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BOUNDEDNESS, RATE OF CONVERGENCE AND GLOBAL BEHAVIOR OF A HIGHER-ORDER SYSTEM OF DIFFERENCE EQUATIONS

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This paper is dedicated to Professor Mehmed Nurkanović on the occasion of his 65^{th} *birthday*

ABSTRACT. In this paper, we conduct a comprehensive exploration of the dynamical characteristics of a higher-order non-symmetric system of difference equations. Our investigation covers various fundamental aspects, including the existence of equilibria, persistence, periodic points, boundedness, local behavior at equilibria, convergence rate, and global dynamics. Our results significantly extend and improve upon existing findings in the literature. Finally, theoretical findings are illustrated numerically.

1. INTRODUCTION

1.1. Motivation and review of the literature

One of the fundamental concepts of mathematics is the concept of difference, which are used to build a real-life mathematical model to demonstrate and analyze the behaviors and characteristics of models at different times. Basically, difference equations form discrete mathematics which describes changes quantitatively over small-time intervals, helpful in understanding many fields such as biology, ecology, economics, physics and their evolution. The history, meaning, and numerous uses of difference equations are addressed in this introduction, along with their fundamental implication in analyzing complex dynamical systems.

During the 17th century many mathematicians used difference equations to explore different dynamical systems. Two notables are Sir Isaac Newton and Pierre de Fermat who studied and used them first. Yet, the basis for present-day difference equations was laid out by the significant research of the French mathematician Abraham de Moivre in the 18th century. De Moivre studied recursive sequences, which are important in probability. His work helped us to understand how systems

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behave when recurring occurs. Later, other mathematicians like George Boole, George Polya, and George Udny Yule used difference equations based on his ideas and developed the new theory of difference equations. This made it easier to use these ideas in many fields. The study of difference equations includes many mathematical concepts, from simple first-order equations to more complicated ones.

Higher-order difference equations are very important for modeling complex problems that involve non-linearity and feedback. These higher-order difference equations help to understand how different variables interact and change over time, offering a better view of system behavior than lower-order equations. This shows the significance of how advanced mathematical techniques are essential for solving real-world problems more accurately [21]. Difference equations are used in many scientific fields because they provide unique ways to model and analyze situations. In biology and ecology, these are easy and important for studying how populations change and its impact, how species interact, and how diseases spread over time. By modeling these factors like birth rates, death rates, and migration, difference equations help us to predict population growth, risks of extinction, and disease spread, which aids in making better decisions for conservation of people's lives.

In medical sciences, problems related to drug pharmacokinetics, physiological processes, and disease progression are prescribed by using difference equations, which interprets the procedure of drugs behavior in the diseased body, and how diseases progress and its required control. These manufactured mathematical models help us to improve many factors like healthcare plans, medicines dosing, and patient cure by analyzing the treatments and interventions. One can predict and analyze the disease outbreaks, how patients respond to treatment and evaluate how healthcare resources are beneficial for the patient through these models. The healthcare system has become more efficient and effective by studying the prediction and analysis, leading to better patient treatments and overall health facilities for the population.

In engineering, many phenomena are calculated by using difference equations, that is, modeling dynamic systems of engineering, signal processing and control theory. For improving and redesigning the complex structures like electrical circuits, machines, and communication networks, engineers have been using mathematical models. Through these models, engineers ensure reliable final structure of their projects by simulating how these systems work in different inputs, they can predict how their design will behave in different circumstances depending upon necessary factors. They bring required changes to ensure the reliable work.

Considering the field of economy, difference equations are very important due to discrete behavior in many models, predicting financial trends and analyzing existing policies for development in economics. These modeled equations can help economists deal with many key economic factors, forecast future changes by considering GDP growth, inflation, and unemployment. Policy-makers by using difference equations make decisions for stability in the economy through interpreting the main factors such as adjusting interest rates or spending.

Similarly, in physics especially in quantum mechanics and computational physics, difference equations help to model many dynamic systems. Continuous mathematical models are converted into discrete form, so that these models make it easier to run simulations, analyze theories, and predict experimental results. Other than this these models are studied for simulating how fluid moves over time, different particles interact, gravity, and modeling quantum systems. Difference equations basically provide a flexible, trusted, and easy way to find the laws of nature and produce understanding of the universe's mysteries [5,29].

In conclusion, many scientific fields study the behavior of dynamic systems by using difference equations as a very powerful mathematical tool. Complex real-life problems have been solved from ancient time to modern era by these discrete dynamical models, that encourage us to understand and get remarkable and mind-blowing development in many fields specifically science, engineering, and economics. By connecting these theories with practical work, these discrete models enable policy-makers, researchers, and engineers to deal with real-life problems by taking smart decisions and using them improve society. That is why, many scientists have been exploring different aspects of difference equations, that is, boundedness, bifurcation analysis, persistence, periodic solutions, two-period solutions, stability and many more. For instance, Elsayed [8] explored the behavior of the difference equation in the following form:

$$\beta_{n+1} = \mathfrak{B}_1 + \frac{\mathfrak{B}_2\beta_{n-1} + \mathfrak{B}_3\beta_{n-m}}{\mathfrak{B}_4\beta_{n-1} + \mathfrak{B}_5\beta_{n-m}}, \tag{1.1}$$

where $\mathfrak{B}_\upsilon\,(\upsilon = 1,2,\cdots,5)$ and $\beta_\upsilon\,(\upsilon = -m,\cdots,0)$ are positive. Khan and El-Metwally [14] studied the behavior of the difference equation:

$$\beta_{n+1} = \mathfrak{B}_\mathfrak{n} + \frac{\beta_n^p}{\beta_{n-1}^p}, \tag{1.2}$$

with positive $\beta_\upsilon\,(\upsilon = -1,0)$. Li & Li [18] explored the dynamics of the difference equation:

$$\beta_{n+1} = \frac{\mathfrak{B}_1 + \mathfrak{B}_2\beta_n}{1 + \mathfrak{B}_3\beta_{n-m}}, \tag{1.3}$$

where $\mathfrak{B}_\upsilon\,(\upsilon=1,\cdots,3)$ and $\beta_\upsilon(\upsilon=-m,\cdots,0)$ are positive. Khan & Qureshi [15] explored the behavior of difference equation systems of the form:

$$\begin{aligned}\beta_{n+1}=&\frac{\mathfrak{B}_1\beta_{n-m}}{\mathfrak{B}_2+\mathfrak{B}_3\gamma_{n-m+1}^2},\gamma_{n+1}=\frac{\mathfrak{B}_4,\gamma_{n-m}}{\mathfrak{B}_5+\mathfrak{B}_6\beta_{n-m+1}^2},\\ \beta_{n+1}=&\frac{\mathfrak{B}_7\gamma_{n-m}}{\mathfrak{B}_8+\mathfrak{B}_9\beta_{n-m+1}^2},\gamma_{n+1}=\frac{\mathfrak{B}_{10}\beta_{n-m}}{\mathfrak{B}_{11}+\mathfrak{B}_{12}\gamma_{n-m+1}^2},\end{aligned}\tag{1.4}$$

with positive $\mathfrak{B}_\upsilon\,(\upsilon=1,\cdots,12)$ and $\beta_\upsilon,\gamma_\upsilon\,(\upsilon=-m,\cdots,0)$. Oğul et al. [24] examined the dynamics of difference equation of the type:

$$\beta_{n+1}=\frac{\beta_{n-17}}{\pm1\pm\beta_{n-2}\beta_{n-5}\beta_{n-8}\beta_{n-11}\beta_{n-14}\beta_{n-17}},\tag{1.5}$$

with positive $\beta_\upsilon\,(\upsilon=-17,\cdots,0)$. Taşdemir [31] examined the dynamics of a second-order system of the form:

$$\beta_{n+1}=\beta_{n-1}\gamma_n-1,\gamma_{n+1}=\gamma_{n-1}\beta_n-1,\tag{1.6}$$

where initial conditions are considered to be positive. Bešo et al. [3] examined the dynamics of a second-order difference equation:

