \in S$.

Also, in accordance with (J3), we have:

Lemma 3.2. *Any pre-L-ideal in a L-algebra A is a subalgebra in A.*

Example 3.1. *Let $A = \{1,x,y,z\}$ as in Example 2.1. The relation $\leqslant$ in this L-algebra A is given by*

$$\leqslant = \{((1,1),(x,1),(x,x),(y,1),(y,x),(y,y),(z,1),(z,x),(z,z)\}.$$

The subsets $J_0 = \{1\}$ and A are pre-L-ideals in A. The subset $K = \{1,x\}$ is not a weak pre-L-ideal in A because, for example, $x \in K$ and $x \cdot z = x \in K$ hold but $z \notin K$. The subset $T = \{1,x,y\}$ is also not a weak pre-L-ideal in A because, for example, $x \in T$ and $x \cdot z = x \in T$ hold but $z \notin T$. It can be shown that the subset $V = \{1,x,z\}$ is not a weak pre-L-ideal in A, either, because, for example, we have that $z \in V$, $z \cdot y = x \in V$ is valid but $y \notin V$. □

The following example shows that the concepts of weak pre-L-ideals, pre-L-ideals and L-ideals are mutually independent.

Example 3.2. *Let $A = \{1,x,y,z\}$ be as in Examples 2.4. The sets $\{1,x\}$ and $\{1,y\}$ are weak pre-L-ideals in A but they are not L-ideals in A. For example, for the pre-L-ideal $J = \{1,x\}$ we have $x \in J$ but $(x \cdot z) \cdot z = y \cdot z = z \notin J$ which means that the condition ($J2_R$) is not satisfied. These two weak pre-L-ideals in A are not pre-L-ideals in A. For the first of them $J = \{1,x\}$, we have $x \in J$ but $z \cdot x = z \notin J$. For the second weak pre-L-ideal $I = \{1,y\}$, we have $y \in I$ but $z \cdot y = z \notin I$. The subset $K = \{1,z\}$ is not a weak pre-L-ideal in A because, for example, $z \in K$ and $z \cdot x = z \in K$ is valid but $x \notin K$. Also, the subsets $\{1,x,y\}$ and $\{1,x,z\}$ are not weak pre-L-ideals in A.* □

It is easy to show that for a weak pre-L-ideal J in a pre-L-algebra A holds:

Lemma 3.3. *If J is a weak pre-L-ideal in a L-algebra A, then the following*

(J4) $(\forall x,y \in A)((x \in J \wedge x \leqslant y) \implies y \in J)$.

is valid.

Proof. Let $x, y \in L$ such that $x \in J$ and $x \leqslant y$. Then $x \in J$ and $x \cdot y = 1 \in J$ by (L0). Thus $y \in J$ according to (J1). □

In [9], Proposition 3.6, it is shown that a subset K of a self-similar L-algebra L (The definition of self-similarity, see in [7], pp. 2329) is an ideal in L if and only if

(0) $1 \in K$,

(i) $(\forall x, y \in L)((x \in K \wedge y \in K) \Longleftrightarrow x \cdot y \in K)$ and

(ii) $(\forall x, y \in L)(x \in K \Longrightarrow ((x \cdot y) \cdot (y \cdot x) \in K \wedge (y \cdot x) \cdot (x \cdot y) \in K))$.

Let us show that the first parts of the second and third requirements mentioned above are satisfied if K is a pre-L-ideal in an L-algebra L.

Proposition 3.1. Any pre-L-ideal J in an L-algebra A satisfies the conditions

(iii) $(\forall x, y \in A)((x \in J \wedge y \in J) \Longrightarrow (x \cdot y \in J \wedge y \cdot x \in J))$ and

(iv) $(\forall x, y \in A)(x \in J \Longrightarrow (x \cdot y) \cdot (y \cdot x) \in J)$.

If the condition

(S) $(\forall x, y \in A)(x \cdot y \leqslant y)$

is valid formula in the L-algebra A, then the reverse implication of the implication *(iii)* is also valid in A.

Proof. Let J be a pre-L-ideal in L-algebra A. Since $x \in J$ and $y \in J$, (J3) gives $x \cdot y \in J$ and $y \cdot x \in J$. It is obvious that the reverse implication is valid according to (J4) if (S) is valid in L.

