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CUBIC EIGENVALUE PROBLEMS

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Dedicated to Professor Mehmed Nurkanović on the occasion of his 65^{th} *birthday*

ABSTRACT. In this paper, we observe cubic eigenvalue problems, which belong to a special class of nonlinear eigenvalue problems. The degree of a cubic eigenvalue problem is relatively small, which allows us to determine some important properties of these problems. We present an algorithm to determine whether a cubic pencil is hyperbolic or not. Also, a definite cubic pencil will be considered. We use a variational characterization as a tool for solving cubic eigenvalue problems and compare the results with the application of the linearization method.

1. INTRODUCTION

The cubic eigenvalue problem is a special case among polynomial eigenvalue problems. We will start by defining a polynomial eigenvalue problem as:

$$P_l(\lambda)x = 0,\ x \neq 0 \tag{1.1}$$

where

$$P_l(\lambda) = \sum_{j=0}^{l} \lambda^j A_j,\ A_j \in \mathbb{C}^{n\times n},\ A_l \neq 0,\ A_j^H = A_j. \tag{1.2}$$

The most common way for solving polynomial eigenvalue problems is linearization (see [7]). In [2], the authors proved that Hermitian matrices from equation (4.2) which allow definite linearization, are characterized with the property that there exists $\mu \in \mathbb{R}\cup\{\infty\}$ so that $P_l(\mu)$ is positive definite and for each $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x} \neq \mathbf{0}$ the scalar polynomial

$$f(\lambda;\mathbf{x}) = \mathbf{x}^H P_l(\lambda)\mathbf{x} \tag{1.3}$$

has l distinct roots in $\mathbb{R}\cup\{\infty\}$. In mathematical literature, these Hermitian matrix polynomials are called definite. More about polynomial eigenvalue problems can be found in [10]. Most commonly observed polynomial eigenvalue problems are quadratic eigenvalue problems, since for solving these problems we can use

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linearization and variational characterization. Variational characterization for quadratic eigenvalue problems can be easily applied because of the simplicity of defining the related functionals.

The cubic eigenvalue problem is defined as

$$P_3(\lambda)\mathbf{x} = 0,\ \mathbf{x} \neq \mathbf{0} \tag{1.4}$$

where

$$P_3(\lambda) = \lambda^3 A_3 + \lambda^2 A_2 + \lambda A_1 + A_0,\ \ A_j \in \mathbb{C}^{n\times n},\ A_j \neq 0,\ j = \overline{0,3}, \tag{1.5}$$

where A_j are Hermitian.

Tsung-Min Hwang et al. [4] proposed multiple Jacob-Davidson type methods for computing interior eigenpairs of large-scale cubic eigenvalue problems.

In this paper, we will observe cubic eigenvalue problems and their properties. The paper is organized as follows: In Section 2, in order to define functionals we will be using Vieta's formulas. In Section 3, we consider variational characterization, in Section 4, we observe the hyperbolic cubic eigenvalue problem and algorithm for determining whether the cubic eigenvalue problem is hyperbolic or not. Definite cubic pencils are considered in Section 5. We present numerical results in Section 6, and the Conclusion in Section 7.

2. Roots of Cubic Polynomials

Multiplying the equation (1.4) on the left with $\mathbf{x}^H$ we obtain the cubic equation with real coefficients:

$$\mathbf{x}^H P_3(\lambda)\mathbf{x} = \lambda^3 \mathbf{x}^H A_3 \mathbf{x} + \lambda^2 \mathbf{x}^H A_2 \mathbf{x} + \lambda \mathbf{x}^H A_1 \mathbf{x} + \mathbf{x}^H A_0 \mathbf{x} = 0, \tag{2.1}$$

by observing Vieta's formula and the corresponding discriminant of the cubic equation. Using a proper change of variables, we obtain a cubic equation in one variable as follows:

$$a\lambda^3 + b\lambda^2 + c\lambda + e = 0,\ a \neq 0, \tag{2.2}$$

and the discriminant of a cubic equation is defined by

$$d = -4b^3 e + b^2 c^2 - 4ac^3 + 18abce - 27a^2 e^2. \tag{2.3}$$

Roots of the cubic equation with real coefficients depend on the sign of the discriminant: if $d < 0$ there is one real and two complex roots, if $d > 0$ then there are three different real roots and if $d = 0$ then equation has three solutions with at least two being equal.

These properties hold only for equations with real coefficients. Cubic equations were known to the ancient Babylonians, Greeks, Chinese, Indians and Egyptians [1,3,12]. Although these equations were known earlier, only after finding Cardano formulas, we have found the solutions to these equations.

Through linear Tschirnhaus-Transformation

$$\lambda = z - \frac{b}{3a}, \tag{2.4}$$

the cubic equation is reduced to the short reduced form $z^3 + pz + q = 0$ that can be solved in many ways. If we go back to the equation (2.1), for determining eigenvalues it is enough to observe the following equation:

$$z^3 x^H x + z x^H C_1 x + x^H C_0 x = 0,\ x \neq 0. \tag{2.5}$$

From equation (2.4) it is easy to calculate the corresponding λ. The corresponding discriminant of the reduced form is

$$d = -4c^3 - 27e^2. \tag{2.6}$$

3. Variational Characterization and Sylvester's Law of Inertia

One of the most common tools for solving nonlinear eigenvalue problems is a minmax characterization or a variational characterization. Let us observe the nonlinear eigenvalue problem:

$$T(\lambda)x = 0, \tag{3.1}$$

where $T(\lambda) \in \mathbb{C}^{n\times n}$, $\lambda \in J$, is a family of Hermitian matrices that depend continuously on the paramater $\lambda \in J$, and J is a real open interval which may be unbounded.