$$\beta_{n+1}=\mathfrak{B}_1+\mathfrak{B}_2\frac{\beta_n}{\beta_{n-1}^2},\tag{1.7}$$

with positive $\mathfrak{B}_\upsilon\,(\upsilon=1,2)$ and $\beta_\upsilon\,(\upsilon=-1,0)$. Taşdemir [33] and Taşdemir et al. [34] explored the behavior of the following difference equation, which is a natural extension of the work of Bešo et al. [3]:

$$\beta_{n+1}=\mathfrak{B}_1+\mathfrak{B}_2\frac{\beta_n}{\beta_{n-m}^2},\tag{1.8}$$

and

$$\beta_{n+1}=\mathfrak{B}_1+\mathfrak{B}_2\frac{\beta_{n-m}}{\beta_n^2},\tag{1.9}$$

with positive $\mathfrak{B}_\upsilon\,(\upsilon=1,2)$ and $\beta_\upsilon\,(\upsilon=-m,\cdots,0)$. Furthermore, Taşdemir [35] reported the global stability of following system, which is an extension of the work of Bešo et al. [3] for two-species:

$$\beta_{n+1}=\mathfrak{B}_1+\mathfrak{B}_2\frac{\gamma_n}{\gamma_{n-1}^2},\ \gamma_{n+1}=\mathfrak{B}_3+\mathfrak{B}_4\frac{\beta_n}{\beta_{n-1}^2},\tag{1.10}$$

with positive $\mathfrak{B}_\upsilon\,(\upsilon=1,\cdots,4)$ and $\beta_\upsilon,\gamma_\upsilon\,(\upsilon=-1,0)$. For more interesting results regarding dynamical characteristics of difference equations and their high-order systems, we refer the reader to [1, 4, 6, 7, 9, 10, 19, 20, 27, 28, 30, 32, 36]. However, in the past few years, many mathematicians have explored the dynamical characteristics of discrete and continuous mathematical models described by systems of difference and differential equations in mathematical biology. For instance, Naik [22] explored the global dynamics of a SIR epidemic model with a Holling type-II treatment rate and a Crowley-Martin type functional response. Ghori et al. [13] analyzed the bifurcation and global dynamics of a fractional-order SEIR epidemic model with a saturation incidence rate. Naik et al. [23] examined the behavior of a SIR epidemic model in discrete form. Farman et al. [11] studied a Hepatitis B model using an evolutionary approach. Alshaikh et al. [2] investigated the dynamics of a discrete infection model. M.Kulenović et al. [16] investigated asymptotic behavior of a discrete-time density-dependent SI epidemic model with constant recruitment. A fractional model's mathematical modelling and dynamics were examined by Saadeh et al. [26].

Also, from a numerical point of view, the rate of convergence of solutions of difference equations and systems of difference equations is important, [12] and [17].

1.2. **Main findings**

Motivated by the aforementioned studies, the aim of this paper is to explore the global dynamics of the following difference equation system, which is a natural extension of the work of Taşdemir [35] to higher-order systems. Specifically, the $(m+1)$-order difference equation system of the following form will be investigated:

$$\beta_{n+1} = \mathfrak{B}_1 + \mathfrak{B}_2 \frac{\gamma_n}{\gamma_{n-m}^2}, \ \gamma_{n+1} = \mathfrak{B}_3 + \mathfrak{B}_4 \frac{\beta_n}{\beta_{n-m}^2}. \tag{1.11}$$

Alternatively, (1.11) takes the following form:

$$x_{n+1} = 1 + \Delta_1 \frac{y_n}{y_{n-m}^2}, \ y_{n+1} = 1 + \Delta_2 \frac{x_n}{x_{n-m}^2}, \tag{1.12}$$

by taking $x_n = \frac{\beta_n}{\mathfrak{B}_1}$, $y_n = \frac{\gamma_n}{\mathfrak{B}_3}$ where $\Delta_1 = \frac{\mathfrak{B}_2}{\mathfrak{B}_1 \mathfrak{B}_3} > 0$, $\Delta_2 = \frac{\mathfrak{B}_4}{\mathfrak{B}_1 \mathfrak{B}_3} > 0$. By replacing, x by β and y by γ, (1.12) can also be written as:

$$\beta_{n+1} = 1 + \Delta_1 \frac{\gamma_n}{\gamma_{n-m}^2}, \ \gamma_{n+1} = 1 + \Delta_2 \frac{\beta_n}{\beta_{n-m}^2}. \tag{1.13}$$

Specifically, the following are our primary results in this paper:

- Persistence and boundedness of the positive solution of higher-order discrete system (1.13),
- Local dynamics at equilibria of higher-order discrete system (1.13),
- Existence of periodic points,
- Global dynamics and rate of convergence of higher-order discrete system (1.13),
- Validation of theoretical results numerically.

1.3. **Paper layout**

The rest of the paper is organized as follows: Section 2 explores the linearized form and existence of equilibria for system (1.13). In Section 3, the persistence and boundedness of positive solutions for the discrete system (1.13) will be examined. Periodic points, global analysis, and convergence rate are discussed in Sections 4 and 5, respectively. Section 6 presents numerical simulations by using Wolfram Mathematica to verify the theoretical results. The paper is finally concluded and future work is outlined in Section 7.

2. Linearized form and existence of equilibria

Theorem 2.1. *The higher-order discrete system (1.13) has equilibria*

$$\Gamma_1=\left(\tfrac{1+\Delta_1-\Delta_2+\Psi}{2},\tfrac{1-\Delta_1+\Delta_2+\Psi}{2}\right)\quad \textit{and}\quad \Gamma_2=\left(\tfrac{1+\Delta_1-\Delta_2-\Psi}{2},\tfrac{1-\Delta_1+\Delta_2-\Psi}{2}\right),$$

where $\Psi=\sqrt{4\Delta_2+(1+\Delta_1-\Delta_2)^2}$.

Proof. If the higher-order discrete system (1.13) has equilibrium $\Gamma=(\bar{\beta},\bar{\gamma})$ then

$$\bar{\beta}=1+\Delta_1\frac{\bar{\gamma}}{\bar{\gamma}^2},\ \bar{\gamma}=1+\Delta_2\frac{\bar{\beta}}{\bar{\beta}^2}, \tag{2.1}$$

which further takes the following form:

$$\bar{\gamma}=\frac{\Delta_1}{\bar{\beta}-1}, \tag{2.2}$$

and

$$\bar{\beta}=\frac{\Delta_2}{\bar{\gamma}-1}. \tag{2.3}$$

Now using (2.2) into (2.3), one gets:

$$\bar{\beta}=\frac{\Delta_2}{\frac{\Delta_1}{\bar{\beta}-1}-1}, \tag{2.4}$$

which further takes the following form:

$$\bar{\beta}^2-(1+\Delta_1-\Delta_2)\,\bar{\beta}-\Delta_2=0, \tag{2.5}$$

whose roots are:

$$\bar{\beta}=\frac{1+\Delta_1-\Delta_2+\Psi}{2}, \tag{2.6}$$

and

$$\bar{\beta}=\frac{1+\Delta_1-\Delta_2-\Psi}{2}. \tag{2.7}$$

Now using (2.6) into (2.2), one gets:

$$\bar{\gamma} = \frac{\Delta_1}{\frac{1+\Delta_1-\Delta_2+\Psi}{2} - 1}. \tag{2.8}$$

After some simplifications, from (2.8) one gets:

$$\bar{\gamma} = \frac{1-\Delta_1+\Delta_2+\Psi}{2}. \tag{2.9}$$

So, from (2.6) and (2.9), the one equilibrium solution of higher-order discrete system (1.13) is $\Gamma_1 = \left(\frac{1+\Delta_1-\Delta_2+\Psi}{2}, \frac{1-\Delta_1+\Delta_2+\Psi}{2}\right)$. Similarly, substituting (2.7) into (2.2), after doing routine calculations the desired second equilibrium point of higher-order discrete system (1.13) is $\Gamma_2 = \left(\frac{1+\Delta_1-\Delta_2-\Psi}{2}, \frac{1-\Delta_1+\Delta_2-\Psi}{2}\right)$. □