Applying (J3) to $x \in J$ twice, we get $(x \cdot y) \cdot (y \cdot x) \in J$. □

If we denote by $\mathfrak{p}\mathfrak{J}(A)$ the family of all (weak) pre-L-ideals in an L-algebra A, it is obvious that $\{1\} \in \mathfrak{p}\mathfrak{J}(A)$ and $A \in \mathfrak{p}\mathfrak{J}(L)$ holds. Therefore, the family $\mathfrak{p}\mathfrak{J}(L)$ is not empty.

For a (weak) pre-L-ideal J in an L-algebra A we say that it is a minimal (weak) pre-L-ideal in A if $K \subseteq J \Longrightarrow K = J$ holds for every other (weak) pre-L-ideal K in A.

Theorem 3.1. *The family* $\mathfrak{p}\mathfrak{J}(A)$ *forms a complete lattice.*

Proof. Let $\{J_i\}_{i \in I}$ be a family of (weak) pre-L-ideals in an L-algebra A. It is easy to conclude that $\bigcap_{i \in I} J_i$ is a (weak) pre-L-ideal in A. Let $\mathfrak{Y}$ be the family of all (weak) pre-L-ideals in L-algebra A that contain the set $\bigcup_{i \in I} J_i$. Then $\cap\mathfrak{Y}$ is a minimal (weak) pre-L-ideal in A that contains $\bigcup_{i \in I} J_i$. If we put $\sqcup_{i \in I} J_i = \cap\mathfrak{Y}$ and $\sqcap_{i \in I} J_i = \bigcap_{\in I} J_i$, then $(\mathfrak{p}\mathfrak{J}(A), \sqcup, \sqcap)$ is a complete lattice. □

Corollary 3.1. *For every subset* X *in an L-algebra* A *there is a minimal (weak) pre-L-ideal containing* X.

Corollary 3.2. *Let A be an L-algebra. For every element $x \in A$ there is a minimal (weak) pre-L-ideal J_x that contains x.*

For what follows, we need the following definition: For an equivalence ρ on an L-algebra A, we say that it is a left congruence on A, or that ρ is left compatible with the operation in A, if

$$(\forall x, y, z \in A)((x, y) \in \rho \implies (z \cdot x, z \cdot y) \in \rho).$$

Theorem 3.2. *Any pre-L-ideal J in an L-algebra A defines an equivalence ρ on A left compatible with the operation in A as follows*

$$(\forall x, y \in A)((x, y) \in \rho \iff (x \cdot y \in J \wedge y \cdot x \in J)).$$

Proof. From (J0) and (L1) we get the following $(x,x) \in \rho$, which shows that ρ is a reflexive relation. ρ is a symmetric relation by definition.

Let us prove transitivity. Let $x, y, z \in A$ be such that $(x, y) \in \rho$ and $(y, z) \in \rho$. This means $x \cdot y \in J \wedge y \cdot x \in J \wedge y \cdot z \in J \wedge z \cdot y \in J$. From $y \cdot z \in J$ follows $(y \cdot x)(y \cdot z) \in J$ according to (J3). Thus $(x \cdot y) \cdot (x \cdot z) \in J$ using (L3). Now, from $x \cdot y \in J$ and $(x \cdot y) \cdot (x \cdot z) \in J$ we get $x \cdot z \in J$ by applying (J1). We have that $y \cdot x \in J$ implies $(y \cdot z) \cdot (y \cdot x) \in J$ by (J3). Then $(z \cdot y) \cdot (z \cdot x) \in J$ according (L3). Thus, from this and from $z \cdot y \in J$, according to (J1), we get $z \cdot x \in J$. Therefore, $(x, z) \in \rho$.

