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SURVEY ON THE KAKUTANI PROBLEM IN P-ADIC ANALYSIS I

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ABSTRACT. Let $\mathbb{K}$ be a complete ultrametric algebraically closed field and let A be the Banach $\mathbb{K}$-algebra of bounded analytic functions in the "open" unit disk D of $\mathbb{K}$ provided with the Gauss norm. Let $Mult(A, \| \cdot \|)$ be the set of continuous multiplicative semi-norms of A provided with the topology of pointwise convergence, let $Mult_m(A, \| \cdot \|)$ be the subset of the $\phi \in Mult(A, \| \cdot \|)$ whose kernel is a maximal ideal and let $Mult_1(A, \| \cdot \|)$ be the subset of the $\phi \in Mult(A, \| \cdot \|)$ whose kernel is a maximal ideal of the form $(x-a)A$ with $a \in D$. By analogy with the Archimedean context, one usually calls *ultrametric Corona problem, or ultrametric Kakutani problem* the question whether $Mult_1(A, \| \cdot \|)$ is dense in $Mult_m(A, \| \cdot \|)$. In order to recall the study of this problem that was made in several successive steps, here we first recall how to characterize the various continuous multiplicative semi-norms of A, with particularly the nice construction of certain multiplicative semi-norms of A whose kernell is neither a null ideal nor a maximal ideal, due to J. Araujo. Here we prove that multbijectivity implies density. The problem of multbijectivity will be described in a further paper.

CONTENTS

1. INTRODUCTION AND RESULTS.

Let $T = H^\infty(B)$ be the unital Banach algebra of bounded analytic functions on the open unit disk B in the complex plane. Each $a \in B$ defines a multiplicative linear functional ϕ_a on T by "point evaluation" i.e. $\phi_a(f) = f(a)$. If a function f lies in the kernel of all the ϕ_a then clearly $f = 0$. This tells us that the set of all the ϕ_a is dense in the set $\Xi(T)$ of all non-zero multiplicative linear functionals on T in the hull-kernel topology which is lifted from the kernels of the functionals, which are the maximal ideals of T (each maximal ideal, being of codimension 1, is the kernel of a multiplicative linear functional).

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The Corona Conjecture of Kakutani was that one also has density with respect to the *weak* topology (or Gelfand topology) which is the topology of pointwise convergence on T, defined on the space $\Xi(T)$. This was famously proved by Carleson in 1962 [4]. The key fact is that if $f_1,...,f_n$ belong to T and if there exists $d>0$ such that, for all $a\in B$ we have

$$|f_1(z)|+......+|f_n(z)|>d$$

then the ideal generated by the $f_1,....,f_n$ is the whole of T. People often transfer the name ”Corona Statement” to this key fact. Indeed, this Corona Statement implies that the Corona Conjecture is true, thanks to the fact that all maximal ideals of a $\mathbb{C}$-Banach algebra are of codimension 1.

Now consider the situation in the non-archimedean context.

Notations. Let $\mathbb{K}$ be an algebraically closed field complete with respect to an ultrametric absolute value $|\ .\ |$. Given $a\in\mathbb{K}$ and $r>0$, we denote by $d(a,r)$ the disk $\{x\in\mathbb{K}\mid |x-a|\le r\}$, by $d(a,r^-)$ the disk $\{x\in K\mid |x-a|<r\}$, by $C(a,r)$ the circle $\{x\in\mathbb{K}\mid |x-a|=r\}$ and set $D=d(0,1^-)$.

Let $a\in D$. Given $r,\ s\in]0,1]$ such that $0<r<s$ we set $\Gamma(a,r,s)=\{x\in\mathbb{K}\ |r<|x-a|<s\}$.

Let A be the $\mathbb{K}$-algebra of bounded power series converging in D which is complete with respect to the Gauss norm defined as $\|\sum_{n=1}^{\infty}a_nx^n\|=\sup_{n\in\mathbb{N}}|a_n|$: we know that this norm actually is the norm of uniform convergence on D [5].

In [16] the Corona problem was considered in a similar way as it is on the field $\mathbb{C}$ [4]: the author asked the question whether the set of maximal ideals of A defined by the points of D (which are well known to be of the form $(x-a)A$), is *dense* in the whole set of maximal ideals with respect to a so-called ”Gelfand Topology”. In fact, as explained in [8], this makes no sense because the maximal ideals which are not of the form $(x-a)A$ are of infinite codimension [8]. Consequently, a *Corona problem* should be defined in a different way, as explained in [8]. Actually, one can't define a relevant topology on the maximal spectrum of a Banach $\mathbb{K}$-algebra having maximal ideals of infinite codimension: the only spectrum we have to consider is Guennebaud's spectrum of Continuous multiplicative semi-norms [13], which is at the basis of Berkovich theory [2].

However, in [16] a ”Corona Statement” similar to that mentioned above was shown in our algebra A and it is useful in the present paper as it was in [8]. Roughly, the ”Corona Statement” shows that each maximal ideal is just the ideal of elements of the algebra T, vanishing along an ultrafilter, on the domain D. Therefore, on $\mathbb{C}$, $f(z)$ has a limit along the ultrafiter and the limit defines a character which, by definition, lies in the closure of the set of characters defined by points of D. And there are no other characters. On the field $\mathbb{K}$, although a similar ”Corona Statement” remains true [16], we can't manage the problem in the same way because

$f(x)$ has no limit along an ultrafilter (the field is not locally compact). But we may consider continuous multiplicative semi-norms and then $|f(x)|$ has a limit along an ultrafilter, which defines again a continuous multiplicative semi-norm. The role of ultrafilters then appears essential to define the multiplicative semi-norms of A. But do we get all continuous multiplicative semi-norms whose kernel is a maximal ideal, in that way? That is the problem (easily solved when the field is strongly valued [9] and next, solved when $\mathbb{K}$ is spherically complete [9]). Here we will recall that if A is multbijective, then each continuous multiplicative semi-norms whose kernel is a maximal ideal lies in the closure of the set of multiplicative semi-norms defined by points of D. The hard problem of multbijectivity will be examined in a further article.

On the other hand, we will show that certain continuous multiplicative semi-norms have a kernel that is neither null nor a maximal ideal: followin the method due to J. Araujo, we call them Araujo's semi-norms [1].

Recall classical results on analytic elements. Let E be a closed bounded subset of $\mathbb{K}$. We denote by $R(E)$ the algebra of rational functions having no pole in E and we denote by $H(E)$ the Banach $\mathbb{K}$-algebra completion of $R(E)$ with respect to the norm of uniform convergence $\|\ .\ \|_E$ which is defined on $R(E)$ because every rational function is bounded on such a set E.

Theorem I.1 summerizes a few classical properties of analytic elements in a disk $d(a,R)$ [5].

Theorem I.1. *Let $a \in \mathbb{K}$ and let $r > 0$. Then $H(d(a,r))$ is the set of power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ such that $\lim_{n\to+\infty} |a_n|r^n = 0$. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n \in H(d(a,R))$ and let q be the biggest integer such that $|a_q|r^q = \max_{n\in\mathbb{N}} |a_n|r^n$. Then the number of zeros of f in $d(a,r)$ is equal to q.*

Given a maximal ideal of $H(d(a,R))$, it is of the form $(x-b)H(d(a,R))$.

We have defined by A the $\mathbb{K}$-algebra of power series $f = \sum_{n=0}^{\infty} a_n x^n$ such that $\sup_{n\in\mathbb{N}} |a_n| < +\infty$. Then each element of A converges in the disk D. Now, fixing $a \in D$, then f is also equal to a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ which also converges in D. We denote by $\|\ .\ \|$ the norm of uniform convergence on D.