Next, the linearized form of a system of difference equation (1.13) about $\Gamma = (\bar{\beta}, \bar{\gamma})$ under the map $(\beta_{n+1}, \beta_n, \beta_{n-1}, \cdots, \beta_{n-m+1}, \gamma_{n+1}, \gamma_n, \cdots, \gamma_{n-m+1}) \mapsto (f_1, f_2, \cdots, f_{n-m}, g_1, g_2, \cdots, g_{n-m})$ is

$$\varpi_{n+1} := J_\Gamma \varpi_n, \tag{2.10}$$

where:

$$\varpi_n = \begin{pmatrix} \beta_n \\ \beta_{n-1} \\ \vdots \\ \beta_{n-m} \\ \gamma_n \\ \gamma_{n-1} \\ \vdots \\ \gamma_{n-m} \end{pmatrix}, \tag{2.11}$$

$$J_\Gamma = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{\Delta_1}{\bar{\gamma}^2} & 0 & \dots & 0 & -\frac{2\Delta_1}{\bar{\gamma}^2} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{\Delta_2}{\bar{\beta}^2} & 0 & \dots & 0 & -\frac{2\Delta_2}{\bar{\beta}^2} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \tag{2.12}$$

and

$$\begin{aligned} f_1 &= 1+\Delta_1\frac{\gamma_n}{\gamma_{n-m}^2}, f_2 = \beta_n, \cdots, f_{n-m} = \beta_{n-m+1}, \\ g_1 &= 1+\Delta_2\frac{\beta_n}{\beta_{n-m}^2}, g_2 = \gamma_n, \cdots, g_{n-m} = \gamma_{n-m+1}. \end{aligned} \tag{2.13}$$

3. Boundedness and persistence

Theorem 3.1. *Solution* $\{(\beta_n,\gamma_n)\}_{n=-m}^{\infty}$ *of a higher-order system (1.13) is bounded and persisted if*

$$\Delta_1\Delta_2 < 1. \tag{3.1}$$

Proof. If the higher-order system (1.13) has a solution of the form $\{(\beta_n,\gamma_n)\}_{n=-m}^{\infty}$ then

$$\beta_n \geq 1,\ \gamma_n \geq 1. \tag{3.2}$$

Further, from (1.13) and (3.2), we have:

$$\beta_{n+1} \leq 1+\Delta_1+\Delta_1\Delta_2\beta_{n-1}, \gamma_{n+1} \leq 1+\Delta_2+\Delta_1\Delta_2\gamma_{n-1}. \tag{3.3}$$

From the first inequality of (3.3), one has:

$$u_{n+1} = 1+\Delta_1+\Delta_1\Delta_2 u_{n-1}, \tag{3.4}$$

whose solution is:

$$u_n = \frac{1+\Delta_1}{1-\Delta_1\Delta_2} + c_1\left(\Delta_1\Delta_2\right)^{\frac{n}{2}} + c_2(-1)^n\left(\Delta_1\Delta_2\right)^{\frac{n}{2}}, \tag{3.5}$$

where $c_\upsilon(\upsilon = 1,2)$ depends on $u_\upsilon(\upsilon = -1,0)$. Now, the second inequality of (3.3) yields:

$$v_{n+1} = 1+\Delta_2+\Delta_1\Delta_2 v_{n-1}, \tag{3.6}$$

whose solution is:

$$v_n = \frac{1+\Delta_2}{1-\Delta_1\Delta_2} + c_3\left(\Delta_1\Delta_2\right)^{\frac{n}{2}} + c_4(-1)^n\left(\Delta_1\Delta_2\right)^{\frac{n}{2}}, \tag{3.7}$$

where $c_\upsilon(\upsilon = 3,4)$ depends on $v_\upsilon(\upsilon = -1,0)$. Now, if one consider the solution in which $u_{-1} = \beta_{-1}$, $u_0 = \beta_0$, $v_{-1} = \gamma_{-1}$, $v_0 = \gamma_0$ and additionally, if $\Delta_1\Delta_2 < 1$ then from (3.3), (3.5) and (3.7), we have:

$$\beta_n \leq \frac{1+\Delta_1}{1-\Delta_1\Delta_2},\ \gamma_n \leq \frac{1+\Delta_2}{1-\Delta_1\Delta_2}. \tag{3.8}$$

Finally, from (3.2) and (3.8), one has the following:

$$1 \leq \beta_n \leq \frac{1+\Delta_1}{1-\Delta_1\Delta_2}, 1 \leq \gamma_n \leq \frac{1+\Delta_2}{1-\Delta_1\Delta_2}. \tag{3.9}$$

□

Theorem 3.2. *The invariant rectangle for the higher-order system (1.13) is*

$$\left[1, \frac{1+\Delta_1}{1-\Delta_1\Delta_2}\right] \times \left[1, \frac{1+\Delta_2}{1-\Delta_1\Delta_2}\right].$$

Proof. If higher-order system (1.13) has a solution of the form: $\{(\beta_n, \gamma_n)\}_{n=-m}^{\infty}$ with $\beta_{-m}, \cdots, \beta_0 \in \left[1, \frac{1+\Delta_1}{1-\Delta_1\Delta_2}\right]$ and $\gamma_{-m}, \cdots, \gamma_0 \in \left[1, \frac{1+\Delta_2}{1-\Delta_1\Delta_2}\right]$, then

$$\begin{aligned} 1 \le \beta_1 = 1 + \Delta_1 \frac{\gamma_0}{\gamma_{-m}^2} \le \frac{1+\Delta_1}{1-\Delta_1\Delta_2}, \\ 1 \le \gamma_1 = 1 + \Delta_2 \frac{\beta_0}{\beta_{-m}^2} \le \frac{1+\Delta_2}{1-\Delta_1\Delta_2}. \end{aligned} \tag{3.10}$$

From (3.10), it can be concluded that $\beta_1 \in \left[1, \frac{1+\Delta_1}{1-\Delta_1\Delta_2}\right]$ and $\gamma_1 \in \left[1, \frac{1+\Delta_2}{1-\Delta_1\Delta_2}\right]$.

Additionally, it can be deduced by induction that $\beta_{k+1} \in \left[1, \frac{1+\Delta_1}{1-\Delta_1\Delta_2}\right]$ and $\gamma_{k+1} \in \left[1, \frac{1+\Delta_2}{1-\Delta_1\Delta_2}\right]$ if $\beta_k \in \left[1, \frac{1+\Delta_1}{1-\Delta_1\Delta_2}\right]$ and $\gamma_k \in \left[1, \frac{1+\Delta_2}{1-\Delta_1\Delta_2}\right]$. □

4. Global dynamic behavior and periodic points

Theorem 4.1. *The equilibrium point Γ_1 of system (1.13) is stable if*

$$\frac{8\Delta_1}{(1-\Delta_1+\Delta_2+\Psi)^2 - 4\Delta_1} < 1, \quad \frac{8\Delta_2}{(1+\Delta_1-\Delta_2+\Psi)^2 - 4\Delta_2} < 1. \tag{4.1}$$

Proof. From Γ_1, (2.10) gives

$$\varpi_{n+1} := J|_{\Gamma_1} \varpi_n, \tag{4.2}$$

where from (2.12), one has:

$$J|_{\Gamma_1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{4\Delta_1}{(1-\Delta_1+\Delta_2+\Psi)^2} & 0 & \dots & 0 & \frac{-8\Delta_1}{(1-\Delta_1+\Delta_2+\Psi)^2} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{4\Delta_2}{(1+\Delta_1-\Delta_2+\Psi)^2} & 0 & \dots & 0 & \frac{-8\Delta_2}{(1+\Delta_1-\Delta_2+\Psi)^2} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{4.3}$$