Let us show that ρ is left compatible with the operation in A, that is, show that the implication $(\forall x, y, z \in A)((x, y) \in \rho \implies (z \cdot x, z \cdot y) \in \rho)$ is valid. If $x \cdot y \in J$, then $(x \cdot z) \cdot (x \cdot y) \in J$ by (J3). Thus $(z \cdot x) \cdot (z \cdot y) \in J$ according to (L3). Similarly, from $y \cdot x \in J$ follows $(y \cdot z) \cdot (y \cdot x) \in J$ according to (J3). From here, according to (L3), we get $(z \cdot y)(z \cdot x) \in J$. This proves that $(z \cdot x, z \cdot y) \in \rho$ holds. □

Remark 3.1. As can be seen in the proof of the previous theorem, it is questionable whether the assertion of the theorem is valid for weak pre-L-ideals in an L-algebra.

If we denote by $\mathfrak{lC}(A)$ the family of all left congruences on the L-algebra A, then without much difficulty it can be proved that:

Theorem 3.3. *The family $\mathfrak{lC}(A)$ forms a complete lattice.*

Remark 3.2. Let ρ be an equivalence left compatible with the operation in L-algebra A. Let us determine the properties of the set $[1]_\rho =: \{u \in A : (u, 1) \in \rho\}$. It is obvious that $1 \in [1]_\rho$ because $(1, 1) \in \rho$ since ρ is a reflexive relation. Let $x \in A$ be an arbitrary element such that $x \in [1]_\rho$. Them $(x, 1) \in \rho$. Thus $(y \cdot x, y \cdot 1) = (y \cdot x, 1) \in \rho$ due to the left compatibility of the relation ρ with the operation in L and (L2). So, $y \cdot x \in [1]_\rho$, which shows that the set $[1]_\rho$ satisfies the condition (J3). Since ρ is not right compatible with the operation in A, it is not possible to prove that $[1]_\rho$ satisfies the condition (J1).

Therefore, there is no bi-unique correspondence between the family $\mathfrak{pJ}(L)$ and the family $\mathfrak{lC}(A)$ of all equivalence relations on A left compatible with the operation in A.

However, if the relation ρ is right-cancellative in the sense

$$(\forall x, y, z \in A)((x \cdot z, y \cdot z) \in \rho \implies (x, y) \in \rho),$$

then the set $[1]_\rho$ is a pre-L-ideal in A. Indeed, let $x, y \in A$ be such that $x \in [1]_\rho$ and $x \cdot y \in [1]_\rho$. This means $(x, 1) \in \rho$ and $(x \cdot y, 1) \in \rho$. Then $(x \cdot y, 1) = (x \cdot y, y \cdot y) \in \rho$ implies $(x, y) \in \rho$. Now from this and from $(x, 1) \in \rho$ it follows $(y, 1) \in \rho$ which gives $y \in [1]_\rho$. This proves that the condition (J1) is valid. We conclude that:

Theorem 3.4. *There is a correspondence between the family of all left congruences on L-algebra A that are right cancellative and the family $\mathfrak{pJ}(A)$ of all pre-L-ideals in A.*

Example 3.3 ([9], Proposition 8.2)**.** *Let $f : A \longrightarrow B$ be a morphism of L-algebras. If K is a (weak) pre-L-ideal in B, then the set $f^{-1}(K)$ is a (weak) pre-L-ideal in A. If f is surjective, then for every (weak) pre-L-ideal J in A, the set $f(J)$ is a (weak) pre-L-ideal in B.* □

Example 3.4. *Consider $L = \{x_0 = 1, x_1, x_2, ...\}$ as a countable set and define the operation on L as follows:*

$$x_i \cdot y_j = \begin{cases} 1 & \text{if } j \leqslant i, \\ x_j & \text{if } i < j \end{cases}$$

With a little effort, it can be shown that $(L, \cdot, 1)$ is an L-algebra. The set $J(n) = \{1, x_1, ..., x_n\}$ is a pre-L-ideal in L for any $n \geqslant 1$. Indeed, it is clear that $1 \in J(n)$. Let $x_i, x_j \in L \setminus \{1\}$ be such that $x_i \in J(n)$ and $x_i \cdot x_j \in J(n)$. This mean $i \leqslant n$. If $j \leqslant i \leqslant n$, we have $x_i \cdot x_j = 1 \in J(n)$. If $i < j$, then $x_i \cdot x_j = x_j \in J(n)$ so $j \leqslant n$. Thus, $x_j \in J(n)$. If $x_i \in J(n)$, then $i \leqslant n$. Therefore, for each $x_j \in L$ we have $x_i \cdot x_j = 1 \in J(n)$ or $x_i \cdot x_j = x_j \in J(n)$. This shows that $J(n)$ is a pre-L-ideal in L. □