Notations. Let S be a unital commutative normed algebra whose norm is denoted by $\|\ .\ \|$. We denote by $Mult(S,\|\ .\ \|)$ the set of continuous multiplicative semi-norms of S. For each $\phi \in Mult(S,\|\ .\ \|)$, we denote by $Ker(\phi)$ the closed prime ideal of the $f \in S$ such that $\phi(f) = 0$. The set of the $\phi \in Mult(S,\|\ .\ \|)$ such that $Ker(\phi)$ is a maximal ideal is denoted by $Mult_m(S,\|\ .\ \|)$, the set of the $\phi \in Mult(T,\|\ .\ \|)$ such that $Ker(\phi)$ is a maximal ideal of codimension 1 is denoted by $Mult_1(S,\|\ .\ \|)$ [2], [13].

First, we will characterize all continuous multiplicative norms on A. Next, recalling Araujo's contruction, we will present continuous multiplicative semi-norms whose kernel is a prime closed ideal that is neither null nor maximal [1].

The ultrametric Corona problem may be viewed at two levels:
1) Is $Mult_1(A, \| \,.\, \|)$ dense in $Mult_m(A, \| \,.\, \|)$ (with respect to the topology of pointwise convergence)?
2) Is $Mult_1(A, \| \,.\, \|)$ dense in $Mult(A, \| \,.\, \|)$ (with respect to the same topology)?

In a further article, we will try to solve Question 1). Actually, this way to set the Corona problem on an ultrametric field is not really different from the original problem once considered on $\mathbb{C}$ because on a commutative unital Banach $\mathbb{C}$-algebra S, all continuous multiplicative semi-norms are known to be of the form $|\chi|$ where χ is a character of S. Thus the Corona problem was equivalent to show that the set of multiplicative semi-norms defined by the points of the open disk of center 0 and radius 1 was dense inside the whole set of continuous multiplicative semi-norms, with respect to the topology of pointwise convergence.

Let us recall some classical results on multiplicative semi-norms in ultrametric Banach algebras. Let B be a commutative unital Banach $\mathbb{K}$-algebra. We know that for every $\mathcal{M} \in Max(B)$, there exists at least one $\phi \in Mult_m(B, \| \,.\, \|)$ such that $Ker(\phi) = \mathcal{M}$ but in certain cases, there exist infinitely many $\phi \in Mult_m(B, \| \,.\, \|)$ such that $Ker(\phi) = \mathcal{M}$, [6], [7].

A maximal ideal $\mathcal{M}$ of B is said to be *univalent* if there is only one $\phi \in Mult_m(B, \| \,.\, \|)$ such that $Ker(\phi) = \mathcal{M}$ and the algebra B is said to be *multbijective* if every maximal ideal is univalent. It was proven that non-multbijective commutative unital Banach $\mathbb{K}$-algebras with unity do exist [6], [7]. The question whther A is multbijective here appears to be crucial.

Remark. Given a filter $\mathcal{G}$, if for every $f \in A$, $|f(x)|$ admits a limit $\varphi_{\mathcal{G}}(f)$ along $\mathcal{G}$, the function $\varphi_{\mathcal{G}}$ obviously belongs to $Mult(A, \| \,.\, \|)$. Moreover, it clearly lies in the closure of $Mult_1(A, \| \,.\, \|)$. Consequently, if we can prove that every element of $Mult_m(A, \| \,.\, \|)$ is of the form $\varphi_{\mathcal{G}}$, with $\mathcal{G}$ a certain filter on D, Question 1) is solved.

Thus it is important to know the nature of continuous multiplicative semi-norms on A. Unfortunately, we can't give a complete characterization.

In the proof of Theorems we shall need several basic results. Lemma I.2 is immediate and Lemma I.3 is well known [9]:

Lemma I.2. *Let $\sum_{n=0}^{\infty} u_n$ be a converging series with positive terms. There exists a sequence of strictly positive integers $t_n \in \mathbb{N}$ satisfying*

$t_n \leq t_{n+1},\ n \in \mathbb{N},$

$\lim_{n\to\infty} t_n = +\infty,$

$\sum_{m=0}^{\infty} t_m u_m < +\infty.$

Definition. *An element $f \in A$ will be said to be quasi-invertible if it factorizes in A in the form $P(x)g(x)$ where P is a polynomial whose zeros lie in D and g is an invertible element of A.*

Lemma I.3. *Let $f \in A$ be not quasi-invertible and let $(a_n)_{n\in \mathbb{N}}$ be the sequence of its zeros with respective multiplicity q_n. Then the series $\sum_{n=0}^{\infty} q_n \log(|a_n|)$ converges to $\log(|f(0)|) - \log \|f\|$.*

Now, we have the following Theorem (Theorem 14.6 and Corollary 14.10 in [10]):

Theorem I.4. *Given $f = \sum_{n=0}^{\infty} a_n x^n \in A$, we have $\|f\| = \sup_{n\in \mathbb{N}} |a_n| = \sup\{\phi(f) \mid \phi \in Mult(A, \| \, . \, \|)$. The norm $\|f\|$ is multiplicative and every $f \in A$ is uniformly continuous in D. For every $r \in]0,1[$, f has finitely many zeros in $d(0,r^-)$. Let $a \in C(0,r)$. If f has no zero in $d(a,r^-)$ then $|f(x)| = |f|(r)$ $\forall x \in d(a,r^-)$.*

Moreover, the three following statements are equivalent,

1) f has no zero in D,

2) f is invertible in A,

3) $\|f\| = |a_0|$.

4) $|f(x)|$ is a constant in D.

Corollary I.4.1. *An element $f \in A$ is quasi-invertible if and only if it has finitely many zeros.*

Lemmas I.5, I.6 are also classical and particularly, are given in [5] and particularly in Theorem 28.1 of [10].

Lemma I.5. *Let $f, g \in A$ be such that every zero a of f is a zero of g of order superior or equal to its order as a zero of f. Then there exists $h \in A$ such that $g = fh$.*

Lemma I.6. *Let $f \in A$. Then $|f'|(r) \leq \frac{|f|(r)}{r}$ $\forall r < 1$.*

By classical results on analytic functions we know this lemma (for instance Lemma I.7 is Theorem 22.26 in [10]):

Lemma I.7. *Let $r', r'' \in]0,1[$ and let $f \in A$ admit zeros $a_1, ... a_q$ of respective order k_j, $j = 1, ..., q$ in $\Gamma(0, r', r'')$ and no zero in $d(0, r')$. Then*

$$|f|(r'') = |f|(r') \prod_{j=1}^{q} (\frac{r''}{|a_j|})^{k_j}.$$

Lemma I.8. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \in H(C(0,r))$ and assume that f has a unique zero α, of order 1, in $C(0,r)$. Then $|f'(\alpha)| = |f'|(r)$.*

Proof. By hypothesis, $f(x)$ is of the form $(x-\alpha)h(x)$ with $h \in H(C(0,r))$, having no zero in $C(0,r)$. Then $|f|(r) = r|h|(r)$. Moreover, since h has no zero in $C(0,r)$, we have $|h(\alpha)| = |h|(r)$. And by Lemma I.6, $|f'|(r) \leq \frac{|f|(r)}{r}$. Therefore, we have $|f'(\alpha)| \leq |f'|(r) \leq \frac{|f|(r)}{r} = |h|(r) = |h(\alpha)| = |f'(\alpha)|$ and hence $|f'(\alpha)| = |f'|(r)$.

Definitions and notation. We call *circular filter of center a and diameter R on D* the filter $\mathcal{F}$ which admits as a generating system the family of sets $\Gamma(\alpha, r', r'') \cap D$ with $\alpha \in d(a,R), r' < R < r''$, i.e. $\mathcal{F}$ is the filter which admits for basis the family of sets of the form $D \cap \left(\bigcap\limits_{i=1}^{q} \Gamma(\alpha_i, r'_i, r''_i)\right)$ with $\alpha_i \in d(a,R), r'_i < R < r''_i \quad (1 \leq i \leq q\ ,\ q \in \mathbb{N})$.

Recall that the field $\mathbb{K}$ is said to be *spherically complete* if every decreasing sequence of disks has a non-empty intersection. Each field such as $\mathbb{K}$ admits an algebraically closed spherical completion (see Theorems 7.4 and 7.6 in [10].