If the eigenvalues of $J|_{\Gamma_1}$ are λ_l $(l = 1, \cdots, 2m+2)$ and

$$P = diag(\zeta_1, \zeta_2, \cdots, \zeta_{m+1}, \zeta_{m+2}, \cdots, \zeta_{2m+2})$$

is a diagonal matrix where:

$$\begin{aligned} \zeta_1 = \zeta_{m+2} = 1, \\ \zeta_{k+1} = \zeta_{m+2+k} = 1 - k\varepsilon, 1 \le k \le m \text{ where } 0 < \varepsilon < 1, \end{aligned} \tag{4.4}$$

and

$$0 < \varepsilon < \min\left\{\frac{1}{m}\left(1-\frac{8\Delta_1}{(1-\Delta_1+\Delta_2+\Psi)^2-4\Delta_1}\right), \frac{1}{m}\left(1-\frac{8\Delta_2}{(1+\Delta_1-\Delta_2+\Psi)^2-4\Delta_2}\right)\right\}, \tag{4.5}$$

then

$$\Lambda = \begin{pmatrix} 0 & 0 \ \dots & 0 & 0 & \frac{4\Delta_1\zeta_1\zeta_{m+2}^{-1}}{(1-\Delta_1+\Delta_2+\Psi)^2} & 0 \ \dots & 0 & \frac{-8\Delta_1\zeta_1\zeta_{2m+2}^{-1}}{(1-\Delta_1+\Delta_2+\Psi)^2} \\ \zeta_2\zeta_1^{-1} & 0 \ \dots & 0 & 0 & 0 & 0 \ \dots & 0 & 0 \\ \vdots & \vdots \ \ddots & \vdots & \vdots & \vdots & \ddots \ \vdots & \vdots & \vdots \\ 0 & 0 \ \dots & \zeta_{m+1}\zeta_m^{-1} & 0 & 0 & 0 \ \dots & 0 & 0 \\ \frac{4\Delta_2\zeta_{m+2}\zeta_1^{-1}}{(1+\Delta_1-\Delta_2+\Psi)^2} & 0 \ \dots & 0 & \frac{-8\Delta_2\zeta_{m+2}\zeta_{m+1}^{-1}}{(1+\Delta_1-\Delta_2+\Psi)^2} & 0 & 0 \ \dots & 0 & 0 \\ 0 & 0 \ \dots & 0 & 0 & \zeta_{m+3}\zeta_{m+2}^{-1} & 0 \ \dots & 0 & 0 \\ \vdots & \vdots \ \ddots & \vdots & \vdots & \vdots & \ddots \ \vdots & \vdots & \vdots \\ 0 & 0 \ \dots & 0 & 0 & 0 & 0 \ \dots & \zeta_{2m+2}\zeta_{2m+1}^{-1} & 0 \end{pmatrix}, \tag{4.6}$$

where $\Lambda = PJ|_{\Gamma_1}P^{-1}$. Also,

$$\begin{aligned} &0 < \zeta_{m+1} < \cdots < \zeta_2 < \zeta_1, \\ &0 < \zeta_{2m+2} < \cdots < \zeta_{m+2}. \end{aligned} \tag{4.7}$$

From (4.7), one gets:

$$\zeta_2\zeta_1^{-1} < 1, \cdots, \zeta_{m+1}\zeta_m^{-1} \text{ and } \zeta_{m+3}\zeta_{m+2}^{-1} < 1, \cdots, \zeta_{2m+2}\zeta_{2m+1}^{-1} < 1. \tag{4.8}$$

Equations (4.4) and (4.5) yield:

$$\begin{aligned} &\frac{4\Delta_1\zeta_1\zeta_{m+2}^{-1}}{(1-\Delta_1+\Delta_2+\Psi)^2} + \frac{8\Delta_1\zeta_1\zeta_{2m+2}^{-1}}{(1-\Delta_1+\Delta_2+\Psi)^2} = \frac{4\Delta_1}{(1-\Delta_1+\Delta_2+\Psi)^2}\left(1+\frac{2}{1-m\varepsilon}\right) < 1, \\ &\frac{4\Delta_2\zeta_{m+2}\zeta_1^{-1}}{(1+\Delta_1-\Delta_2+\Psi)^2} + \frac{8\Delta_2\zeta_{m+2}\zeta_{m+1}^{-1}}{(1+\Delta_1-\Delta_2+\Psi)^2} = \frac{4\Delta_2}{(1+\Delta_1-\Delta_2-\Psi)^2}\left(1+\frac{2}{1-m\varepsilon}\right) < 1. \end{aligned} \tag{4.9}$$

Finally, equations (4.8) and (4.9) yield:

$$\begin{aligned} \max_{1\le k\le 2m+2}|\lambda_k| = \left\|PJ|_{\Gamma_1}P^{-1}\right\| = \max\Bigg\{ &\zeta_2\zeta_1^{-1}, \cdots, \zeta_{m+1}\zeta_m^{-1}, \frac{4\Delta_1\zeta_1\zeta_{m+2}^{-1}}{(1-\Delta_1+\Delta_2+\Psi)^2} \\ &+ \frac{8\Delta_1\zeta_1\zeta_{2m+2}^{-1}}{(1-\Delta_1+\Delta_2+\Psi)^2}, \zeta_{m+3}\zeta_{m+2}^{-1}, \cdots, \zeta_{2m+2}\zeta_{2m+1}^{-1}, \\ &\frac{4\Delta_2\zeta_{m+2}\zeta_1^{-1}}{(1+\Delta_1-\Delta_2+\Psi)^2} + \frac{8\Delta_2\zeta_{m+2}\zeta_{m+1}^{-1}}{(1+\Delta_1-\Delta_2+\Psi)^2}\Bigg\} < 1. \end{aligned} \tag{4.10}$$

□

Theorem 4.2. Γ_2 of system (1.13) is stable if

$$\begin{aligned} \frac{8\Delta_1}{(1-\Delta_1+\Delta_2-\Psi)^2-4\Delta_1} &< 1, \\ \frac{8\Delta_2}{(1+\Delta_1-\Delta_2-\Psi)^2-4\Delta_2} &< 1. \end{aligned} \tag{4.11}$$

Proof. Using Γ_2 in (2.10), one gets:

$$\varpi_{n+1} := J|_{\Gamma_2}\varpi_n, \tag{4.12}$$

where from (2.12), one has:

$$J|_{\Gamma_2} = \begin{pmatrix} 0 & 0\dots 0 & 0 & \frac{4\Delta_1}{(1-\Delta_1+\Delta_2-\Psi)^2} & 0\ \dots 0 & \frac{-8\Delta_1}{(1-\Delta_1+\Delta_2-\Psi)^2} \\ 1 & 0\dots 0 & 0 & 0 & 0\ \dots 0 & 0 \\ \vdots & \vdots\ddots\vdots & \vdots & \vdots & \ddots\ \vdots\ \vdots & \vdots \\ 0 & 0\dots 1 & 0 & 0 & 0\ \dots 0 & 0 \\ \frac{4\Delta_2}{(1+\Delta_1-\Delta_2-\Psi)^2} & 0\dots 0 & \frac{-8\Delta_2}{(1+\Delta_1-\Delta_2-\Psi)^2} & 0 & 0\ \dots 0 & 0 \\ 0 & 0\dots 0 & 0 & 1 & 0\ \dots 0 & 0 \\ \vdots & \vdots\ddots\vdots & \vdots & \vdots & \ddots\ \vdots\ \vdots & \vdots \\ 0 & 0\dots 0 & 0 & 0 & 0\ \dots 1 & 0 \end{pmatrix}. \tag{4.13}$$