In what follows, we need the following lemma:

Lemma 3.4 ([3], Theorem 5.1)**.** *The direct product $\prod_{i\in I} L_i$ of a family $\{L_i\}_{i\in I}$ of L-algebras is an L-algebra.*

We conclude this section with the following theorem:

Theorem 3.5. *Let $\{L_i\}_{i\in I}$ be a family of L-algebras and let $\{J_j\}_{j\in I}$ be family of pre-L-ideals in $L_i =: (L_i, \cdot_i, 1_i)$ respectively. Then the direct production $\prod_{i\in I} J_i$ is a pre-$\prod_{i\in I} L_i$-ideal in $\prod_{i\in I} L_i$.*

Proof. As is well known, $\prod_{i\in I} L_i$ is the set of all maps $f : I \longrightarrow \bigcup_{i\in I} L_i$ such that $f(i) \in L_i$ for all $i \in I$. If 1_i and $\cdot_i$ are the logical unit and the operation in L-algebra

L_i, we consider the map $1 : I \longrightarrow \bigcup_{i\in I} L_i$ defined by $1(i) = 1_i$. For any $f,g \in \prod_{i\in I} L_i$ define the operation $\cdot$ on $\prod_{i\in I} L_i$ by $(f \cdot g)(i) = f(i) \cdot_i g(i)$, for all $i \in I$. It is easy to check that $(\prod_{i\in I} L_i, \cdot, 1)$ is an L-algebra, let's check that $\prod_{i\in I} J_i$ satisfies the conditions (J0), (J1) and (J3):

It is obvious, according to the determination of the function 1, that $1 \in \prod_{i\in I} J_i$ holds.

Let $x,y \in \prod_{i\in I} L_i$ be arbitrary elements such that $x \in \prod_{i\in I} J_i$ and $x \cdot y \in \prod_{i\in I} J_i$. This means $(\forall i \in I)(x(i) \in J_i)$ and $(\forall \in I)((x\cdot y)(i) = x(i) \cdot_i y(i) \in J_i)$. Then $(\forall i \in I)(y(i) \in J_i)$ according to (J1). So, $y \in \prod_{i\in I} J_i$.

Let $x,y \in \prod_{i\in I} L_i$ be arbitrary elements such that $x \in \prod_{i\in I} J_i$. Then, we have $(\forall i \in I)(x(i) \in J_i)$. Thus $(\forall \in I)(y(i) \cdot_i x(i) \in J_i)$ according to (J3). This means $x \cdot y \in \prod_{i\in I} J_i$. □

In the following two examples we give left congruences on an L-algebra $\prod_{i\in I} L_i$.

Example 3.5. *Let I be a set and $\mathfrak{J}$ be a subfamily of $\mathcal{P}(I)$ such that*

$$\emptyset \in \mathfrak{J}, A \subseteq B \wedge B \in \mathfrak{J} \Longrightarrow A \in \mathfrak{J}, A \in \mathfrak{J} \wedge B \in \mathfrak{J} \Longrightarrow A \cup B \in \mathfrak{J}.$$

If $\{L_i\}_{i\in i}$ is a family of L-algebras, then the relation q on $\prod_{i\in I} L_i$, defined by

$$(x,y) \in q \Longleftrightarrow \{i \in I : x(i) \neq y(i)\} \in \mathfrak{J},$$

is an equality relation on the product $\prod_{i\in I} L_i$ left compatible with the operation in $\prod_{i\in I} L_i$.