Important conditions for application of variational characterization, that must be satisfied, are:

(A_1) for every fixed $x \in \mathbb{C}^n$, $x \neq 0$, the scalar real equation

$$f(\lambda; x) := x^H T(\lambda)x = 0 \tag{3.2}$$

has at most one solution $\lambda =: p(x) \in J$.

(A_2) for every $x \in \mathcal{D}$ and every $\lambda \in J$ with $\lambda \neq p(x)$,

$$(\lambda - p(x))f(\lambda; x) > 0. \tag{3.3}$$

If p is defined on $\mathcal{D} = \mathbb{C}^n \setminus \{0\}$, then the problem (3.1) is called overdamped.

Important theorem that refers to minmax characterization is:

Theorem 3.1. *[11] Let $T(\lambda) \in \mathbb{C}^{n\times n}$, $\lambda \in J$, be a family of Hermitian matrices depending continuously on the parameter $\lambda \in J$, where J is an open interval in $\mathbb{R}$, such that conditions (A_1) and (A_2) are satisfied. Then the following statements hold:*

(i) For every $l \in \mathbb{N}$ there is at most one lth eigenvalue of $T(\cdot)$ which can be characterized by

$$\lambda_l = \min_{V \in H_l,\ V \cap \mathcal{D} \neq 0}\ \sup_{v \in V \cap \mathcal{D}} p(v). \tag{3.4}$$

(ii) *If*

$$\lambda_l := \inf_{V \in H_l,\ V \cap \mathcal{D} \neq 0} \ \sup_{v \in V \cap \mathcal{D}} p(v) \in J$$

for some $l \in \mathbb{N}$*, then* λ_l *is the lth eigenvalue of* $T(\cdot)$ *in J, and* (3.4) *holds.*

(iii) *If there exist the kth and the lth eigenvalue* λ_k *and* λ_l *in* J $(k < l)$*, then J contains the jth eigenvalue* $\lambda_j (k \leqslant j \leqslant l)$ *as well with* $\lambda_k \leqslant \lambda_j \leqslant \lambda_l$*.*

(iv) *Let* $\lambda_1 = \inf_{x \in \mathcal{D}} p(x) \in J$ *and* $\lambda_l \in J$*. If the minimum in* (3.4) *is attained for an l-dimensional subspace V, then* $V \subset \mathcal{D} \cup \{0\}$*, and* (3.4) *can be replaced by*

$$\lambda_l = \min_{V \in H_l,\ V \subset \mathcal{D} \cup \{0\}} \ \sup_{v \in V, v \neq 0} p(v).$$

(v) $\tilde{\lambda}$ *is an lth eigenvalue if and only if* $\mu = 0$ *is teh lth largest value of the linear eigenproblem* $T(\tilde{\lambda})x = \mu x$*.*

(vi) *The minimum in* (3.4) *is attained for the invariant subspace of* $T(\lambda_l)$ *corresponding to its l larges eigenvalues.*

Theorem 3.2. *[5] Assume that* $T : J \to \mathbb{C}^{n \times n}$ *satisfies the conditions of the minmax characterization in Theorem 3.1, and assume that the nonlinear eigenvalue problem* (3.1) *is overdamped, i.e., for every* $x \neq 0$*, Equation* (3.2) *has a unique solution* $p(x) \in J$*.*
For $\sigma \in J$*, let* (n_p, n_n, n_z) *be the inertia of* $T(\sigma)$*. Then the nonlinear eigenproblem* (3.1) *has n eigenvalues in J,* n_p *of which are less than* σ*,* n_n *exceed* σ*, and for* $n_z > 0$*,* σ *is an eigenvalue of geometric multiplicity* n_z*.*

Let us observe the following condition

(A_2') for every $x \in \mathcal{D}$ and every $\lambda \in J$ with $\lambda \neq p(x)$,

$$(\lambda - p(x)) f(\lambda; x) < 0, \tag{3.5}$$

instead of the condition (A_2). Then Sylvester's law is as follows:

Theorem 3.3. *[5] Assume that* $T : J \to \mathbb{C}^{n \times n}$ *satisfies the conditions of the minmax characterization* (A_1) *and* (A_2')*, and assume that the nonlinear eigenvalue problem* (3.1) *is overdamped, i.e., for every* $x \neq 0$*, Equation* (3.2) *has a unique solution* $p(x) \in J$*.*
For $\sigma \in J$*, let* (n_p, n_n, n_z) *be the inertia of* $T(\sigma)$*. Then the nonlinear eigenvalue problem* (3.1) *has n eigenvalues in J,* n_p *that exceed* σ*,* n_n *are less than* σ*, and for* $n_z > 0$*,* σ *is an eigenvalue of geometric multiplicity* n_z*.*

4. Hyperbolic Cubic Pencils

The cubic matrix polynomial

$$P_3(\lambda) = \sum_{j=0}^{3} \lambda^j A_j,\ A_j^H = A_j,\ j = \overline{0,3}, \tag{4.1}$$

where A_3 is a positive definite matrix,is hyperbolic, if $f(\lambda;x) := x^H P_3(\lambda)x = 0$ for $x \neq 0$ has exactly 3 distinct real roots.