Now if the eigenvalues of $J|_{\Gamma_2}$ are λ_l $(l = 1,\cdots,2m+2)$ and by considering the diagonal matrix $P = diag(\zeta_1,\zeta_2,\cdots,\zeta_{m+1},\zeta_{m+2},\cdots,\zeta_{2m+2})$ along with (4.4) holds. Moreover,

$$\begin{aligned} 0<\varepsilon<\min\Bigg\{ &\frac{1}{m}\left(1-\frac{8\Delta_1}{(1-\Delta_1+\Delta_2-\Psi)^2-4\Delta_1}\right), \\ &\frac{1}{m}\left(1-\frac{8\Delta_2}{(1+\Delta_1-\Delta_2-\Psi)^2-4\Delta_2}\right)\Bigg\}, \end{aligned} \tag{4.14}$$

then

$$\widehat{\Lambda} = \begin{pmatrix} 0 & 0\dots & 0 & 0 & \frac{4\Delta_1\zeta_1\zeta_{m+2}^{-1}}{(1-\Delta_1+\Delta_2-\Psi)^2} & 0\ \dots & 0 & \frac{-8\Delta_1\zeta_1\zeta_{2m+2}^{-1}}{(1-\Delta_1+\Delta_2-\Psi)^2} \\ \zeta_2\zeta_1^{-1} & 0\dots & 0 & 0 & 0 & 0\ \dots & 0 & 0 \\ \vdots & \vdots\ddots & \vdots & \vdots & \vdots & \ddots\ \vdots & \vdots & \vdots \\ 0 & 0\dots & \zeta_{m+1}\zeta_m^{-1} & 0 & 0 & 0\ \dots & 0 & 0 \\ \frac{4\Delta_2\zeta_{m+2}\zeta_1^{-1}}{(1+\Delta_1-\Delta_2-\Psi)^2} & 0\dots & 0 & \frac{-8\Delta_2\zeta_{m+2}\zeta_{m+1}^{-1}}{(1+\Delta_1-\Delta_2-\Psi)^2} & 0 & 0\ \dots & 0 & 0 \\ 0 & 0\dots & 0 & 0 & \zeta_{m+3}\zeta_{m+2}^{-1} & 0\ \dots & 0 & 0 \\ \vdots & \vdots\ddots & \vdots & \vdots & \vdots & \ddots\ \vdots & \vdots & \vdots \\ 0 & 0\dots & 0 & 0 & 0 & 0\ \dots & \zeta_{2m+2}\zeta_{2m+1}^{-1} & 0 \end{pmatrix}, \tag{4.15}$$

where $\widehat{\Lambda} = PJ|_{\Gamma_2}P^{-1}$. Now, if (4.8) and (4.14) hold, then:

$$\frac{4\Delta_1\zeta_1\zeta_{m+2}^{-1}}{(1-\Delta_1+\Delta_2-\Psi)^2} + \frac{8\Delta_1\zeta_1\zeta_{2m+2}^{-1}}{(1-\Delta_1+\Delta_2-\Psi)^2} = \frac{4\Delta_1}{(1-\Delta_1+\Delta_2-\Psi)^2}\left(1+\frac{2}{1-m\varepsilon}\right) < 1,$$

$$\frac{4\Delta_2\zeta_{m+2}\zeta_1^{-1}}{(1+\Delta_1-\Delta_2-\Psi)^2} + \frac{8\Delta_2\zeta_{m+2}\zeta_{m+1}^{-1}}{(1+\Delta_1-\Delta_2-]\Psi)^2} = \frac{4\Delta_2}{(1+\Delta_1-\Delta_2-\Psi)^2}\left(1+\frac{2}{1-m\varepsilon}\right) < 1. \tag{4.16}$$

From (4.8) and (4.16), we get:

$$\max_{1\le k\le 2m+2}|\lambda_k| = \left\|PJ|_{\Gamma_2}P^{-1}\right\| = \max\left\{\zeta_2\zeta_1^{-1},\cdots,\zeta_{m+1}\zeta_m^{-1},\frac{4\Delta_1\zeta_1\zeta_{m+2}^{-1}}{(1-\Delta_1+\Delta_2-\Psi)^2}+\frac{8\Delta_1\zeta_1\zeta_{2m+2}^{-1}}{(1-\Delta_1+\Delta_2-\Psi)^2},\zeta_{m+3}\zeta_{m+2}^{-1},\cdots,\zeta_{2m+2}\zeta_{2m+1}^{-1},\frac{4\Delta_2\zeta_{m+2}\zeta_1^{-1}}{(1+\Delta_1-\Delta_2-\Psi)^2}+\frac{8\Delta_2\zeta_{m+2}\zeta_{m+1}^{-1}}{(1+\Delta_1-\Delta_2-\Psi)^2}\right\} < 1. \tag{4.17}$$

□

Theorem 4.3. Γ_1 *of a higher-order system (1.13) is a global attractor if* $\Delta_1,\Delta_2 \in \left(0,\frac{1}{2}\right)$.

Proof. If higher-order system (1.13) has a solution $\{(\beta_n,\gamma_n)\}_{n=-m}^{\infty}$ such that $\liminf_{n\to\infty}\beta_n = \mathfrak{L}_1$, $\liminf_{n\to\infty}\gamma_n = \mathfrak{L}_2$, $\limsup_{n\to\infty}\beta_n = \mathfrak{B}_1$, and $\limsup_{n\to\infty}\gamma_n = \mathfrak{B}_2$, then:

$$1 < \mathfrak{L}_1 = \liminf_{n\to\infty}\beta_n,\ 1 < \mathfrak{L}_2 = \liminf_{n\to\infty}\gamma_n,$$
$$\mathfrak{B}_1 = \limsup_{n\to\infty}\beta_n < \infty,\ \mathfrak{B}_2 = \limsup_{n\to\infty}\gamma_n < \infty. \tag{4.18}$$

From (1.13) and (4.18), we have:

$$\mathfrak{B}_1 \le 1+\Delta_1\frac{\mathfrak{B}_2}{\mathfrak{L}_2{}^2},\ \mathfrak{L}_1 \ge 1+\Delta_1\frac{\mathfrak{L}_2}{\mathfrak{B}_2{}^2}, \tag{4.19}$$

and

$$\mathfrak{B}_2 \le 1+\Delta_2\frac{\mathfrak{B}_1}{\mathfrak{L}_1{}^2},\ \mathfrak{L}_2 \ge 1+\Delta_2\frac{\mathfrak{L}_1}{\mathfrak{B}_1{}^2}. \tag{4.20}$$

The first and second inequalities of (4.19) and (4.20) yield:

$$\mathfrak{B}_1+\Delta_2\frac{\mathfrak{L}_1}{\mathfrak{B}_1} \le \mathfrak{B}_1\mathfrak{L}_2 \le \mathfrak{L}_2+\Delta_1\frac{\mathfrak{B}_2}{\mathfrak{L}_2}. \tag{4.21}$$

Similarly, the second and first inequalities of (4.19) and (4.20) yield:

$$\mathfrak{B}_2+\Delta_1\frac{\mathfrak{L}_2}{\mathfrak{B}_2} \le \mathfrak{B}_2\mathfrak{L}_1 \le \mathfrak{L}_1+\Delta_2\frac{\mathfrak{B}_1}{\mathfrak{L}_1}. \tag{4.22}$$

From (4.21) and (4.22), one gets:

$$\mathfrak{B}_1 + \Delta_2 \frac{\mathfrak{L}_1}{\mathfrak{B}_1} + \mathfrak{B}_2 + \Delta_1 \frac{\mathfrak{L}_2}{\mathfrak{B}_2} \leq \mathfrak{L}_1 + \Delta_2 \frac{\mathfrak{B}_1}{\mathfrak{L}_1} + \mathfrak{L}_2 + \Delta_1 \frac{\mathfrak{B}_2}{\mathfrak{L}_2}, \tag{4.23}$$

which implies that:

$$(\mathfrak{B}_1 - \mathfrak{L}_1)\left(1 - \Delta_2\left(\frac{1}{\mathfrak{L}_1} + \frac{1}{\mathfrak{B}_1}\right)\right) + (\mathfrak{B}_2 - \mathfrak{L}_2)\left(1 - \Delta_1\left(\frac{1}{\mathfrak{B}_2} + \frac{1}{\mathfrak{L}_2}\right)\right) \leq 0. \tag{4.24}$$

If $\Delta_1, \Delta_2 \in \left(0, \frac{1}{2}\right)$ then from (4.24), one can observe the following:

$$1 - \Delta_2\left(\frac{1}{\mathfrak{B}_1} + \frac{1}{\mathfrak{L}_1}\right) > 0,\ 1 - \Delta_1\left(\frac{1}{\mathfrak{B}_2} + \frac{1}{\mathfrak{L}_2}\right) > 0, \tag{4.25}$$

and finally, from (4.24) one gets $\mathfrak{B}_1 = \mathfrak{L}_1$ and $\mathfrak{B}_2 = \mathfrak{L}_2$. □

Hereafter, it is shown that $\Gamma_{1,2}$ of (1.13) are periodic points of period-$1, 2, \cdots, n$.