Let $x \in \prod_{i\in I} L_i$ be arbitrary element. Then $\{i \in I : x(i) \neq x(i)\} = \emptyset \in \mathfrak{J}$. This means $(x,x) \in q$ for every $x \in \prod_{i\in I} L_i$, that is, the relation q is reflexive. It is obvious that q is a symmetric relation. Let us prove that q is transitive. Let $x,y,z \in \prod_{i\in I} L_i$ be such that $(x,y) \in q$ and $(y,z) \in q$. This means $\{i \in I : x(i) \neq y(i)\} \in \mathfrak{J}$ and $\{i \in I : y(i) \neq z(i)\} \in \mathfrak{J}$. Then

$$\{i \in I : x(i) \neq z(i)\} \subseteq \{i \in I : x(i) \neq y(i)\} \cup \{i \in I : y(i) \neq z(i)\} \in \mathfrak{J}.$$

Thus $\{i \in I : x(i) \neq z(i)\} \in \mathfrak{J}$. Hence $(x,z) \in q$. So, q is an equivalence on $\prod_{i\in I} L_i$.

Let $x,y,z \in \prod_{i\in I} L_i$ be arbitrary elements such that $(x,y) \in q$, i.e. $\{i \in I : x(i) \neq y(i)\} \in \mathfrak{J}$. Recall that holds

$$(\forall i \in I)(x(i) = y(i) \Longrightarrow z(i) \cdot_i x(i) = z(i) \cdot_i y(i))$$

since the operation $\cdot$ is extensive with respect to the equality $=$, and that the contraposition also holds

$$(\forall i \in I)((z\cdot x)(i) \neq (z\cdot y)(i) \Longrightarrow x(i) \neq y(i)).$$

Now, we have

$$\{i \in I : (z\cdot x)(i) \neq (z\cdot y)(i)\} \subseteq \{i \in I : x(i) \neq y(i)\} \in \mathfrak{J}.$$

Thus $\{(z\cdot x)(i)\neq(z\cdot y)(i)\}\in\mathfrak{J}$. *Therefore* $(z\cdot x,z\cdot y)\in q$. $\square$

Example 3.6. *Let* I *be a set and* $\mathfrak{K}$ *be a subfamily of* $\mathcal{P}(I)$ *such that*

$$I\in\mathfrak{K}, A\subseteq B\wedge A\in\mathfrak{K}\Longrightarrow B\in\mathfrak{K}, A\in\mathfrak{K}\wedge B\in\mathfrak{K}\Longrightarrow A\cap B\in\mathfrak{K}.$$

If $\{L_i\}_{i\in i}$ *is a family of L-algebras, then the relation* q *on* $\prod_{i\in I}L_i$*, defined by*

$$(x,y)\in q\iff\{i\in I: x(i)=y(i)\}\in\mathfrak{K},$$

is an equality relation on the product $\prod_{i\in I}L_i$ *left compatible with the operation in* $\prod_{i\in I}L_i$. $\square$

3.2. **Atoms in L-algebras**

This subsection introduces the concept of atoms in (K)L-algebras. Also, this concept is connected with the concept of weak pre-L-ideals, which justifies the introduction of this last concept here.

Definition 3.2. *An element* $a\,(\neq 1)$ *of an L-algebra* A *is said to be an atom in* A *if the following holds*

(A) $(\forall x\in A)(a\leqslant x\Longrightarrow(x=a\vee x=1))$.

We denote the set of all atoms in the L-algebra A *by* $L(A)$.

In the general case, the L-algebra A does not have to have atoms. So, it can be $L(A)=\emptyset$. For example, in the L-algebra A in Example 2.3 is $L(A)=\emptyset$.

Example 3.7. *Let* $A=\{1,x,y,z\}$ *as in Example 2.1. Then the element* x *is an atom in* A*, so* $L(A)=\{x\}$. $\square$

Proposition 3.2. Let A be an l-algebra A and $(1\neq)\,a\in A$. If the subset $\{1,a\}$ is a weak pre-L-ideal in A, then a is an atom in A.

Proof. Let x be an arbitrary element in A such that $a\leqslant x$. This means $a\in\{1,a\}$ and $a\cdot x=1\in\{1,a\}$. From here, according to (J1), we have $x\in\{1,a\}$. Therefore, $x=1$ or $x=a$ which means that a is an atom in A. $\square$

Remark 3.3. The converse of this proposition need not be valid. For example, in the L-algebra $A=\{1,x,y,z\}$ in Example 2.1, the element x is an atom in A (see, Example 3.7) while, on the other hand, the set $\{1,x\}$ is not a weak pre-L-ideal in A (see, Example 3.1).