Remark 4.1. For a positive definite matrix A we will use the notation $A > 0$, and for a negative definite matrix A we will use the notation $A < 0$. Positive semidefinitness of a matrix A, we will denote with $A \geq 0$, and negative semidefinitness of matrix A, we will denote with $A \leq 0$.

For a hyperbolic pencil there are 3 disjoint open intervals $\Delta_j \subset \mathbb{R},\ j = \overline{1,3}$, such that the equation $P_3(\lambda)x = 0$ has exactly n eigenvalues in each interval Δ_j which allow for a minmax characterization. In order to fix the numeration let $\sup \Delta_j < \inf \Delta_{j-1}$.

The corresponding functionals are denoted as follows:

$$p_{+-}(x) \text{ for } 3p_{+-}(x)^2 x^H A_3 x + 2p_{+-}(x) x^H A_2 x + x^H A_1 x > 0, \text{ and } p_{+-}(x) \in \Delta_3, \tag{4.2}$$

$$p_{-}(x) \text{ for } 3p_{-}(x)^2 x^H A_3 x + 2p_{-}(x) x^H A_2 x + x^H A_1 x < 0, \text{ and } p_{-}(x) \in \Delta_2, \tag{4.3}$$

$$p_{+}(x) \text{ for } 3p_{+}(x)^2 x^H A_3 x + 2p_{+}(x) x^H A_2 x + x^H A_1 x > 0, \text{ and } p_{+}(x) \in \Delta_1. \tag{4.4}$$

The question of hyperbolicity is not that simple. Every hyperbolic pencil must have all real eigenvalues, but the converse does not hold. The following example shows a cubic pencil that has all real eigenvalues, but it is not hyperbolic.

Example 4.1. *Let us observe a cubic pencil*

$$\lambda^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \lambda^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix} + \lambda \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 4 & -4 \\ 0 & -4 & 4 \end{pmatrix} = 0.$$

The corresponding eigenvalues are:

$$\begin{aligned} &-2.019593030949422e+00 \\ &-1.414213562373098e+00 \\ &-1.357332128074832e+00 \\ &-2.645960456369809e-01 \\ &2.473012331625537e-01 \\ &1.393476608283088e+00 \\ &1.414213562373066e+00 \\ &2.000000000000002e+00 \\ &4.000743363215627e+00. \end{aligned}$$

Hence, this problem is not hyperbolic, because for $x = [0.231; 0.8907; 0.1286]$ *it follows that* $d(x) = -17.0309$*, which means that for the vector x the equation has one real and two complex solutions.*

Now we will present an algorithm for determining whether a cubic matrix polynomial is hyperbolic or not. This algorithm is a modification of the algorithm from [9], that determines whether a quadratic matrix polynomial is hyperbolic or not. This modification is not as simple as it may seem at first glance. It is not enough to change the discriminant $d(x_k)$, because this problem reduces to a cubic equation that always has a real solution. Therefore, in addition to the largest eigenvalue and the corresponding eigenvector of the matrix $P_3(\lambda)$, we must also observe the smallest eigenvalue and the corresponding eigenvector. To simplify the presentation of the algorithm we will define

$$\begin{aligned} d(x) \quad = \quad & -4(x^H A_2 x)^3 (x^H A_0 x) + (x^H A_2 x)^2 (x^H A_1 x)^2 \\ & -4(x^H A_3 x)(x^H A_1 x)^3 + 18(x^H A_3 x)(x^H A_2 x)(x^H A_1 x)(x^H A_0 x) \\ & -27(x^H A_3 x)^2 (x^H A_0 x)^2 < 0. \end{aligned}$$

Figure 1 shows that the function $f(\theta)$, which has the same sign as the discriminant $d(x)$ can have negative values, that is it intersects the plane $z = 0$, if the algorithm modification mentioned above is not introduced, and the algorithm without the modification can show hyperbolicity even when that is not the case.

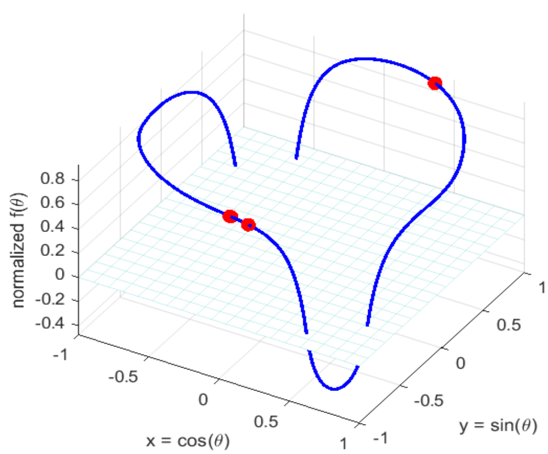

FIGURE 1. Eigenvectors corresponding to σ_k as they converge using the modification of algorithm 2 from [9]