In a field which is not spherically complete, one has to consider decreasing sequences of disks (D_n) with an empty intersection. We call *circular filter with no center, of canonical basis* (D_n) the filter admitting for basis the sequence (D_n) and the number $\lim\limits_{n\to\infty} diam(D_n)$ is called *diameter of the filter*.

Finally the filter of neighborhoods of a point $a \in D$ is called *circular filter of the neighborhoods of a on D* and its diameter is 0. Given a circular filter $\mathcal{F}$, its diameter is denoted by $diam(\mathcal{F})$.

Given $a \in \mathbb{K}$ and $r > 0$, we denote by $\Phi(a,r)$ the set of circular filters secant with $d(a,r)$ i.e. the circular filters of center $b \in d(a,r)$ and radius $s \in [0,r]$.

Here, we will denote by $\mathcal{W}$ the circular filter on D of center 0 and diameter 1 and by $\mathcal{Y}$ the filter admitting for basis the family of sets of the form $\Gamma(0,r,1) \setminus \left(\bigcup_{n=0}^{\infty} d(a_n, r_n^-)\right)$ with $a_n \in D,\ r_n \leq |a_n|$ and $\lim_{n\to\infty} |a_n| = 1$.

On $\mathbb{K}[x]$, circular filters on $\mathbb{K}$ are known to characterize multiplicative semi-norms by associating to each circular filter $\mathcal{F}$ the multiplicative semi-norm $\varphi_{\mathcal{F}}$ defined as $\varphi_{\mathcal{F}}(f) = \lim_{\mathcal{F}} |f(x)|$ [12], [13], [9], [11].

We know that every $f \in A$ is an analytic element in each disk $d(a,r)$ whenever $r \in]0,1[$ [5]. Consequently, by classical results [5], several properties of polynomials have continuation to analytic elements and to A.

Definitions and notation. Let $a \in D$ and let $R \in]0,1]$. Given $r,\ s \in \mathbb{R}$ such that $0 < r < s$ we set $\Gamma(a,r,s) = \{x \in \mathbb{K}\ |r < |x-a| < s\}$.

We call *circular filter of center a and diameter R on D* the filter $\mathcal{F}$ which admits as a generating system the family of sets $\Gamma(\alpha, r', r'') \cap D$ with $\alpha \in d(a,R), r' < R < r''$, i.e. $\mathcal{F}$ is the filter which admits for basis the family of sets of the form
$D \cap \left(\bigcap_{i=1}^{q} \Gamma(\alpha_i, r'_i, r''_i)\right)$ with $\alpha_i \in d(a,R), r'_i < R < r''_i \ \ (1 \leq i \leq q\ ,\ q \in \mathbb{N})$.

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We know that every $f \in A$ is an analytic element in each disk $d(a,r)$ whenever $r \in]0,1[$ [5]. Consequently, by classical results [5], several properties of polynomials have continuation to analytic elements and to A.

Thus, by results on analytic elements, we have Theorem I.9 [5], and Theorem 13.1 in [10]:

Theorem I.9. *Let $a \in \mathbb{K}$ and $R \in]0,1[$ and let $r \in [0,R]$. For each circular filter $\mathcal{F} \in \Phi(a,r)$, for each element f of $H(d(a,r))$ (resp. $f \in H(d(a,r^-))$), $|f(x)|$ has a limit $\varphi_{a,r}(f) = \varphi_{\mathcal{F}}(f)$ along $\mathcal{F}$. Moreover, the mapping $\varphi_{\mathcal{F}}$ defined on $H(d(a,R))$ (resp. on $H(d(a,r^-))$) is a multiplicative semi-norm continuous with respect to the norm $\| \, . \, \|_{d(a,r))}$ (resp. $\| \, . \, \|_{d(a,r^-))}$) and is a norm if and only if $r > 0$.*

Next, if $b \in d(a,r)$ (resp. $f \in d(a,r^-)$), then $\varphi_{a,r}(f) = \varphi_{b,r}(f)$. Further, the mapping associating to each circular filter $\mathcal{F} \in \Phi(a,r)$ secant with $d(a,R)$ the continuous multiplicative semi-norm $\varphi_{\mathcal{F}}$ is a bijection from $\Phi(a,r)$ onto $Mult(H(d(a,r)), \| \, . \, \|_{d(a,R)})$.

Now, we will denote by $\mathcal{W}$ the circular filter on D of center 0 and diameter 1 and by $\mathcal{Y}$ the filter admitting for basis the family of sets of the form $\Gamma(0,r,1) \setminus \left(\bigcup_{n=0}^{\infty} d(a_n, r_n^-)\right)$ with $a_n \in D$, $r_n \leq |a_n|$ and $\lim_{n\to\infty} |a_n| = 1$.

Next, $\varphi_{\mathcal{W}}$ defines the Gauss norm on $\mathbb{K}[x]$ because, given a polynomial $P(x) = \sum_{j=0}^{q} a_j x^j$, we have $\varphi_{\mathcal{W}}(P) = \max\limits_{0 \leq j \leqq} |a_j|$. Therefore $\varphi_{\mathcal{W}}$ admits a natural continuation to A as $\| \sum_{n=1}^{\infty} a_n x^n \| = \sup_{n \in \mathbb{N}} |a_n|$. However, by [6] we know that this continuation is far from unique.

So, the problem is first to determine whether a multiplicative semi-norms defined on $\mathbb{K}[x]$ by circular filters on D, other than the Gauss norm, have a unique continuation to A.

Consequently, given a circular filter $\mathcal{F}$ on D of diameter < 1, according to Theorem I.9, for every $f \in A$, $|f(x)|$ has a limit along $\mathcal{F}$ denoted by $\varphi_{\mathcal{F}}(f)$ and then $\varphi_{\mathcal{F}}$ is a continuous multiplicative semi-norm on A. In particular, given $a \in D$ and $r \in]0,1[$, if we consider the circular filter $\mathcal{F}$ of center a and diameter r, we denote by $\varphi_{a,r}$ the multiplicative semi-norm $\varphi_{\mathcal{F}}$ which actually is defined by $\varphi_{a,r}(f) = \lim\limits_{|x-a| \to r} |f(x)|$ and is a norm whenever $diam(\mathcal{F}) > 0$. For convenience, if $\mathcal{F}$ is the circular filter of center 0 and diameter r, we set $|f|(r) = \varphi_{\mathcal{F}}(f)$.

Definitions and notations. Let E be a subset of $\mathbb{K}$. If E is bounded, of diameter r, we denote by $\overline{\overline{E}}$ the disk $d(a,r),\ a\in E$.

For each $a\in E$, we denote by I_a the mapping from E to $\mathbb{R}_+$ defined as $I_a(x)=|x-a|$.

A subset E of $\mathbb{K}$ is said to be *infraconnected* if for every $a\in E$, the closure in $\mathbb{R}$ of $I_a(E)$ is an interval.

By classical results ([5], Lemma 2.1) we have the following description:

Lemma I.10. *Let E be a closed bounded subset of $\mathbb{K}$, of diameter R. Then $\overline{\overline{E}}\setminus E$ admits a unique partition by a family of maximal disks $(d(b_j,r_j^-))_{j\in I}$.*

Definitions and notations. Let E be a closed bounded subset of $\mathbb{K}$ and let $(d(b_j,r_j^-))_{j\in I}$ be the partition of $\overline{\overline{E}}\setminus E$ shown in Lemma I.10. The disks $d(b_j,r_j^-)),\ j\in I$ are called *the holes of* E.

We can now recall the famous Mittag-Leffler Theorem for analytic elements due to Marc Krasner [14] and [5], Theorem 15.1:

Theorem I.11. (M. Krasner) *Let E be a closed and bounded infraconnected subset of $\mathbb{K}$ and let $f\in H(E)$. There exists a unique sequence of holes $(T_n)_{n\in\ \mathbb{N}^*}$ of E and a unique sequence $(f_n)_{n\in\mathbb{N}}$ in $H(E)$ such that $f_0\in H(\overline{\overline{E}}),\ f_n\in H_0(\mathbb{K}\setminus T_n)\ (n>0),\ \lim_{n\to\infty} f_n=0$ satisfying*

(1) $f=\sum_{n=0}^{\infty} f_n$ *and* $\|f\|_D=\sup_{n\in\mathbb{N}}\|f_n\|_E$.