Example 3.8. *Let* $A=\{1,x,y,z\}$ *as in Example 2.4. The elements* x *and* y*, according to Proposition 3.2, are atoms in* A *because the subsets* $\{1,x\}$ *and* $\{1,y\}$ *are weak pre-L-ideals in* A *(see, Example 3.2). Also, the element* z *is an atom in* A *even though the subset* $\{1,z\}$ *is not a weak pre-L-ideal in* A *(see, Example 3.2). So* $L(A)=\{x,y,z\}$. $\square$

Let us note that in the L-algebra A in the previous example, $L(A) = A \setminus \{1\}$ holds. Thus, there exist L-algebras in which every element, except 1, is an atom in them.

One of the main characteristics of the set L(A) of all atoms of an L-algebra A is given by the following theorem:

Theorem 3.6. *The set $L(A)$ of all atoms in an L-algebra A is an anti-chain.*

Proof. Let a and b be atoms in an L-algebra A such that $a \neq b$. Suppose $a \leqslant b$. Then it would be $a = b$ or $b = 1$, according to (A). As both options are impossible, we conclude that it must be $a \nleqslant b$. In the same way, it can be proved that $b \nleqslant a$. □

Theorem 3.7. *If A is a KL-algebra, then it holds*
$(\forall a, b \in L(A))(a \cdot b = b \wedge b \cdot a = a)$.

Proof. Let A be a KL-algebra and let $a, b \in L(A)$. Then $a \leqslant b \cdot a$ and $b \leqslant a \cdot b$. This $b \cdot a = a$ or $b \cdot a = 1$, and $a \cdot b = a$ or $a \cdot b = 1$. Since the options $b \cdot a = 1$ and $a \cdot b = 1$ are not possible according to Theorem 3.6, we get $b \cdot a = a$ and $a \cdot b = b$. □

Corollary 3.3. *If A is a KL-algebra, then $L(A) \cup \{1\}$ is a subalgebra in A.*

Example 3.9. *Let $A = \{1, a, b, c, d, e\}$ and let the operation $\cdot$ on A be determined as follows*

$\cdot$	1	a	b	c	d	e
1	1	a	b	c	d	e
a	1	1	b	c	d	e
b	1	1	1	c	d	e
c	1	a	b	1	d	e
d	1	a	b	c	1	e
e	1	a	b	c	d	1

Then A is a KL-algebra. The order relation on A is given by

$$\leqslant = \{(1,1),(a,1),(a,a),(a,1),(b,a),(b,b),(c,1),(c,c),(d,1),(d,d),(e,1),(e,e)\}.$$

The subsets $J_1 = \{1, a\}$, $J_2 = \{1, c\}$, $J_3 = \{1, d\}$ and $J_4 = \{1, e\}$ are pre-L-ideals in A. Therefore, a, c, d, e are atoms in A according to Proposition 3.2. Hence $L(A) = \{a, c, d, e\} \subset A \setminus \{1\}$. □

It is easy to see that this L-algebra illustrates Theorem 3.7 and Corollary 3.3. Subset $L(A) \cup \{1\}$ is a proper subalgebra in L-algebra A.

3.3. **Extension of L-algebras**

In this subsection, we will show the construction of the (K)L-algebra $(A \cup \{a\}, *, 1)$ obtained by the extension of given (K)L-algebra $(A, \cdot, 1)$ with an element $a (\notin A)$ such that the element a is an atom in $A \cup \{a\}$.

Theorem 3.8. *Let $(A, \cdot, 1)$ be a (K)L-algebra and $a \notin A$. We can create a (K)L-algebra $(A \cup \{a\}, *, 1)$ that the element a is an atom in $A \cup \{a\}$.*

Proof. Let A be a (K)L-algebra and $a \notin A$. Let us put $B = A \cup \{a\}$ and design the operation $*$ in B as follows

$$x * y = \begin{cases} x \cdot y & \text{for } x \in A \ \wedge \ y \in A, \\ 1 & \text{for } x \in A \setminus \{1\} \ \wedge \ y = a, \\ a & \text{for } x = 1 \ \wedge \ y = a, \\ y & \text{for } x = a \ \wedge \ y \in A \setminus \{1\}, \\ 1 & \text{for } x = a \ \wedge \ y = 1, \\ 1 & \text{for } x = a \ \wedge \ y = a. \end{cases}$$