Theorem 4.4. *Γ_1 and Γ_2 of (1.13) are periodic points of period-$1, 2, \cdots, n$.*

Proof. From (1.13), we denote:

$$\Psi := (f_1, f_2), \tag{4.26}$$

with f_1 and f_2 as defined in (2.13). From (4.26), one obtains:

$$\begin{aligned}
&\Psi|_{\Gamma_1} = \Gamma_1, \\
&\Psi^2 = \left(1 + \Delta_1 \frac{f_2}{f_2^2}, 1 + \Delta_2 \frac{f_1}{f_1^2}\right) && \Rightarrow \Psi^2|_{\Gamma_1} = \Gamma_1, \\
&\Psi^3 = \left(1 + \Delta_1 \frac{f_2^2}{(f_2^2)^2}, 1 + \Delta_2 \frac{f_1^2}{(f_1^2)^2}\right) && \Rightarrow \Psi^3|_{\Gamma_1} = \Gamma_1, \\
&\quad \vdots && \quad \vdots \\
&\Psi^n = \left(1 + \Delta_1 \frac{f_2^{n-1}}{\left(f_2^2\right)^{n-1}}, 1 + \Delta_2 \frac{f_1^{n-1}}{\left(f_1^2\right)^{n-1}}\right) && \Rightarrow \Psi^n|_{\Gamma_1} = \Gamma_1.
\end{aligned} \tag{4.27}$$

So, from (4.27) one can obtain that Γ_1 is a periodic point of period-$1, 2, \cdots, n$. Moreover, the first equation of (4.27) implies that Γ_1 of (1.13) is a periodic point of prime period-1. By a similar procedure one can prove that Γ_2 is also a periodic point of period-$1, 2, \cdots, n$. □

5. Rate of convergence

Theorem 5.1. *If $\{(\beta_n,\gamma_n)\}_{n=-m}^{\infty}$ is the solution of a higher-order system (1.13) such that $\lim_{n\to\infty}\beta_n=\bar{\beta}$ and $\lim_{n\to\infty}\gamma_n=\bar{\gamma}$ then:*

$$\varphi_n=\begin{pmatrix}\varphi_n^1\\ \varphi_{n-1}^1\\ \vdots\\ \varphi_{n-m}^1\\ \varphi_n^2\\ \varphi_{n-1}^2\\ \vdots\\ \varphi_{n-m}^2\end{pmatrix}, \tag{5.1}$$

satisfying

$$\lim_{n\to\infty}\sqrt[n]{||\varphi_n||}=|\lambda J|_{\Gamma}|,\ \lim_{n\to\infty}\frac{||\varphi_{n+1}||}{||\varphi_n||}=|\lambda J|_{\Gamma}|, \tag{5.2}$$

where the norm of φ_n is defined by

$$||\varphi_n||=\sqrt{(\varphi_n^1)^2+\cdots+\left(\varphi_{n-m}^1\right)^2+(\varphi_n^2)^2\cdots+\left(\varphi_{n-m}^2\right)^2}.$$

Proof. If $\lim_{n\to\infty}\beta_n=\bar{\beta}$ and $\lim_{n\to\infty}\gamma_n=\bar{\gamma}$ then:

$$\begin{aligned}\beta_{n+1}-\bar{\beta}&=\Delta_1\frac{\gamma_n}{\gamma_{n-m}^2}-\Delta_1\frac{1}{\bar{\gamma}},\\ &=\frac{\Delta_1}{\gamma_{n-m}^2}(\gamma_n-\bar{\gamma})-\frac{\Delta_1(\gamma_{n-m}+\bar{\gamma})}{\bar{\gamma}\gamma_{n-m}^2}(\gamma_{n-m}-\bar{\gamma}),\\ \gamma_{n+1}-\bar{\gamma}&=\Delta_2\frac{\beta_n}{\beta_{n-m}^2}-\Delta_2\frac{1}{\bar{\beta}},\\ &=\frac{\Delta_2}{\beta_{n-m}^2}\left(\beta_n-\bar{\beta}\right)-\frac{\Delta_2\left(\beta_{n-m}+\bar{\beta}\right)}{\bar{\beta}\beta_{n-m}^2}\left(\beta_{n-m}-\bar{\beta}\right).\end{aligned} \tag{5.3}$$

After taking:

$$\varphi_n^1=\beta_n-\bar{\beta},\ \ \varphi_n^2=\gamma_n-\bar{\gamma}. \tag{5.4}$$

From (5.4) and (5.3), one has:

$$\begin{aligned}\varphi_{n+1}^1&=\alpha_{11}\varphi_n^2+\alpha_{12}\varphi_{n-m}^2,\\ \varphi_{n+1}^2&=\alpha_{21}\varphi_n^1+\alpha_{22}\varphi_{n-m}^1,\end{aligned} \tag{5.5}$$

where

$$\alpha_{11}=\frac{\Delta_1}{\gamma_{n-m}^2},\ \alpha_{12}=-\frac{\Delta_1(\gamma_{n-m}+\bar{\gamma})}{\bar{\gamma}\gamma_{n-m}^2},\ \alpha_{21}=\frac{\Delta_2}{\beta_{n-m}^2},\ \alpha_{22}=-\frac{\Delta_2\left(\beta_{n-m}+\bar{\beta}\right)}{\bar{\beta}\beta_{n-m}^2}. \tag{5.6}$$

From (5.6), one gets:

$$\lim_{n\to\infty}\alpha_{11}=\frac{\Delta_1}{\bar{\gamma}^2},\ \lim_{n\to\infty}\alpha_{12}=-\frac{2\Delta_1}{\bar{\gamma}^2},\ \lim_{n\to\infty}\alpha_{21}=\frac{\Delta_2}{\bar{\beta}^2},\ \lim_{n\to\infty}\alpha_{22}=-\frac{2\Delta_2}{\bar{\beta}^2}, \tag{5.7}$$

that is:

$$\alpha_{11}=\frac{\Delta_1}{\bar{\gamma}^2}+\sigma_{11},\ \alpha_{12}=-\frac{2\Delta_1}{\bar{\gamma}^2}+\sigma_{12},\ \alpha_{21}=\frac{\Delta_2}{\bar{\beta}^2}+\sigma_{21},\ \alpha_{22}=-\frac{2\Delta_2}{\bar{\beta}^2}+\sigma_{22}, \tag{5.8}$$

where $\sigma_{11},\sigma_{12},\sigma_{21},\sigma_{22}\to 0$ as $n\to\infty$. From existing literature [25], one obtains

$$\varphi_{n+1}=(A+B_n)\varphi_n, \tag{5.9}$$

where

$$A=\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{\Delta_1}{\bar{\gamma}^2} & 0 & \dots & 0 & -\frac{2\Delta_1}{\bar{\gamma}^2}\\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0\\ \frac{\Delta_2}{\bar{\beta}^2} & 0 & \dots & 0 & -\frac{2\Delta_2}{\bar{\beta}^2} & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \tag{5.10}$$

and

$$B_n=\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \sigma_{11} & 0 & \dots & 0 & \sigma_{12}\\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0\\ \sigma_{21} & 0 & \dots & 0 & \sigma_{22} & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{5.11}$$