Moreover for every hole $T_n=d(a_n,r_n^-)$, we have

(2) $\|f_n\|_E=\|f_n\|_{\mathbb{K}\setminus T_n}=\varphi_{a_n,r_n}(f_n)\le\varphi_{a_n,r_n}(f)\le\|f\|_E$.

If $\overline{\overline{E}}=d(a,r)$ we have

(3) $\|f_0\|_E=\|f_0\|_{\overline{\overline{E}}}=\varphi_{a,r}(f_0)\ \le\ \varphi_{a,r}(f)\ \le\ \|f\|_E$.

Let $F=\overline{\overline{E}}\setminus\Big(\bigcup_{n=1}^{\infty}T_n\Big)$. Then f belongs to $H(F)$ and its decomposition in $H(F)$ is given again by (1) *and then f satisfies $\|f\|_F=\|f\|_E$.*

Let us recall the following results [7], [13]:

Theorem I.12. *Let B be a unital commutative ultrametric Banach $\mathbb{K}$-algebra. Then $\sup\{\phi(f)\mid\phi\in Mult(B,\|\ .\ \|)\}=\lim_{n\to\infty}(\|f^n\|)^{\frac{1}{n}}\ \forall f\in B$. On the other hand, $Mult(B,\|\ .\ \|)$ is provided with the topology of pointwise convergence and is compact for this topology.*

Let us now recall some general results on maximal ideals [7] (Theorems 15.6 and 27.3):

Theorem I.13. *Let B be a unital cmmutative Banach $\mathbb{K}$-algebra. Each maximal ideal of B is the kernel of at least one continuous multiplicative semi-norm. If $\mathbb{K}$ has a non-countable residue class field or a non-countable value group then each maximal ideal of B is the kernel of a unique continuous multiplicative semi-norm.*

Corollary I.13.1. *If* $\mathbb{K}$ *has a non-countable residue class field or a non-countable value group then every unital commutative Banach* $\mathbb{K}$*-algebra is multbijective.*

Definition. *The field* $\mathbb{K}$ *is said to be strongly valued if one of the following two sets are not countable:*

1) the residue field of $\mathbb{K}$,

2) the set $|\mathbb{K}| = \{|x| \mid x \in \mathbb{K}\}$.

Remark. When the algebraically closed complete field is not strongly valued, there exist Banach $\mathbb{K}$-algebras which are not multbijective [6], [7].

2. Multiplicative spectrum and maximal ideals

Let us first notice the following basic result which is indispensable in the sequel:

Theorem II.1. *Let* $\mathcal{F}$ *be a filter on* D *such that for all* $f \in A$, $|f(x)|$ *have a limit along* $\mathcal{F}$. *Then the mapping* ϕ *defined on* A *as* $\phi(f) = \lim_{\mathcal{F}} |f(x)|$ *belongs to the closure of* $Mult_1(A, \|\ .\ \|)$ *in* $Mult(A, \|\ .\ \|)$.

Proof. Let us first notice that given an element $\phi \in Mult(A, \|\ .\ \|)$ of the form $\varphi_{\mathcal{F}}$ with $\mathcal{F}$ a filter on D, then ϕ clearly belongs to the closure of $Mult_1(A, \|\ .\ \|)$ in $Mult(A, \|\ .\ \|)$. More precisely, take $f_1, \ldots, f_q \in A$ and $\varepsilon > 0$. For each $j = 1, \ldots, q$ therer exists $B_j \in \mathcal{F}$ such that $\Big| |f_j(x)| - \varphi_{\mathcal{U}}(f_j) \Big|_\infty \leq \varepsilon \ \forall x \in B_j$. Let $B = \bigcap_{j=1}^{q} B_j$. Then $\Big| |f_j(x)| - \varphi_{\mathcal{U}}(f_j) \Big|_\infty \leq \varepsilon \ \forall x \in B,\ \forall j = 1, \ldots, q$, which ends the proof. □

We will now apply to the algebra A all results already known concerning algebras $H(d(a,R))$ and $H(d(a,R^-))$.

Now, when studying the set of multiplicative semi-norms of the algebra A, we have to consider coroner ultrafilters.

Definitions and notations. An ultrafilter $\mathcal{U}$ on D will be called *coroner ultrafilter* if it is thinner than $\mathcal{W}$. Similarly, a sequence (a_n) on D will be called *a coroner sequence* if its filter is a coroner filter, i.e. if $\lim_{n\to+\infty} |a_n| = 1$.

Two coroner ultrafilters $\mathcal{F}$, $\mathcal{G}$ are said to be *contiguous* if for every subsets $F \in \mathcal{F}$, $G \in \mathcal{G}$ of D the distance from F to G is null.

Let $\psi \in Mult(A, \|\ .\ \|)$ be different from $\|\ .\ \|$. Then ψ will be said to be *coroner* if its restriction to $\mathbb{K}[x]$ is equal to $\|\ .\ \|$.

In [8] regular ultrafilters were defined. Let $(a_n)_{n\in\mathbb{N}}$ be a coroner sequence in D. The sequence is called *a regular sequence* if $\inf_{j\in\mathbb{N}} \prod_{\substack{n\in\mathbb{N}\\ n\neq j}} |a_n - a_j| > 0$.

An ultrafilter $\mathcal{U}$ is said to be *regular* if it is thinner than a regular sequence. Thus, by definition, a regular ultrafilter is a coroner ultrafilter.

Now, given an ultrafilter $\mathcal{U}$ on D, the function $|f(x)|$ from D to $[0, \|f\|]$ has a limit $\varphi_{\mathcal{U}}(f)$ which clearly defines an element of $Mult(A, \|\ .\ \|)$. We can then derive Theorem II.2.

Given a filter $\mathcal{F}$ on D, we will denote by $\mathcal{J}(\mathcal{F})$ the ideal of the $f \in A$ such that $\lim_{\mathcal{F}} f(x) = 0$.

Theorem II.2. *Let $\mathcal{U}$ be an ultrafilter on D. For every $f \in A$, $|f(x)|$ admits a limit $\varphi_{\mathcal{U}}(f)$ along $\mathcal{U}$. Moreover, the mapping $\varphi_{\mathcal{U}}$ from A to $\mathbb{R}_+$ belongs to $Mult(A, \| \,.\, |)$ and $Ker(\varphi_{\mathcal{U}}) = \mathcal{J}(\mathcal{U})$. Given two contiguous ultrafilters $\mathcal{U}_1,\ \mathcal{U}_2$ on D, $\varphi_{\mathcal{U}_1} = \varphi_{\mathcal{U}_2}$.*

Proof. Let θ be the function defined in D, by $\theta(x) = |f(x)|$. For each $f \in A$, θ takes values in the compact $[0, \|f\|]$. Clearly, θ admits a limit $\varphi_{\mathcal{U}}$ along every ultrafilter $\mathcal{U}$ on D. Consequently, $\varphi_{\mathcal{U}}$ defines a continuous multiplicative semi-norm on A whose kernel is $\mathcal{J}(\mathcal{U})$. Finally, since every function $f \in A$ is uniformly continuous, it is easily seen that $\lim_{\mathcal{U}_1} |f(x)| = \lim_{\mathcal{U}_2} |f(x)|\ \forall f \in A$, hence $\varphi_{\mathcal{U}_1}(f) = \varphi_{\mathcal{U}_2}(f)\ \forall f \in A$, hence $\varphi_{\mathcal{U}_1} = \varphi_{\mathcal{U}_2}$. □

Remark. Contrary to the context of uniformly continuous bounded functions [11], it seems very hard to know whether two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on D such that $\varphi_{\mathcal{U}} = \varphi_{\mathcal{V}}$ are contiguous.