In order to understand the meaning of the function $f(\theta)$ and the need to modify the algorithm let us observe the following example: For $n = 2$ we can present the graph of the function $d(x)$, where $x = (x_1, x_2)$ and $\|x\| = 1$, as a curve in space, as in Figure 1. In the previous figure, we have plotted the function $\frac{d(x)}{1+|d(x)|}$. All values $x=(x_1,x_2)$, assuming $\|x\|$ can be represented as $x=(\cos\theta,\sin\theta)$. Hence the blue curve shown in the Figure 1 is a plot of a function $f(\theta)=\frac{d(\theta)}{1+|d(\theta)|}$, where $d(\theta)=d(x), x=(\cos\theta,\sin\theta)$. The red points $((\cos\theta_{k-1},\sin\theta_{k-1}), f(\theta_{k-1}))$, where $(\cos\theta_{k-1},\sin\theta_{k-1})$ is the eigenvector corresponding to the largest eigenvalues of $P_3(\sigma_{k-1})$, where σ_{k-1} is given in Algorithm 1. All red points are above the plane $z=0$ and we cannot use them to determine whether the pencil is hyperbolic. That is why in the, following, Algorithm 1 we need to add a new condition which checks the smallest eigenvalues and corresponding eigenvectors and their convergence.

Algorithm 1 Detecting hyperbolic cubic pencils

Require: initial vector $x_0 \neq 0$

if $d(x_0) < 0$ **then**
  STOP: $P_3(\lambda)$ is not hyperbolic.
end if
Determine $\sigma_0 = p_+(x_0)$.
for $k = 1,2,\ldots$ until convergence **do**
  Determine eigenvector x_k of $P_3(\sigma_{k-1})$ corresponding to its largest eigenvalue.
  if $d(x_k) < 0$ **then**
    STOP: $P_3(\lambda)$ is not hyperbolic.
  end if
  Determine $\sigma_k = p_+(x_k)$.
  if $\sigma_k \geq \sigma_{k-1}$ **then**
    STOP: $P_3(\lambda)$ is not hyperbolic.
  end if
  Determine eigenvector y_k of $P_3(\sigma_{k-1})$ corresponding to its smallest eigenvalue.
  if $d(y_k) < 0$ **then**
    STOP: $P_3(\lambda)$ is not hyperbolic.
  end if
  Determine $\tau_k = p_{+-}(y_k)$.
  if $\tau_k \geq \tau_{k-1}$ **then**
    STOP: $P_3(\lambda)$ is not hyperbolic.
  end if
end for

According to [8] the following characterization of cubic hyperbolic problems holds:

Theorem 4.1. *Let $P_3(\lambda)$ be a Hermitian matrix polynomial of degree $l = 3$ with a positive definite A_3. Then $P_3(\lambda)$ is hyperbolic if and only if there exist $\gamma_j \in \mathbb{R}$ such that $\gamma_1 > \gamma_2$ and $(-1)^j P_3(\gamma_j) > 0,\ j = 1,2$.*

Because of the possibility of suitable transformation of the cubic hyperbolic equation as we mentioned in Section 2, it is enough to observe the following cubic pencil:

$$C(\lambda) = \lambda^3 I + \lambda C_1 + C_0, \tag{4.5}$$

(C_0, C_1 - Hermitian matrices, I - identity matrix) and the corresponding eigenvalue problem. Let us observe some properties of matrices from the cubic pencil (4.5) and the corresponding eigenvalue problem

$$C(\lambda)x = 0,\ x \neq 0. \tag{4.6}$$

Analogously to Theorem 4.1 for the reduced form of the cubic pencil (4.5) and eigenvalue problem (4.6) we obtain Theorem 4.2.

Theorem 4.2. *Let $C(\lambda)$ be a cubic hyperbolic Hermitian matrix polynomial, then there exist $\gamma_0 > \gamma_1 > \gamma_2 > \gamma_3$ and $(-1)^j C(\gamma_j) > 0,\ j = 0,1,2,3,$ with $\gamma_0 > 0$, $\gamma_3 < 0$.*

Proof. The existence of γ_1 and γ_2 with the corresponding properties follows from Theorem 4.1. For each $x \neq 0$ we have:

$$\lim_{\lambda\to+\infty} x^H C(\lambda)x = \lim_{\lambda\to+\infty} (\lambda^3 x^H x + x^H C_1 x + x^H C_0 x) = +\infty,$$

and analogously

$$\lim_{\lambda\to-\infty} x^H C(\lambda)x = -\infty$$

i.e. existence of γ_0 and γ_3 with property $C(\gamma_0) > 0$ and $C(\gamma_3) < 0$. □

Let us denote $f(\lambda;x) = \lambda^3 x^H x + \lambda x^H C_1 x + x^H C_0 x$. In [6], the influence of the properties of matrices in the quadratic pencil on the eigenvalue problem was observed. Now, we will try to observe the influence of properties of matrices C_1 and C_0 on the corresponding cubic eigenvalue problem.