Therefore, about Γ, one has:

$$\begin{pmatrix} \varphi^1_{n+1}\\ \varphi^1_n\\ \vdots\\ \varphi^1_{n-m+1}\\ \varphi^2_{n+1}\\ \varphi^2_n\\ \vdots\\ \varphi^2_{n-m+1} \end{pmatrix}=\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{\Delta_1}{\bar{\gamma}^2} & 0 & \dots & 0 & -\frac{2\Delta_1}{\bar{\gamma}^2}\\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0\\ \frac{\Delta_2}{\bar{\beta}^2} & 0 & \dots & 0 & -\frac{2\Delta_2}{\bar{\beta}^2} & 0 & 0 & \dots & 0 & 0\\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}\begin{pmatrix} \varphi^1_n\\ \varphi^1_{n-1}\\ \vdots\\ \varphi^1_{n-m}\\ \varphi^2_n\\ \vdots\\ \varphi^2_{n-1}\\ \varphi^2_{n-m} \end{pmatrix}, \tag{5.12}$$

which is same as J_Γ at Γ. Particularly, about Γ_1 and Γ_2, (5.12) becomes:

$$\varphi_{n+1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{4\Delta_1}{\left(1-\Delta_1+\Delta_2+\Psi\right)^2} & 0 & \dots & 0 & \frac{-8\Delta_1}{\left(1-\Delta_1+\Delta_2+\Psi\right)^2} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{4\Delta_2}{\left(1+\Delta_1-\Delta_2+\Psi\right)^2} & 0 & \dots & 0 & \frac{-8\Delta_2}{\left(1+\Delta_1-\Delta_2+\Psi\right)^2} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \varphi_n, \quad (5.13)$$

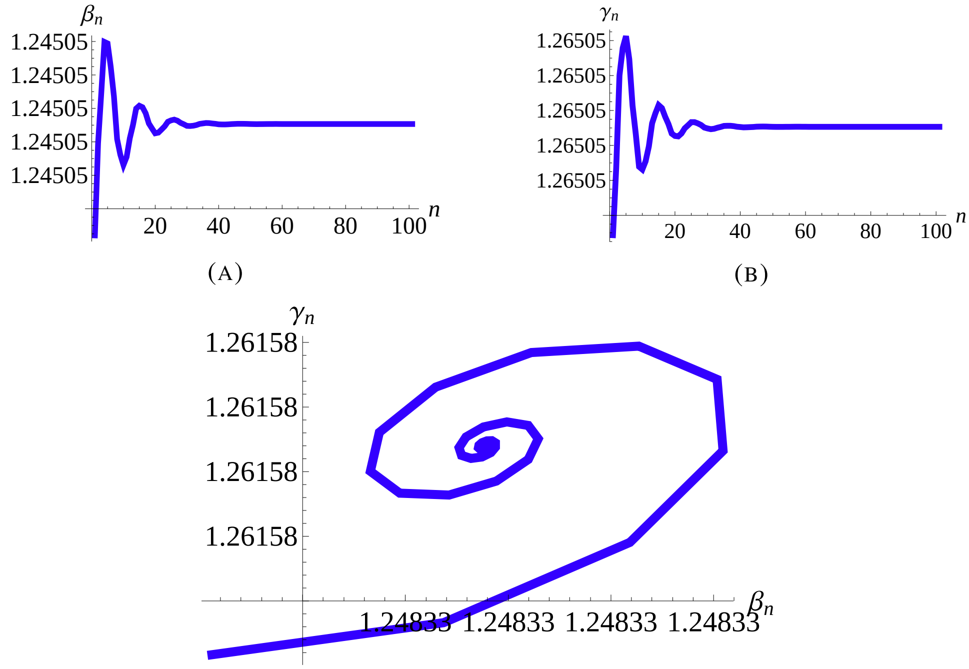
$$\varphi_{n+1} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \frac{4\Delta_1}{\left(1-\Delta_1+\Delta_2-\Psi\right)^2} & 0 & \dots & 0 & \frac{-8\Delta_1}{\left(1-\Delta_1+\Delta_2\Psi\right)^2} \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \frac{4\Delta_2}{\left(1+\Delta_1-\Delta_2-\Psi\right)^2} & 0 & \dots & 0 & \frac{-8\Delta_2}{\left(1+\Delta_1-\Delta_2-\Psi\right)^2} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \varphi_n, \quad (5.14)$$

which are the same as $J|_{\Gamma_1}$ and $J|_{\Gamma_2}$ about Γ_1 and Γ_2, respectively. □

6. NUMERICAL SIMULATIONS

Case a: If $m = 2$ and $\Delta_1 = 0.41, \Delta_2 = 0.43 \in \left(0, \frac{1}{2}\right)$, then from (3.1) one gets $\Delta_1\Delta_2 = 0.17629999999999998 < 1$, and consequently, from (3.9), the conditions for occurrence of a bounded solution, that is, $1 < \beta_n < \frac{1+\Delta_1}{1-\Delta_1\Delta_2} = 1.7117882724292823$ and $1 < \gamma_n < \frac{1+\Delta_2}{1-\Delta_1\Delta_2} = 1.7360689571445913$ hold. Furthermore, parametric conditions, as shown in (4.1), which determine the equilibrium point $\Gamma_1 = \left(\frac{1+\Delta_1-\Delta_2+\Psi}{2}, \frac{1-\Delta_1+\Delta_2+\Psi}{2}\right) = (1.3085963596303125, 1.3285963596303128)$ of the higher-order discrete system (1.13) is a sink also hold, i.e., $\frac{8\Delta_1}{(1-\Delta_1+\Delta_2\Psi)^2-4\Delta_1} = 0.6050909012359067 < 1$ and $\frac{8\Delta_2}{(1+\Delta_1-\Delta_2+\Psi)^2-4\Delta_2} = 0.6706048155720664 < 1$. So, Figure 1 (A), (B) implies that $\Gamma_1 = (1.3085963596303125, 1.3285963596303128)$ of a third-order system (1.13) is stable whereas Figure 1 (C) indicates that the equilibrium is an attractor globally. So, the simulation agrees with the results achieved in Theorems 4.1 and 4.3.

Case b: If $m = 4$ and $\Delta_1 = 0.31, \Delta_2 = 0.33 \in \left(0, \frac{1}{2}\right)$, then from (3.1) one gets $\Delta_1\Delta_2 = 0.10231$, and therefore, from (3.9), the conditions for the existence of a boundedness solution, i.e., $1 < \beta_n < \frac{1+\Delta_1}{1-\Delta_1\Delta_2} = 1.4592848390330848$ and $1 < \gamma_n < \frac{1+\Delta_2}{1-\Delta_1\Delta_2} = 1.4815639968809182$ hold. Additionally, parametric conditions (4.1) under which the positive fixed point $\Gamma_1 = \left(\frac{1+\Delta_1-\Delta_2+\Psi}{2}, \frac{1-\Delta_1+\Delta_2+\Psi}{2}\right) = (1.2450496672$



(C)

FIGURE 1. Behavior of higher-order system (1.13) with $\beta_\upsilon, \gamma_\upsilon(\upsilon = -4, \cdots, 0)$ are $0.9, 0.7, 0.9, 0.7, 0.9, 0.4, 0.4, 0.4, 0.4, 0.7$, respectively.

$405067, 1.2650496672405067)$ of the discrete-time system (1.13) is stable also hold, that is, $\frac{8\Delta_1}{(1-\Delta_1+\Delta_2+\Psi)^2-4\Delta_1} = 0.48048954360883683 < 1$ and $\frac{8\Delta_2}{(1+\Delta_1-\Delta_2+\Psi)^2-4\Delta_2}$ $= 0.5409176882459323 < 1$. Hence, Figure 1 (A), (B) implies that $\Gamma_1 = (1.245049$ $6672405067, 1.2650496672405067)$ of the fifth-order system (1.13) is stable whereas Figure 1 (C) shows that the equilibrium is an attractor globally. So, the simulation agrees with the results achieved in Theorems 4.1 and 4.3.