Notation. We will denote by $\mathcal{Y}$ the filter on D admitting for basis the family of sets of the form

$$d(0, 1^-) \setminus \bigcup_{n=1}^{\infty} \Big(\bigcup_{j=1}^{q_n} d(a_{n,j}, r_n^-) \Big)$$

with $|a_{j,n}| = r_n < r_{n+1} < 1\ \forall n \in \mathbb{N}$ and $\lim\limits_{n \to \infty} r_n = 1$.

Proposition II.3. *For every $f \in A$, $\|f\| = \lim\limits_{\mathcal{Y}} |f(x)|$.*

Proof. Let $f \in A$, let $(C(0, r_n))_{n \in \mathbb{N}}$ be the sequence of circles of center 0 containing zeros of f and for every $n \in \mathbb{N}$, and let $a_{n,1}, \dots, a_{n,q_n}$ be the zeros of f in $C(0, r_n)$. Then we can see that $|f(x)| = |f|(|x|)\ \forall x \notin d(0, 1^-) \setminus \bigcup_{n=1}^{\infty} \Big(\bigcup_{j=1}^{q_n} d(a_{n,j}, r_n^-) \Big)$.

Next, by definition of the norm $\| \,.\, \|$, we have $\|f\| = \lim\limits_{r \to 1} |f|(r)$, which ends the proof. □

Proposition II.4. *Let $\phi \in Mult(A, \| \,.\, \|)$ satisfy $\phi(P) = \|P\|\ \forall P \in \mathbb{K}[x]$. Every quasi-invertible element $f \in A$ also satisfies $\phi(f) = \|f\|$.*

Proof. First suppose $f \in A$ invertible in A. Then $1 = \phi(f)\phi(f^{-1})$. But $\phi(f) \leq \|f\|$, $\phi(f^{-1}) \leq \|f^{-1}\|$, hence both inequalities must be equalities. Now, let $f = Pg \in A$ be quasi-invertible, with $P \in \mathbb{K}[x]$ a polynomial having all zeros in D and $g \in A$, invertible in A. Then $\phi(f) = \phi(P)\phi(g) = \|P\|\|g\| = \|Pg\| = \|f\|$. □

Let us now look at the maximal spectrum of A.

Theorem II.5. *Let $\mathcal{M}$ be an ideal of A. Then $\mathcal{M}$ is a maximal ideal of codimension 1 if and only if it is of the form $(x - a)A$ with $a \in D$.*

Proof. Given $a \in D$, the ideal $(x - a)A$ is obviously a maximal ideal of codimension 1 because the mapping χ_a from A to $\mathbb{K}$ defined as $\chi_a(f) = f(a)$ maps A onto $\mathbb{K}$.

Now, let $\mathcal{M}$ be a maximal ideal of codimension 1 and let θ be the $\mathbb{K}$-algebra homorphism from A onto $\mathbb{K}$ admitting $\mathcal{M}$ for kernel. Let $a = \theta(x)$. Since $\theta(x-a) = \theta(x) - a$ is not invertible, we have $|a| < 1$ because if $|a| \geq 1$, then $\dfrac{1}{x-a}$ belongs to $H(D)$ and hence to A. Thus, a belongs to D. We know that all characters of a Banach $\mathbb{K}$-algebra are continuous (see for instance Theorem 6.19 in [7]), hence so is θ. Consequently, $\theta(f) = f(a)\ \forall f \in A$ and hence $Ker(\theta) = (x-a)A$. □

Notation. We will denote by $Mult_1(A, \|\ .\ \|)$ the set of maximal ideals of codimension 1.

Remark. A admits maximal ideals of infinite codimension.

Theorem II.6. *Let M be a maximal ideal of A. The following statements are equivalent:*

(i) there exists $a \in D$ such that $M = (x-a))A$,
(ii) M is principal,
(iii) M is of finite type,
(iv) M is of codimension 1.
(v) M contains a quasi-invertible element.

Proof. Suppose *(i)* is satisfied. Then so are *(ii)* and *(iii)* and by Theorem I.4, so is *(iv)*. Moreover, by*(i)*, $x-a$ belongs to A, hence *(v)* is satisfied. Suppose now that *(v)* is satisfied and let $P(x)g(x) \in M$ be quasi-invertible, with P a polynomial whose zeros lie in D and g an invertible element of A. Then P belongs to M. Let $P(x) = \prod_{j=1}^{q}(x-a_j)$. Since M is prime, one of the $x-a_j$ belongs to D and hence *(i)* is satisfied, which ends the proof. □

Corollary II.6.1. *An element φ of $Mult(A, \|\ .\ \|)$ belongs to $Mult_1(A, \|\ .\ \|)$ if and only if there exists $\alpha \in D$ such that $\varphi(f) = |f(\alpha)|, \forall f \in A$.*

Notation. Given an ideal I of A we will denote by $\mathcal{G}_I$ the filter generated by the sets $E(f, \varepsilon),\ f \in I, \varepsilon > 0$. By definition, $\mathcal{G}_I$ is minimal, with respect to the relation of thinness, among the filters $\mathcal{H}$ such that $\lim_{\mathcal{H}} f(x) = 0\ \forall f \in I$.

Theorem II.7. *Let M be a non-principal maximal ideal of A. Then $M = \mathcal{I}(\mathcal{G}_M)$.*

Proof. By definition, we have $M \subset \mathcal{I}(\mathcal{G}_M)$. On the other hand, $\mathcal{I}(\mathcal{G}_M) \neq A$ because by Theorem II.5 all elements of $\mathcal{I}(\mathcal{G}_M)$ are non-quasi-invertible. Consequently, $M = \mathcal{I}(\mathcal{G}_M)$. □

Corollary II.7.1: *Let M be a non-principal maximal ideal of A. For every ultrafilter $\mathcal{U}$ thinner than $\mathcal{G}_M$, $\mathcal{I}(\mathcal{U}) = M$.*

Corollary II.7.2. *For every maximal ideal M of A, there exist ultrafilters $\mathcal{U}$ such that $M = \mathcal{I}(\mathcal{U})$.*

Proof. Indeed, either M is principal, of the form $(x-a)A$ or M is not principal and then the answer comes from Corollary II.7.1. □

Definition. A maximal ideal $\mathcal{M}$ of A will be said to be *coroner* (resp. *regular*) if there exists a coroner (resp. regular) ultrafilter $\mathcal{U}$ such that $\mathcal{M} = \mathcal{I}(\mathcal{U})$.

Notation. Let $\mathcal{H}$ be the $\mathbb{K}$-algebra of (bounded or not) analytic functions in D.

Theorem II.8 also is classical ([10], Theorem 22.26):

Theorem II.8. *Let $f \in \mathcal{H}$ and let $r_1, r_2 \in]0,1[$ satisfy $r_1 < r_2$. If f admits exactly q zeros in $d(O, r_1)$ (taking multiplicity into account) and t different zeros α_j, of respective multiplicity order m_j $(1 \leq j \leq t)$ in $\Gamma(0, r_1, r_2)$, then f satisfies*

$$\frac{|f|(r_2)}{|f|(r_1)} = \Big(\prod_{j=1}^{t}\big(\frac{r_2}{|\alpha_j|}\big)^{m_j}\Big)\big(\frac{r_2}{r_1}\big)^q$$

Corollary II.8.1. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{H}$ have a set of zeros in D that consists of a sequence $(\alpha_n)_{n\in \mathbb{N}}$ such that $\alpha_n \neq 0$ $\forall n \in \mathbb{N}$ and where each α_n is of order u_n. Then $\|f\|_D = |f(0)|\prod_{n=0}^{\infty}\big(\frac{1}{|\alpha_n|}\big)^{u_n}$.*

Corollary II.8.2. *Let $f \in A$ be not quasi-invertible, such that $f(0) = 1$ and let $(a_n)_{n\in \mathbb{N}}$ be the sequence of zeros with respective multiplicity q_n. Then the series $\sum_{n=0}^{\infty} q_n \log\Big(\frac{1}{|a_n|}\Big)$ converges to $\log\|f\|$.*

By Theorem II.6, II.7, and II.8 we can derive Theorem II.9:

Theorem II.9. *A maximal ideal of A is of infinite codimension if and only if it is coroner.*

Proof. Let M be a maximal ideal of A. By Corollary II.7.2, M is of the form $\mathcal{I}(\mathcal{U})$ with $\mathcal{U}$ an ultrafilter on D. If $\mathcal{U}$ is coroner, then M is not of the form $(x-a)A$ $a \in D$, and by Theorem II.6 it contains no quasi-invertible element and it is of infinite codimension. If $\mathcal{U}$ is not coroner, either it is a Cauchy filter of limit $a \in D$ or it is thinner than a circular filter on D of center $a \in D$ and diameter $r \in]0,1[$. If $\mathcal{U}$ is a Cauchy filter, M is the ideal of functions f such that $f(a) = 0$, then by Theorem II.6., M is of codimension 1.