Theorem 4.3. *If the cubic pencil* (4.5) *is hyperbolic, then $C_1 < 0$.*

Proof. A discriminant that gives us the answer whether it is a cubic pencil hyperbolic is:

$$d(x) = -4(x^H x)(x^H C_1 x)^3 - 27(x^H x)^2(x^H C_0 x)^2.$$

If the matrix $C_1 \geqslant 0$ it is enough to take the eigenvector that corresponds to the positive eigenvalue or the zero eigenvalue for the vector x. We immediately get a negative value of the discriminant d. □

Remark 4.2. Theorem 4.3 can be proved in a different way, and the proof sketch is as follows: According to Theorem 4.2 there exists γ_2 and γ_1 with properties $\gamma_1 > \gamma_2$ and

$$C(\gamma_1) < 0 < C(\gamma_2), \tag{4.7}$$

so for each $x \neq 0$

$$\gamma_2^3 x^H x + \gamma_2 x^H C_1 x + x^H C_0 x > 0, \tag{4.8}$$

and

$$\gamma_1^3 x^H x + \gamma_1 x^H C_1 x + x^H C_0 x < 0. \tag{4.9}$$

Subtracting from (4.9) the equation (4.8) we get

$$(\gamma_1^3 - \gamma_2^3) x^H x + (\gamma_1 - \gamma_2) x^H C_1 x < 0. \tag{4.10}$$

Dividing (4.10) with $\gamma_1 - \gamma_2 > 0$ we get

$$(\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2) x^H x + x^H C_1 x < 0. \tag{4.11}$$

From $(\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2) x^H x > 0$ for each $x \neq 0$ and (4.11), $x^H C_1 x < 0$ i.e. $C_1 < 0$ follows.

The following theorem gives us information about the nature of the eigenvalues.

Theorem 4.4. *If the cubic pencil $C(\lambda)$ is hyperbolic, then it has n negative eigenvalues and n positive eigenvalues.*

Proof. Since,

$$\frac{\partial f(\lambda;x)}{\partial \lambda} = 3\lambda^2 x^H x + x^H C_1 x$$

and $x^H C_1 x < 0$, it follows that, for each x, $\frac{\partial f(\lambda;x)}{\partial \lambda}$ has two real zeroes, one negative and one positive. For the fixed x let us define $g(\lambda) = f(\lambda;x)$, therefore $g'(\lambda) = \frac{\partial f(\lambda;x)}{\partial \lambda}$.

According to Rolle's theorem there exists a zero of the first derivative between the zeroes of the function, therefore there are n positive, and n negative eigenvalues. □

Theorem 4.5. *Let $C(\lambda) = \lambda^3 I + \lambda C_1 + C_0$ be hyperbolic, and let (n_p, n_n, n_z) be inertia of $C(\sigma)$ for $\sigma \in \mathbb{R}$. Then the following statements hold:*

(i) If $3\sigma^2 x^H x + x^H C_1 x > 0$, for arbitrary $x \neq 0$, and $\sigma > 0$ the number of eigenvalues greater than σ is n_n. If $3\sigma^2 x^H x + x^H C_1 x > 0$ for arbitrary $x \neq 0$, and $\sigma < 0$ the number of eigenvalues greater than σ is $2n + n_n$.

(ii) If $n_p = n$, $\sigma > 0$ and $3\sigma^2 x^H x + x^H C_1 x > 0$, then there are no eigenvalues greater than σ.

(iii) If $3\sigma^2 x^H x + x^H C_1 x < 0$, then there are $n + n_n$ eigenvalues less than σ and $n + n_p$ eigenvalues greater than σ.

Proof. (i) If $3\sigma^2 x^H x + x^H C_1 x > 0$ for an arbitrary $x \neq 0$ and $\sigma > 0$, then $\sigma \in \Delta_1$. According to Theorem 3.2 there are n_n eigenvalues greater than σ. If $3\sigma^2 x^H x + x^H C_1 x > 0$, for an arbitrary $x \neq 0$ and $\sigma < 0$, then $\sigma \in \Delta_3$. According to Theorem 3.2 there are n_n eigenvalues greater than σ in Δ_3 and $2n$ eigenvalues outside Δ_3. That is a total of $2n + n_n$ eigenvalues greater than σ.

(ii) If $n_p = n$ and $\sigma < 0$ then $\sigma \in \Delta_3$ or $\sigma \in \Delta_2$. If $\sigma \in \Delta_3$ all eigenvalues outside Δ_3 are greater than σ and there are $2n$ of them. If $\sigma \in \Delta_2$ according to Theorem 3.3 there are $n_p = n$ eigenvalues in Δ_2 greater than σ, and all eigenvalues in Δ_1 are greater than the eigenvalues in Δ_2 and therefore greater than σ. If $n_p = n$, $\sigma > 0$ and $3\sigma^2 x^H x + x^H C_1 x > 0$ then $\sigma \in \Delta_1$, so according to Theorem 3.2 there are no eigenvalues greater than σ.

(iii) If $3\sigma^2 x^H x + x^H C_1 x < 0$, then $\sigma \in \Delta_2$. If $\sigma \in \Delta_2$ according to Theorem 3.3 there are $n_p = n$ eigenvalues in Δ_2 greater than σ. All eigenvalues from Δ_1 are greater than σ. Hence, there are $n + n_p$ eigenvalues greater than σ. Therefore there are $n + n_n$ eigenvalues less than σ.

□

The matrix C_0 can give us more information about the eigenvalue problem (4.6), as seen in the previous theorems.