Since it is obvious that the set B satisfies the conditions (L0), (L1), (L2) and (L4), it is necessary to prove that the set B satisfies the condition (L3). This is done by replacing one, two or all three variables in the formula (L3) with the constant a. For the sake of illustration, we list some of those procedures:

For $x = a$, $y = z = 1$, we have $(a \cdot 1) \cdot (a \cdot 1) = 1 \cdot 1 = 1$ and $(1 \cdot a) \cdot (1 \cdot 1) = a \cdot 1 = 1$.
For $x = a$, $y \neq 1$ and $z \neq 1$, we have $(a \cdot y) \cdot (a \cdot z) = y \cdot z$ and $(y \cdot a) \cdot (y \cdot z) = 1 \cdot (y \cdot z) = y \cdot z$.
For $x = a$, $y = a$ and $z \neq 1$, we have $(a \cdot a) \cdot (a \cdot z) = 1 \cdot z = z$.

If A is a KL-algebra, then the procedure for checking the validity of formula (K) in the L-algebra B can be done in the same way.

Let us show that a is an atom in B. If we take $x \in B$ such that $a \leqslant x$, we have $a \cdot x = 1$. On the other hand, $a \cdot x = x$ so $x = 1$, which proves that a is an atom in (K)L-algebra B. $\square$

To illustrate the previous theorem, we will use the L-algebra in Example 2.1.

Example 3.10. *Let $A = \{1, x, y, x\}$ as in Example 2.1. Let's put $B = A \cup \{a\}$ and let the operation $*$ be given by the following table*

$*$	1	x	y	z	a
1	1	x	y	z	a
x	1	1	y	x	1
y	1	1	1	x	1
z	1	1	x	1	1
a	1	x	y	z	1

Then $(B,*,1)$ is an KL-algebra. The relation $\leqslant$ in this L-algebra B is given by $\leqslant = \{(1,1),(x,1),(x,x),(x,a),(y,1),(y,x),(y,y),(y,a),(z,1),(z,x),(z,z),(z,a),(a,1),(a,a)\}$. *Let $u \in B$ be such that $a \leqslant u$. This means $a*u = 1$. On the other hand, we have $a*u = u$ which gives $u = 1$. This shows that the element a is an atom in B. Besides, the subset $\{1,a\}$ is a pre-L-ideal in B.* □

A different extension of the (K)L-algebra $(A,\cdot,1)$ to the (K)L-algebra $(A\cup\{a\},*,1)$ can be done so that $L(B) = L(A)\cup\{a\}$ holds.

Theorem 3.9. *Let $(A,\cdot,1)$ be a (K)L-algebra and $a \notin A$. One can create a (K)L-algebra $(A\cup\{a\},*,1)$ such that $L(A\cup\{a\}) = L(A)\cup\{a\}$ holds.*

Proof. Let A be a (K)L-algebra and $a \notin A$. Let us put $B = A\cup\{a\}$ and design the operation $*$ in B as follows

$$x*y = \begin{cases} x\cdot y & \text{for } x\in A \ \wedge \ y\in A, \\ 1 & \text{for } x\in A\setminus L(A)\cup\{1\} \ \wedge \ y = a, \\ a & \text{for } x\in L(A)\cup\{1\} \ \wedge \ y = a, \\ y & \text{for } x = a \ \wedge \ y\in A\setminus\{1\}, \\ 1 & \text{for } x = a \ \wedge \ y = 1, \\ 1 & \text{for } x = a \ \wedge \ y = a. \end{cases}$$

Since it is obvious that the set B satisfies the conditions (L0), (L1), (L2) and (L4), it is necessary to prove that the set B satisfies the condition (L3). This is done by replacing one, two or all three variables in the formula (B3) with the constant a. Since these procedures are of a technical nature, we omit them. □

Example 3.11. *Let $A = \{1,a,b,c,d,e\}$ as in example 3.9. According to Example 3.9, the set of all atoms of the algebra A is $L(A) = \{a,c,d,e\}$. Let us put $B = A\cup\{w\}$ and let the operation $*$ be given by the following table*