Suppose now that If $\mathcal{U}$ is not a Cauchy filter. Since it is not coroner, it is secant with a disk $d(a,r) \subset D$ and hence it is thinner than a circular filter $\mathcal{F}$ of center $b \in d(a,r)$ and diameter $s \in]0,r]$ so that $\lim_{\mathcal{F}} |f(x)| = 0$ $\forall f \in M$. But since the function $f \in A$ is an element f $H(d(a,r))$, by Theorem I.9 it cannot satisfy $\lim_{\mathcal{F}} |f(x)| = 0$, a contradiction. □

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Theorem III.1. *If A is multbijective, then $Mult_1(A, \|\ .\ \|)$ is dense in $Mult_m(A, \|\ .\ \|)$.*

Proof. By Theorem II.2, each ultrafilter $\mathcal{U}$ on D defines an element $\varphi_{\mathcal{U}}$ of $Mult(A, \|\ .\ \|)$. Conversely, by Corollary II.7.2, every maximal ideal $\mathcal{M}$ is of the form $\mathcal{J}(\mathcal{U})$ with $\mathcal{U}$ an ultrafilter on D. Suppose now that A is multbijective and let $\phi \in Mult_m(A, \|\ .\ \|)$. Then $Ker(\phi)$ is a maximal ideal $\mathcal{M}$ of the form $\mathcal{J}(\mathcal{U})$ with $\mathcal{U}$ an ultrafilter on D. Consequently, $Ker(\varphi_{\mathcal{U}}) = \mathcal{M} = Ker(\phi)$. But since A is multbijective, then $\phi = \varphi_{\mathcal{U}}$. And then, by Theorem II.1, $\varphi_{\mathcal{U}}$ does belong to the closure of $Mult_1(A, \|\ .\ \|)$ in $Mult(A, \|\ .\ \|)$, which ends the proof. □

On the other hand, consequently to Theorem II.8, we can state Theorem III.:

Theorem III.2. *Let $r,\ s,\ R \in]0, +\infty[$ satisfy $0 < r < s < R$ and let $f \in H((0,R))$. Then*

$$\log(|f|(s)) - \log(|f|(r)) \leq \Big(\log(|f|(R)) - \log(|f|(s))\Big)\Big(\frac{\log(s) - \log(r)}{\log(R) - \log(s)}\Big).$$

Proof. Let q be the total number of zeros of f in $d(0,s)$, each counted with its multiplicity. Then by Theorem II.8 we have $\log(|f|)(s)) - \log(|f|)(r)) \leq q(|og(s) - \log(r))$. On the other hand, $\log(|f|(R)) - \log(|f|(s)) \geq q\big(\log(R) - \log(s)\big)$. Consequently,

$$q \leq \frac{\log(|f|(R)) - \log(|f|(s)}{\log(R) - \log(s)}$$

which ends the proof. □

Theorem III.3. *Let $\phi \in Mult(A, \|\ .\ \|)$ and assume that its restriction $\varphi_{\mathcal{F}}$ to $H(D)$ is not $\|\ .\ \|$. Then $\phi(f) = \lim_{\mathcal{F}} |f(x)|\ \forall f \in A$.*

Proof. By hypothesis, $\mathcal{F}$ is a circular filter of diameter $l < 1$. Suppose first that f is invertible. Then $|f(x)|$ is a constant $b > 0$. Consequently, $\|f\| = b = \varphi_{\mathcal{F}}(f)$. Suppose $\phi(f) \neq b$. Then $\phi(f) < b$ because $b = \|f\|$. Now consider $h = \dfrac{1}{f}$. Since $\|\ .\ \|$ is multiplicative, we see that $\phi(h) > \|h\|$, a contradiction. Consequently, $\phi(f) = \varphi_{\mathcal{F}}(f)$.

Suppose now f is quasi-invertible. Then f is of the form Pg with $P \in \mathbb{K}[x]$ and g invertible in A. Then, $\phi(f) = \phi(P)\phi(g) = \varphi_{\mathcal{F}}(P)\varphi_{\mathcal{F}}(g) = \varphi_{\mathcal{F}}(f)$.

We now suppose that f is not quasi-invertible. By Corollary I.4.1, f has a sequence of zeros $(a_n)_{n\in\mathbb{N}}$ in D, each having a multiplicity order u_n. By Corollary II.8.1, we have $\lim\limits_{n\to+\infty} |a_n| = 1$, so we can assume $|a_n| \leq |a_{n+1}|\ \forall n \in \mathbb{N}$. Let $t = \phi(f)$ and $s = \lim_{\mathcal{F}} |f(x)|$. We shall show that $t \leq s$.

Suppose first that $\mathcal{F}$ has a disk $d(a,r)$ which contains none of the a_n. By Corollary II.8.1 we have $\frac{\|f\|_{d(a,r)}}{\|f\|} = \prod_{n=1}^{\infty}\Big(|a_n - a|\Big)^{u_n}$, hence inside the disk $d(a,r)$, $|f(x)|$ is a constant equal to $\|f\|\prod_{n=0}^{\infty}\Big(|a_n - a|\Big)^{u_n}$ and therefore,

$$s=\|f\|\prod_{n=1}^{\infty}\Big(|a_n-a|\Big)^{u_n}. \tag{1}$$

For each $q\in \mathbb{N}$, let $f_q=\dfrac{f}{\prod_{n=0}^{q}(x-a_n)^{u_n}}$ and let $l_q=\sum_{k=0}^{q}u_k$. So, clearly, $\|f_q\|=\|f\|\ \forall q\in \mathbb{N}$.

Now, since $\phi(P)=\varphi_{\mathcal{F}}(P)\ \forall P\in \mathbb{K}[x]$, we have $\phi(\prod_{n=1}^{q}(x-a_n)^{u_n})=\prod_{n=1}^{q}|a_n-a|^{u_n}$, hence $\phi(f_q)=\dfrac{t}{\prod_{n=1}^{q}|a_n-a|^{u_n}}$. But since $\phi(f_q)\leq\|f_q\|$, that yields

$$\frac{t}{\prod_{n=1}^{q}|a_n-a|^{u_n}}\leq\|f_q\|=\|f\|\ \forall q\in \mathbb{N}$$

hence

$$\frac{t}{\prod_{n=1}^{q}|a_n-a|^{u_n}}\leq\|f\|\ \forall q\in \mathbb{N}$$

Since this is true for every $q\in \mathbb{N}$, we can derive

$$t\prod_{n=1}^{\infty}\Big(\frac{1}{|a_n-a|^{u_n}}\Big)\leq\|f\|$$

hence by (1), $\dfrac{t\|f\|}{s}\leq\|f\|$ and therefore $t\leq s$.

Now consider the case when there exists no disk $d(a,r)$ belonging to $\mathcal{F}$, such that none of the a_n lie in $d(a,r)$. Since $\lim\limits_{n\to\infty}|a_n|=1$, $\mathcal{F}$ is a filter admitting a center α. Let ρ be its diameter: of course $\rho<1$ because $\varphi_{\mathcal{F}}$ is not $\|\ .\ \|$. Consequently, $d(\alpha,\rho)$ contains finitely many zeros of f $a_1,\dots,a_s$ (eventually, if $\rho=0$, then $d(\alpha,\rho)$ is reduced to the singelton $\{\alpha\}$).

Suppose first $\rho=0$. Then $\varphi_{\mathcal{F}}(f)=0$ and $\phi(x-\alpha)=0$ therefore $s=t$.