Theorem 4.6. *The cubic eigenvalue problem* (4.6) *has zero as an eigenvalue if and only if C_0 is singular and in that case the eigenvector of matrix $C(\lambda)$ coincides with the eigenvector of matrix C_3 corresponding to the eigenvalue zero.*

Proof. The proof of Theorem 4.6 is trivial. □

Theorem 4.7. *Let $C(\lambda) = \lambda^3 I + \lambda C_1 + C_0$ be hyperbolic, and let $In(C_0) = (n_p, n_n, n_z)$ be the inertia of matrix C_0, then we have in total $n + n_p$ positive eigenvalues, $n + n_n$ negative eigenvalues and n_z zero eigenvalues of the problem* (4.6)*.*

Proof. We have,

$$\frac{\partial f}{\partial x} = 3\lambda^2 x^H x + x^H C_1 x, \tag{4.12}$$

and

$$\frac{\partial f}{\partial x}(0;x) = x^H C_1 x < 0, \tag{4.13}$$

and from Theorem 4.5 *(iii)* the claim of Theorem 4.7 follows.

□

5. Definite cubic pencils

In this section we deal with definite cubic pencils, because the rank of some associated matrices has a great role in the corresponding eigenvalue problems.

Definition 5.1. *A Hermitian matrix polynomial* $P_3(\lambda) = \sum_{j=0}^{3} \lambda^j A_j$ *is called definite if there exists* $\mu \in \mathbb{R} \cup \{\infty\}$ *such that* $P_3(\mu)$ *is positive definite and for every* $x \in \mathbb{C}^n$, $x \neq 0$ *the scalar polynomial* $f(\lambda;x) := x^H P_3(\lambda)x$ *has* 3 *distinct roots in* $\mathbb{R} \cup \{\infty\}$.

If the matrix A_3 from the definite pencil is regular we can apply the linear Tschirnhaus-Transformation and reduce the problem to a short reduced form which gives the hyperbolic pencil. In the case that the matrix A_3 is singular, and matrix A_0 is regular i.e. $0 < rank(A_3) = m < n$, and $rank(A_0) = n$ we can divide the initial problem with λ^3 and get the following equation:

$$\frac{1}{\lambda^3} x^H A_0 x + \frac{1}{\lambda^2} x^H A_1 x + \frac{1}{\lambda} x^H A_2 x + x^H A_3 x = 0. \tag{5.1}$$

Dividing with λ^3 is possible according to Theorem 4.6. We can apply the linear Tschirnhaus-Transformation on (5.1), and get a reduced form of the problem:

$$\frac{1}{\lambda^3} x^H x + \frac{1}{\lambda} x^H C_2 x + x^H C_3 x = 0. \tag{5.2}$$

From equation (5.2) we can calculate $\frac{1}{\lambda}$ i.e. λ. Hence, if A_3 is singular, it will have zero as an eigenvalue, which means that the initial problem will have infinity as an eigenvalue. We have seen that singularity of matrix A_3 influences the existence of infinity as an eigenvalue, and that in the case when matrix A_0 is regular the problem reduces to hyperbolic. Regardless of whether A_0 is regular or singular, the number of infinity eigenvalues gives us the following theorem:

Theorem 5.1. *Let* $P_3(\lambda) = \sum_{j=0}^{3} \lambda^j A_j$ *be definite and matrix* A_3 *singular, so that* $Rank(A_3) = p < n$, *then the corresponding eigenvalue problem has at least* $n - p$ *infinite eigenvalues. Let* $\varepsilon_{0A_3} = Ker(A_3) = \{x |\ A_3 x = 0\}$, *then each* $x \in \varepsilon_{0A_3}$ *is an eigenvector of infinite eigenvalue.*

Proof. An infinite eigenvalue is obtained only in the case of singularity of matrix A_3. From $Rank(A_3) = p < n$ it follows that matrix A_3 has zero as an eigenvalue of multiplicity $n - p$. The number of zeroes as eigenvalues of matrix A_3 is equal to the number of infinity eigenvalues $P_3(\lambda)$. As matrix A_3 is Hermitian it is immediately normal and can be diagonalized. This means, the geometric and the algebraic multiplicity ε_{0A_3} are the same. Let $\{x_1, x_2, \dots, x_{n-p}\}$ be the base of the vector space ε_{0A_3} and let $x \in \varepsilon_{0A_3}$, i.e. $x = \sum_{i=1}^{n-p} \alpha_i x_i$. Then $A_3 x = A_3(\sum_{i=1}^{n-p} \alpha_i x_i) = \sum_{i=1}^{n-p} \alpha_i A_3 x_i = 0$. □

Remark 5.1. Based on Theorem 5.1, $Rank(A_3)$ can influence the existence of infinity as an eigenvalue. However, $Rank(A_2)$ and $Rank(A_1)$ can influence the multiplicity of infinite eigenvalues.