$\cdot$	1	a	b	c	d	e	w
1	1	a	b	c	d	e	w
a	1	1	b	c	d	e	w
b	1	1	1	c	d	e	1
c	1	a	b	1	d	e	w
d	1	a	b	c	1	e	w
e	1	a	b	c	d	1	w
w	1	a	b	c	d	e	1

Then B is a KL-algebra. The order relation $\leqslant$ on A is given by $\leqslant = \{(1,1),(a,1),(a,a),(a,1),(b,a),(b,b),(b,w),(c,1),(c,c),(d,1),(d,d),(e,1),(e,e),(w,1),(w,w)\}$. *Let x be an arbitrary element in B such that $w \leqslant x$. While on the one hand we have $w*x = 1$, on the other hand we have $w*x = x$ which gives $x = 1$. This shows that w*

is an atom in B. Therefore, $L(B) = L(A) \cup \{w\}$. In addition to the previous one, the subset $\{1, w\}$ is a pre-L-ideal of the KL-algebra B. □

4. Final comments

The notion of L-algebras was introduced in 2008 by W. Rump. He also introduced the concept of L-ideals in this class of logical algebras. Since then, this class of logical algebras has been the focus of interest of many researchers. This article introduces the concept of (weak) pre-L-ideals in L-algebras and connects it to the concept of atoms in this class of logical algebras. Also, two types of extensions of the L-algebra A to the L-algebra $A \cup \{a\}$ were created so that a is an atom in $A \cup \{a\}$. The space opened by introducing the concept of (weak) pre-L-ideal as a generalization of the concept of L-ideals and its connection with the concept of atoms in L-algebras should be filled with new knowledge about these classes of substructures in L-algebras.

Acknowledgement The author would like to thank the reviewer for the very useful suggestions that enabled the material presented in this article to be more consistent.

References

[1] M.A. Kologani, *Relations between L-algebras and other logical algebras*, Journal of Algebraic Hyperstructures and Logical Algebras, 4(1)(2023), 27–46. https://doi.org/ 10.52547/HATEF.JAHLA.4.1.3

[2] L.C. Ciungu, *Results in L-algebra*, Algebra Univers., 82(1)(2021), 7–18. https://doi.org/ 10.1007/s00012-020-00695-1

[3] L.C. Ciungu, *The category of L-algebras*, Transactions on Fuzzy Sets and Systems (TFSS), 1(2)(2022), 142–159. DOI: http://doi.org/10.30495/tfss.2022.1959857.1034

[4] C. Dietzel, P. Menchón, and L. Vendramin, *On the enumeration of finite L-algebras*, arXiv:2206.04955v3 [math.LO]

[5] A. Facchini, M. Gran, and M. Pompili, *Ideals and congruences in L-algebras and pre-L-algebras*, arXiv:2305.19042v1 [math.CT]

[6] L. Mao, X. Xin, and S. Zhang, *The extension of L-algebras and states*, Fuzzy Sets and Systems, 455(2023), 35–52. https://doi.org/10.1016/j.fss.2022.10.017

[7] W. Rump, *L-algebras, self-similarity, and L-groups*, J. Algebra, 320(6)(2008), 2328-2348. https://doi.org/10.1016/j.jalgebra.2008.05.033

[8] W. Rump, *L-algebras and three main non-classical logics*, Ann. Pure Appl. Logic., 173(7)(2022), ID: 103121. DOI 10.1016/j.apal.2022.103121

[9] W. Rump, *The category of L-algebras*, Theory Appl. Categ., 39(21)(2023), 598-624.

[10] J. Wang, W. Yali, and Y. C. Yang, *Basic algebras and L-algebras*, Soft Comput, 24(19)(2020), 14327-14332.
https://doi.org/10.1007/s00500-020-05231-w

(Received: August 04, 2024)
(Revised: July 13, 2025)

Daniel Abraham Romano
International Mathematical Virtual Institute
6, Kordunaška ulica, 78000 Banja Luka
Bosnia and Herzegovina
e-mail: *daniel.a.romano@hotmail.com*