Suppose now $\rho>0$. Suppose $|a_j-a|\leq\rho$ whenever $j=1,\dots,q$. We can choose $a\neq a_j\ \forall j=1,\dots,q$. Set $h=\dfrac{f}{\prod_{j=1}^{q}(x-a_j)^{u_j}}$. Then $\varphi_{\mathcal{F}}(h)=\dfrac{s}{\prod_{j=1}^{q}|a_j-\alpha|^{u_j}}$ and $\phi(h)=\dfrac{t}{\prod_{j=1}^{q}|a_j-\alpha|^{u_j}}$. Thus we are led to the same problem with h. Setting $s'=\dfrac{s}{\prod_{j=1}^{q}|a_j-\alpha|^{u_j}}$, $t'=\dfrac{t}{\prod_{j=1}^{q}|a_j-\alpha|^{u_j}}$, we have $t'\leq s'$ hence $t\leq s$ in all cases and therefore we have proven again that

$$\phi(h)\leq\varphi_{\mathcal{F}}(h)\ \forall h\in A. \tag{2}$$

Suppose now that for some $f\in A$, we have $\phi(f)<\varphi_{\mathcal{F}}(f)$. We can take $r\in\ ]l,1[$ such that the disk $d(0,r)$ belongs to $\mathcal{F}$. Let $f(x)=\sum_{n=1}^{\infty}b_nx^n$. For every $q\in \mathbb{N}$, let $g_q(x)=\sum_{n=1}^{q}b_nx^n$. We notice that when q is big enough we have

$\varphi_{\mathcal{F}}(g_q) = \sup_{n\in \mathbb{N}} |b_n|r^n$. Set $w = \sup_{n\in \mathbb{N}} |b_n|r^n$. Now $\varphi_{\mathcal{F}}(f - g_q) \leq \sup_{n>q} |b_n|r^n$, therefore $\lim_{q\to+\infty} \varphi_{\mathcal{F}}(f - g_q) = 0$ and hence, by (2), we have

$$\lim_{q\to+\infty} \phi(f - g_q) = 0. \tag{3}$$

So, we can take q such that $\varphi_{\mathcal{F}}(f - g_q) < \varphi_{\mathcal{F}}(f)$ and hence, by (2), we have $\phi(f - g_q) < \varphi_{\mathcal{F}}(f)$. But since g_q is a polynomial, we have $\phi(g_q) = \varphi_{\mathcal{F}}(g_q)$, hence $\phi(g_q) > \phi(f)$. Consequently, $\phi(f - g_q) = \phi(g_q) = w$ when q is big nenough, a contradiction to (3). □

By Theorem III.3 we now have the following corollaries:

Corollary III.3.1. *Let $\mathcal{F}$ be a circular filter on D of diameter $r \in]0,1[$. Then $\varphi_{\mathcal{F}}$ has extension to a norm that belongs to $Mult(A, \| \, . \, \|)$.*

Proof. Let $s \in]r,1[$ and let $d(a,s)$ be a disk that belongs to $\mathcal{F}$. As an element of $H(d(a,s))$, each element f of A is such that $\varphi_{\mathcal{F}}(f) = \lim_{\mathcal{F}} |f(x)|$ and that defines a multiplicative norm on A. □

Corollary III.3.2. *Let $\phi \in Mult(A, \| \, . \, \|) \setminus Mult_1(A, \| \, . \, \|)$. If the restriction of ϕ to $H(D)$ is of the form $\varphi_{\mathcal{F}}$ with $\mathcal{F}$ a circular filter on D of diameter $r \in]0,1[$, then ϕ is a norm on A.*

Proof. Indeed, given a disk L of diameter $s \in [r,1[$, which belongs to $\mathcal{F}$, $\varphi_{\mathcal{F}}$ is a norm on $H(L)$ which contains A. □

Corollary III.3.3. *Let $\phi \in Mult(A, \| \, . \, \|) \setminus Mult_1(A, \| \, . \, \|)$. If ϕ is not a norm on A, its restriction to $H(D)$ is $\| \, . \, \|$.*

Proof. Indeed, if the restriction of ϕ to $H(D)$ is of the form $\varphi_{\mathcal{F}}$ with $\mathcal{F}$ a circular filter on D of diameter $r \in]0,1[$, then by Corollary III.3.1 ϕ is a norm on A. □

The following Theorem III.4 is Theorem 22.33 in [10]:

Theorem III.4. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$ (resp. $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}(d(0,r^-))$). All zeros of f are of order one and the set of zeros of f is a sequence $(\alpha_n)_{n\in \mathbb{N}}$ such that $|\alpha_n| < |\alpha_{n+1}|$ if and only if the sequence $\left|\frac{a_n}{a_{n+1}}\right|$ is strictly increasing. Moreover, if these properties are satisfied, then the sequence of zeros of f in $\mathbb{K}$ (resp. in $d(0,r^-)$) is a sequence $(\alpha_n)_{n\in \mathbb{N}^*}$ such that $\lim_{n\to+\infty} |\alpha_n| = +\infty$ (resp. $\lim_{n\to+\infty} |\alpha_n| = r$) and $|\alpha_n| = \left|\frac{a_n}{a_{n+1}}\right|$.*

The following Theorem III.5 is also given in [5] as Theorem 25.5:

Theorem III.5. *Let $(a_j)_{j\in\mathbb{N}}$ be a sequence in $d(0,1^-)$ such that $0 < |a_n| \leq |a_{n+1}|$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} |a_n| = r$. Let $(q_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}^*$ and let $B \in]1,+\infty[$. There exists $f \in A(d(0,r^-))$ satisfying*

i) $f(0) = 1$

ii) $\|f\| \leq B\prod_{j=0}^{n}|\frac{a_n}{a_j}|^{q_j}$ *whenever* $n \in \mathbb{N}$

iii) *for each* $n \in \mathbb{N}$ a_n *is a zero of* f *of order* $z_n \geq q_n$.

Corollary III.5.1. *Let* $(a_j)_{j\in\mathbb{N}}$ *be a sequence in* $d(0,r^-)$ *such that* $0 < |a_n| \leq |a_{n+1}|$ *for every* $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} |a_n| = r$ *and let* $(q_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathbb{N}^*$ *such that*

$$\prod_{j=0}^{n}\Big(\frac{|a_n|}{r}\Big)^{q_j} > 0.$$

Let $B \in]1,+\infty[$. *There exists* $f \in A$ *satisfying*

i) $f(0) = 1$

ii) $\|f\| \leq B\prod_{j=0}^{\infty}\Big(\frac{r}{|a_n|}\Big)^{q_j}$ *whenever* $n \in \mathbb{N}$

iii) *for each* $n \in \mathbb{N}$, a_n *is a zero of* f *of order* $z_n \geq q_n$.

Notation. Let $(a_n)_{n\in \mathbb{N}}$ a sequence in D such that $|a_n| \leq |a_{n+1}|$ and $\lim\limits_{n\to+\infty} |a_n| = 1$ and let $(q_n)_{n\in \mathbb{N}}$ be a sequence of integers ($q_n \geq 0$). The family $(a_n,q_n)_{n\in \mathbb{N}}$ is called *a divisor of D*. The definition applies to a divisor where all q_n are null but finitely many.

The family of divisors of D is provided with a natural order: given two divisors $T = (a_n,q_n)_{n\in \mathbb{N}}$ and $E = (a_n,s_n)_{n\in \mathbb{N}}$, we say that $T \leq E$ if $q_n \leq s_n$ $\forall n \in \mathbb{N}$.

Let $f \in \mathcal{H}$ and let $(a_n,q_n)_{n\in \mathbb{N}}$ be the set of zeros of f, each zero a_n being of order q_n. We will denote by $\mathcal{T}(f)$ this family $(a_n,q_n)_{n\in \mathbb{N}}$ and the expression $\mathcal{T}(f)$ is then called *divisor of* f.