Example 5.1. *Let us observe now the cubic pencil* $P_3(\lambda)=\sum_{j=0}^{3}\lambda^j A_j$, *where*
$A_3=\begin{pmatrix}5&3&0\\3&5&0\\0&0&0\end{pmatrix}$, $A_2=\begin{pmatrix}1&0&0\\0&5&3\\0&3&5\end{pmatrix}$, $A_1=\begin{pmatrix}1&2&0\\2&1&0\\0&0&5\end{pmatrix}$ *and* $A_0=\begin{pmatrix}0&0&0\\0&1&2\\0&2&1\end{pmatrix}$,
which has infinity as a single eigenvalue. If we observe now instead of A_2 *the matrix*
$A_2=\begin{pmatrix}0&0&0\\0&5&0\\0&0&0\end{pmatrix}$ *the multiplicity of the infinity eigenavalue is 2. If we additionally change the matrix* A_1, *placing* $A_1=\begin{pmatrix}1&2&0\\1&2&0\\0&0&0\end{pmatrix}$ *we get infinity as an eigenvalue of multiplicity 3.*

This example opens up many possibilities for considering the definite eigenvalue problem depending on the singularity of the matrices A_2 and A_1. Example 5.1 is a motivation for the following theorem:

Theorem 5.2. *Let* $P_3(\lambda)=\sum_{j=0}^{3}\lambda^j A_j$ *be definite and matrices* A_0 *and* A_3 *singular, so that* $Rank(A_3)=p<n$, *then the corresponding eigenvalue problem has at least* $n-p$ *infinite eigenvalues. Let* A_2 *be also singular and* $\varepsilon_{0A_2}=Ker(A_2)=\{x|\ A_2x=0\}$. *The necessary condition for the cubic pencil* $P_3(\lambda)$ *to be definite is* $\varepsilon_{0A_3}\cap\varepsilon_{0A_2}=\emptyset$. *If* $A_i,\ i=0,1,2,3$ *are singular and* $\varepsilon_{0A_i}=Ker(A_i)=\{x|\ A_ix=0\}, i=0,1,2,3$ *the necessary condition for the cubic pencil* $P_3(\lambda)$ *to be definite is* $(\varepsilon_{0A_3}\cap\varepsilon_{0A_2})\cap\varepsilon_{0A_1}=\emptyset$ *and* $((\varepsilon_{0A_3}\cap\varepsilon_{0A_2})\cap\varepsilon_{0A_1})\cap\varepsilon_{0A_0}=\emptyset$.

Proof. According to the Theorem 5.1 it is obvious that $x\in\varepsilon_{0A_3}$ is an eigenvector of the infinity eigenvalue. If $x\in\varepsilon_{0A_3}\cap\varepsilon_{0A_2}$ it is obvious that it is an eigenvector of the infinity eigenvalue of multiplicity 2, then according to the Definition 5.1 the cubic pencil $P_3(\lambda)$ is not definite. If $(\varepsilon_{0A_3}\cap\varepsilon_{0A_2})\cap\varepsilon_{0A_1}\neq\emptyset$ and $((\varepsilon_{0A_3}\cap\varepsilon_{0A_2})\cap\varepsilon_{0A_1})\cap\varepsilon_{0A_0}=\emptyset$, then there is $x\in(\varepsilon_{0A_3}\cap\varepsilon_{0A_2})\cap\varepsilon_{0A_1}$, which is an eigenvector of the infinite eigenvalue of multiplicity 3. If $(\varepsilon_{0A_3}\cap\varepsilon_{0A_2})\cap\varepsilon_{0A_1}\neq\emptyset$ and $((\varepsilon_{0A_3}\cap\varepsilon_{0A_2})\cap\varepsilon_{0A_1})\cap\varepsilon_{0A_0}\neq\emptyset$ then there is a vector $x\in((\varepsilon_{0A_3}\cap\varepsilon_{0A_2})\cap\varepsilon_{0A_1})\cap\varepsilon_{0A_0}$ which is an eigenvector of the zero eigenvalue, multiplicity 3, therefore $P_3(\lambda)$ is not definite. □

6. Numerical Results

In this Section we will give the numerical results. The question of hyperbolicity is not so stable, which we can see in the following example:

Example 6.1. *We observe* $C(\lambda) = \lambda^3 I + \lambda C_1 + C_0$.

$$C_1 = 1000\ rand(1000)$$
$$C_1 = \frac{C_1 + C_1^{'}}{2}, C_1 = -C_1 C_1^{'}$$
$$C_1 = 0.5(C_1 + C_1^{'})$$
$$C_0 = rand(1000)$$
$$C_0 = 0.5(C_0 + C_0^{'}).$$

Experiments in 100 iterations prove that all matrices are hyperbolic. In the next step with for loop we multiply each matrix C_1 *by* $\mu = 2 * rand - 1$ *and we obtain 53 matrices that are not hyperbolic. If we multiply in the following 100 iteration the matrix* C_1 *with* μ_1 *we obtain 23 matrices that are not hyperbolic.*

Example 6.2. *Let us observe* $C(\lambda) = \lambda^3 I + \lambda C_1 + C_0$, *where* I, C_1 *and* C_0 *are* 10×10 *matrices, given by :*

$$\begin{aligned} C_1 &= n * rand(n); \\ C_1 &= (C_1 + C_1')/2; \\ C_1 &= (-1) * C_1' * C_1; \\ C_0 &= rand(n); \\ C_0 &= (C_0 + C_0')/2. \end{aligned}$$