Case c: If $m = 6$ and $\Delta_1 = 0.21, \Delta_2 = 0.43 \in \left(0, \frac{1}{2}\right)$, then from (3.1) we get $\Delta_1\Delta_2 = 0.0903$, and therefore from (3.9), the conditions for the existence of boundedness solution, i.e., $1 < \beta_n < \frac{1+\Delta_1}{1-\Delta_1\Delta_2} = 1.3301088270858525$ and $1 < \gamma_n < \frac{1+\Delta_2}{1-\Delta_1\Delta_2}$ $= 1.5719467956469166$ hold. Additionally, parametric conditions (4.1) under which the unique positive fixed point $\Gamma_1 = \left(\frac{1+\Delta_1-\Delta_2+\Psi}{2}, \frac{1-\Delta_1+\Delta_2+\Psi}{2}\right) = (1.1529547$ $824084991, 1.3729547824084989)$ of (1.13) is stable are also true, i.e.,

$$\frac{8\Delta_1}{(1-\Delta_1+\Delta_2+\Psi)^2-4\Delta_1} = 0.25074554493196555 < 1$$

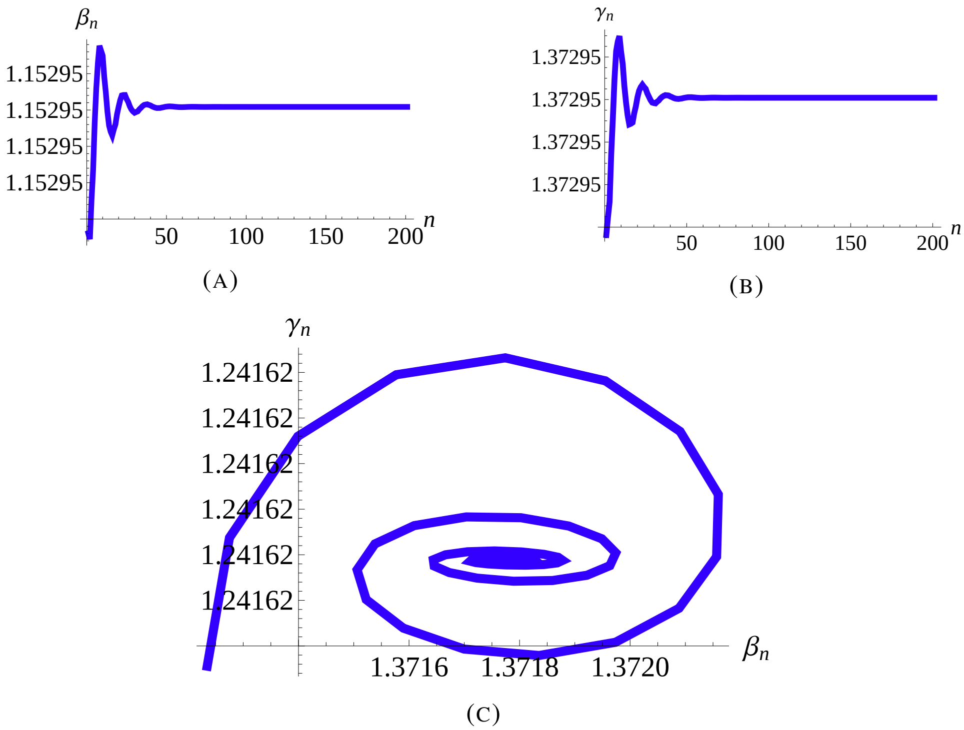


FIGURE 2. Behavior of higher-order system (1.13) with $\beta_\upsilon, \gamma_\upsilon (\upsilon = -6, \cdots, 0)$ are 0.9, 0.7, 0.9, 0.7, 0.9, 0.7, 0.9, 0.4, 0.4, 0.4, 0.4, 0.4, 0.4, 0.7, respectively.

and $\frac{8\Delta_2}{(1+\Delta_1-\Delta_2+\Psi)^2-4\Delta_2} = 0.9562943138679455 < 1$. Hence, Figure 2 (A), (B) implies that $\Gamma_1 = (1.1529547824084991, 1.3729547824084989)$ of the seventh-order system (1.13) is stable whereas Figure 2 (C) shows that the equilibrium is a global attractor. So, the simulation agrees with the results achieved in Theorems 4.1 and 4.3.

Case d: If $m = 2$ and $\Delta_1 = 1.4, \Delta_2 = 1.5$, then Figure 3 shows the fact that the positive fixed point $\Gamma_1 = \left(\frac{1+\Delta_1-\Delta_2+\Psi}{2}, \frac{1-\Delta_1+\Delta_2+\Psi}{2}\right) = (1.7547988350699888, 1.854798835069989)$ of the discrete system (1.13) is unstable.

7. CONCLUSION

This work investigates the dynamics of a non-symmetric system of higher-order difference equations, extending Taşdemir's work [35] to a more complex framework. We focused on exploring the intricate behavior of the system (1.13),

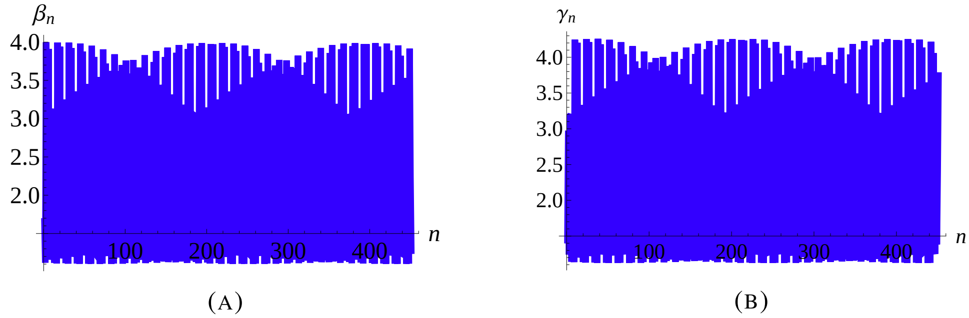


FIGURE 3. Behavior of higher-order system (1.13) with $\beta_\upsilon, \gamma_\upsilon (\upsilon = -2, \cdots, 0)$ are 5.0, 4.0, 0.6, 1.7, 0.4, 2.0, respectively.

analyzing its fixed points, local stability, boundedness, persistence, periodic points, global dynamics, and convergence rate.

Initially, we examined the fixed points of the higher-order system (1.13) and identified all possible fixed points, classifying their stability using linear stability theory. This analysis revealed various stability behaviors of the fixed points depending on parameter values. A key finding of this work is that at every positive solution of the system is bounded and persistent, meaning that the system won't grow or shrink without limit. The system stays within a realistic range, making it more useful in practical situations due to this boundedness.

We have periodic points, which are cycles or repeating patterns in which we looked at how the system evolves. Additionally, we studied the overall behavior of the system over time, explaining how solutions behave beyond just their nearest stable points. By these significant outcomes, one can say that the higher-order system (1.13) can exhibit numerous global behaviors, such as settling into repeating cycles or fixed points, underlining the dynamics complexities.

Likewise, in what way solutions converge to equilibria are studied. Our results suggest that this speed is affected by the system's parameters and initial conditions, giving us a clearer idea of how quickly the system stabilizes.

The main strength of this study is its thorough approach to analyzing the dynamics of a non-symmetric higher-order difference equations system. By building on Taşdemir's work [35], we have expanded the range of models that can be applied and provided a strong framework for studying these systems. Our results extend and improve upon existing literature, offering new insights and methodologies.

Finally, the theoretical discussions are supported by numerical simulations, which visually and empirically validate our findings. These simulations demonstrate various dynamic behaviors, including stability, periodicity, and chaos, bridging theory and practice.

In conclusion, this study makes a significant contribution to the field of discrete dynamical systems by exploring the dynamics of a non-symmetric higher-order difference equations system (1.13). The findings enhance our understanding of the system's behavior and provide a foundation for future research, particularly in applying higher-order difference equations to real-world problems.

7.1. **Future work**

Our next aim is to calculate the forbidden set and periodicity nature of the solution for a higher-order discrete system (1.13).

COMPETING INTERESTS

The authors declare that they have no competing interests.

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