An interesting question was whether certain elements of $Mult(A,\|\,.\,\|)$ may have a kernel that is neither null nor a maximal ideal. The question was solved by Jesus Araujo thanks to this nice example [1].

In the proof of Theorem III.7, we will need the following Theorem III.6 that comes from Theorems 28.14 and 29.6 in [10].

Theorem III.6. *Let* $E = (a_n,q_n)_{n\in \mathbb{N}}$ *be a divisor on D with* $a_n \neq 0$ $\forall n \in \mathbb{N}$ *and let* $\varepsilon > 0$. *There exists* $f \in \mathcal{H}$ *such that* $\mathcal{T}(f) \geq E$, $f(0) = 1$ *and* $|f|(r) \leq |E|(r)(1+\varepsilon)$ $\forall r \in]0,1[$. *Moreover, if* $\mathbb{K}$ *is spherically complete, then there exists* $f \in B$ *such that* $\mathcal{T}(f) = E$.

Corollary III.6.1. *Let* $\mathbb{K}$ *be spherically complete. Let* $(a_j)_{j\in\mathbb{N}}$ *be a coroner sequence such that* $\prod_{n=0}^{\infty}|a_n| > 0$. *There exists* $f \in \mathcal{H}$ *admitting each* a_n *as a zero of order* 1 *and having no other zeros.*

Corollary III.6.2. *Let* $f,\ g \in A$ *be such that* $\mathcal{T}(g) \leq \mathcal{T}(f)$. *There exists* $h \in A$ *such that* $f = gh$.

Theorem III.7. (J. Araujo) *Let* $h(x) = \sum_{n=0}^{\infty} a_n x^n$ *and suppose that the sequence* $\Big(\frac{|a_n|}{|a_{n+1}|}\Big)_{n\in \mathbb{N}}$ *is strictly increasing, of limit* 1. *Then h belongs to A. Moreover,* *putting* $r_n = \dfrac{|a_n|}{|a_{n+1}|}$, $n \in \mathbb{N}$, *h admits a unique zero on each circle* $C(0, r_n)$ *and has no other zero in D.*

Let $\mathcal{N}$ *be an ultrafilter on* $\mathbb{N}$ *and for every* $f \in A$, *let* $\phi(f, n) = \|f\|_{d(\alpha_n, r)}$. *Let* $\varphi_r(f) = \lim_{\mathcal{N}} \phi(f, n)$.

Then φ_r *belongs to* $Mult(A, \|\ .\ \|)$ *and* $Ker(\varphi_r)$ *is neither null nor a maximal ideal of A. Moreover,* $Ker(\varphi_r)$ *does not depend on* $r \in]0, 1[$.

However each so defined semi-norm φ_r *belongs to the closure of* $Mult_1(A, \|\ .\ \|)$ *in* $Mult(A, \|\ .\ \|)$.

Proof. h belongs to A because the sequence (a_n) is bounded. Next, h has a unique zero α_n in each circle $C(0, r_n)$ and no other zero in D by Theorem III.4.

Let $\mathcal{M}$ be the ideal of the $f \in A$ such that $\lim_{\mathcal{N}} |f(\alpha_n)| = 0$. Of course h belongs to $\mathcal{M}$ and $Ker(\varphi_r)$ is strictly included in $\mathcal{M}$. Indeed, since h admits a unique zero in the disk $d(\alpha_n, r)$, it satisfies $\|h\|_{d(\alpha_n,, r)} = |h|(r_n)\dfrac{r_n}{r}$ and therefore $\lim_{\mathcal{N}} \|h\|_{d(\alpha_n, r)}) = \dfrac{1}{r}$, which proves that h does not belong to $Ker(\varphi_r)$.

On the other hand, we will prove that $Ker(\varphi_r)$ is not null. Let $(q_n)_{n \in \mathbb{N}^*}$ be a sequence of positive integers satisfying $q_n \le q_{n+1}$ $\forall n \in \mathbb{N}^*$, $\lim_{n \to +\infty} q_n = +\infty$ and such that the series $\sum_{n=1}^{+\infty} q_n \log(\dfrac{1}{r_n})$ converges: we can easily find the sequence (q_n) since $\lim_{n \to +\infty} r_n = 1$. Now, consider the divisor $(\alpha_n, q_n)_{n \in \mathbb{N}}$ of D. By Theorem III.6 there exists $g \in A$ admitting each α_n as a zero of order $t_n \ge q_n$ and such that $|g|(r_n) \le |T|(r_n) + 1$ $\forall n \in \mathbb{N}^*$. Consequently, g is bounded in D and hence belongs to A. Next, for every $n \in \mathbb{N}^*$, by Corollary II.8.1 we have

$$\|g\|_{d(\alpha_n, r)} \le |g|(r_n)\left(\frac{r}{r_n}\right)^{t_n} \le \|g\|\left(\frac{r}{r_n}\right)^{q_n}.$$

Since the sequence $(q_n)_{n \in \mathbb{N}^*}$ tends to $+\infty$ and the sequence (r_n) is increasing, we have $\lim_{n \to +\infty} \|g\|_{d(\alpha_n, r)} = 0$, which proves that g belongs to $Ker(\varphi_r)$.

Let $f \in Ker(\varphi_r)$ and let $s \in]0, 1[$. If $s < r$, it is obvious that f belongs to $Ker(\varphi_s)$. Now suppose $s > r$. Consider an element L of $\mathcal{N}$ such that $\inf_{n \in L} \|f\|_{d(\alpha_n, r)} = 0$. We will prove that $\inf_{n \in L} \|f\|_{d(\alpha_n, s)} = 0$.

For each $n \in \mathbb{N}^*$, by Theorem III.2, we have

$$\log(\|f\|_{d(\alpha_n, s)}) - \log(\|f\|_{d(\alpha_n, r)}) \tag{1}$$

$$\le \Big(\log(\|f\|_{d(\alpha_n, r_n)}) - \log(\|f\|_{d(\alpha_n, s)})\Big)\Big(\frac{\log(s) - \log(r)}{\log(r_n) - \log(s)}\Big).$$

Suppose that the sequence $\|f\|_{d(\alpha_n, s)}$ does not tend to 0.

There exists a sequence $(u_m)_{m\in\mathbb{N}}$ of $\mathbb{N}^*$ such that $\|f\|_{d(\alpha_{u_m},s)} > b,\ \forall m\in\mathbb{N}$ with $b>0$. But then, we get to a contradiction with (1).

Consequently, $\inf_{n\in L}\|f\|_{d(\alpha_n,s)} = 0$ and therefore f belongs to $Ker(\varphi_s)$, which proves that $Ker(\varphi_s) = Ker(\varphi_r)$.

Consider now a neighborhood $\mathcal{V}(\varphi_r, f_1,...,f_q,\varepsilon)$ of φ_r, where $f_1,...,f_q\in A$ and $\varepsilon>0$, with respect to the topology of pointwise convergence, i.e.
$\mathcal{V}(\varphi_r, f_1,...,f_q,\varepsilon)=\{\phi\in Mult(A,\|.\|)\mid \left|\varphi_r(f_j)-\phi x f_j)\right|_\infty\leq\varepsilon,\ j=1,...,q,\ q\in\mathbb{N}^*\}$.
By definition of that topology, there exists a subset G of $\mathbb{N}$ such that
$|\varphi_r(f_j)-\|f_j\|_{d(\alpha_n,r)}|_\infty\leq\varepsilon\ \forall n\in G,\forall j=1,...,q$. But now, in each disk $d(\alpha_n,r)$, we can take a class $d(b_n,r^-)$ where none of the f_j admits a zero, and hence we have $|f_j(b_n)| = \|f_j\|_{d(\alpha_n,r)}\ \forall j=1,...,q$ and hence $|\varphi_{b_n}(f_j)-\varphi_r(f_j)|_\infty\leq\varepsilon\ \forall j=1,...,q$. Consequently, φ_{b_n} belongs to $\mathcal{V}(\varphi_r, f_1,...,f_q,\varepsilon)$, which proves that φ_r lies in the closure of $Mult_1(A,\|\ .\ \|)$ and that finishes the proof of Theorem III.7. □

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