We checked the hyperbolicity of the cubic pencil, while we got the hyperbolic pencil. Then we have observed the new cubic pencil

$$D(\lambda) = \lambda^3 I + \beta A_1 \lambda + \alpha A_0 \tag{6.1}$$

by taking A_1 *and* A_0 *such that* $C(\lambda)$ *is hyperbolic, and* $\alpha, \beta \in [-1, 1]$ *are randomly selected. If for* (α, β) *the matrix polinomial* (6.1) *is hyperbolic, then that value is shown as a blue dot, in the opposite case it is a red dot in the Figure 2. We observed* 10000 *random pairs, and we get* 6872 *nonhyperbolic pairs and the rest of these pairs were hyperbolic.*

Remark 6.1. Algorithm 1 is very efficient but not cheap in terms of the number of operations. In Examples 6.1 and 6.2, we had a reduced form of the cubic pencil, so the costs are lower. In the k-th step of the algorithm, we have 2 matrix-vector products, which amount to $4n^2$ operations, three dot products with $6n$ operations and the determination of the largest eigenvalue and eigenvector, as well as the determination of the smallest eigenvalue and eigenvectors. This is the most expensive part of the algorithm, and the costs depend on the choice of the method for determining eigenvectors and eigenvalues. It is best to use the Lanczos algorithm because the total number of operations is $O(mn^2)$, $m \ll n$ and m is the number of iterations for the Lanczos algorithm. The Lanczos algorithm is particularly suitable for extreme eigenvalues and eigenvectors.

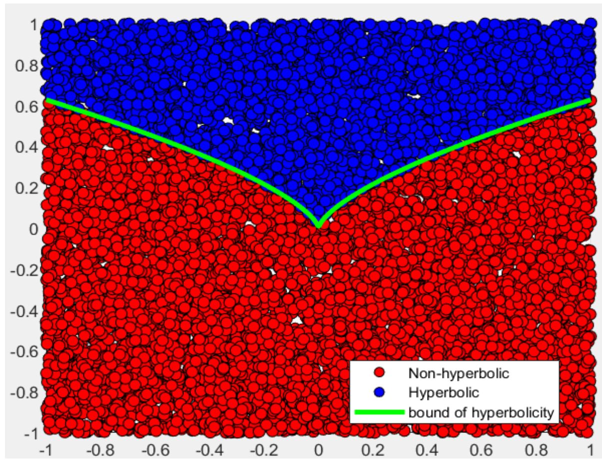

FIGURE 2. Bounds of hyperbolicity

Example 6.3. *We observed* 10000 *hyperbolic matrix pencils* (4.5) *of dimension* 100×100*, and we look for eigenvalues using lineraization and variational characterization. We noticed that the same eigenvalues differ up to* 10^{-6}*. For linearization we used* $L(\lambda)x = \lambda Gx - Hx$ *where* $G, H \in \mathbb{R}$ *are of dimension* 300×300*.*

$$L(\lambda) = \lambda \begin{pmatrix} I_{100} & 0 & 0 \\ 0 & I_{100} & 0 \\ 0 & 0 & I_{100} \end{pmatrix} + \begin{pmatrix} 0 & C_1 & C_0 \\ -I_{100} & 0 & 0 \\ 0 & -I_{100} & 0 \end{pmatrix}.$$

Remark 6.2. Linearization is very important for polynomial eigenvalue problems because they are difficult to solve directly. Linearization enables the application of efficient and stable numerical methods. The cost of finding eigenvalues and eigenvectors through linearization amounts to $O(27n^3)$ operations. The cost of variational characterization is $mnO(n^2)$, where m is the number of iterations for determining all eigenvalues and $m \ll n$.

7. CONCLUSION

In this paper, we consider a cubic eigenvalue problem. We have developed an algorithm for detecting whether the cubic eigenvalue problem is hyperbolic or not. This algorithm differs significantly from the algorithm for a quadratic pencil. Due to the linear Tschirnhaus-Transformation that can be carried out on the corresponding functional, we considered the reduced form of cubic pencils. In addition to hyperbolic cubic pencils and their properties, we also considered definite pencils and their properties. We considered variational characterization for this class of eigenvalue problems, and we compare the obtained results with the results obtained with linearization. In further research, we will deal with some unexplored properties of the cubic pencil. In this paper, we touched upon the question of the hyperbolic limit through Example 6.1 and Example 6.2. In Example 6.1, where only the matrix C_1 is changed, setting the hyperbolic limit is simple. In Example 6.2, to determine the limit of hyperbolicity, we had to include numerical optimization through the gradient ascent function. This provides motivation for further study of this problem. We will try to set the hyperbolicity limit for the pencil (6.1) and consider the possibility of a suitable factorization of the cubic pencil, with the aim of using the corresponding square pencil. We plan to deal with the matrix polynomial of the fourth degree and the corresponding eigenvalue problems. In future research, we will try to apply Algorithm 1 to fourth-degree polynomials, because there is a corresponding determinant d that has the property: if $d = 0$, the corresponding fourth-degree polynomial has equal roots; for $d < 0$, two roots are real and two are complex; and for $d > 0$, all roots are either real or all complex. This presents a new challenge for developing appropriate algorithms for fourth-degree matrix polynomials. Based on the Abel-Ruffini theorem, for $n \geqslant 5$ a general formula in radicals does not exist, so there is also no corresponding determinant. Some specific polynomials of the fifth degree or higher can be solved by radicals, but not all; therefore, polynomials of degree greater than or equal to 5 are a major challenge to study. However, some specific higher-degree polynomials can still be solvable by radicals if their Galois group is solvable.

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