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REVEALING HIDDEN PATTERNS: POWERS OF JACOBSTHAL, PELL AND MERSENNE SEQUENCES VIA THE POWERS OF LOWER TERMS AND SOME APPLICATIONS

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ABSTRACT. In this paper, we investigate the powers of Jacobsthal, Pell, and Mersenne numbers expressed as the sum of the powers of lower terms. Specifically, we derive formulas for the m^{th} power of $\mathrm{A}_{n+1} \in \{\mathrm{J}_{n+1}, \mathrm{P}_{n+1}, \mathrm{M}_{n+1}\}$ in terms of the sum of the m^{th} powers of the lower terms $\mathrm{A}_n, \mathrm{A}_{n-1}, \ldots, \mathrm{A}_{n-m}$. We also present some applications of these powers to determine recurrence relations for the integer sequences of the squares and cubes of Jacobsthal, Pell, and Mersenne numbers. Additionally, a few numerical examples are provided for each case.

1. INTRODUCTION

Integer sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Mersenne, and others are of great interest in the literature. These numbers appear frequently in mathematics and computer science.

The Fibonacci, Jacobsthal, Pell, and Mersenne sequences are defined by the following recurrence relations for $n \geq 2$:

$$\mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2};\ \mathrm{F}_0 = 0, \mathrm{F}_1 = 1, \tag{1.1}$$

$$\mathrm{J}_n = \mathrm{J}_{n-1} + 2\mathrm{J}_{n-2};\ \mathrm{J}_0 = 0, \mathrm{J}_1 = 1, \tag{1.2}$$

$$\mathrm{P}_n = 2\mathrm{P}_n + \mathrm{P}_{n-1};\ \mathrm{P}_0 = 0, \mathrm{P}_1 = 1, \tag{1.3}$$

$$\mathrm{M}_n = 3\mathrm{M}_{n-1} - 2\mathrm{M}_{n-2};\ \mathrm{M}_0 = 0, \mathrm{M}_1 = 1, \tag{1.4}$$

respectively. The first 10 numbers of these sequences are listed in Table 1.

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TABLE 1. First few Fibonacci, Jacobsthal, Pell and Mersenne numbers

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34
J_n	0	1	1	3	5	11	21	43	85	171
P_n	0	1	2	5	12	29	70	169	408	985
M_n	0	1	3	7	15	31	63	127	255	511

Binet's formula for these sequences has the following form:

$$\frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{1.5}$$

Here, α and β are the roots of the corresponding characteristic equation of the sequence. In Table 2, the characteristic equation of each sequence, along with the values of α and β, is provided. For more details and relations concerning these

TABLE 2. Characteristic equations of sequences

Sequence	Characteristic equation	Value of α	Value of β
F_n	$x^2 - x - 1 = 0$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
J_n	$x^2 - x - 2 = 0$	2	-1
P_n	$x^2 - 2x - 1 = 0$	$1+\sqrt{2}$	$1-\sqrt{2}$
M_n	$x^2 - 3x + 2 = 0$	2	1

number sequences see [1, 2, 4, 5, 7–14, 16–19].

Theorem 1.1. *[6] Let $G_n = M_{-n}$ and $H_n = J_{-n}$, then*

$$G_n = -\frac{M_n}{2^n}, \tag{1.6}$$

and

$$H_n = \frac{(-1)^{n+1}}{2^n} J_n, \tag{1.7}$$

where M_n and J_n are the n^{th} Mersenne and Jacobsthal numbers, respectively.

Proposition 1.1. *[12] The Pell numbers with negative indices are as follows*

$$P_{-n} = (-1)^{n+1} P_n. \tag{1.8}$$

Table 3 includes a few Jacobsthal, Pell, and Mersenne numbers with negative indices.

TABLE 3. First few Jacobsthal, Pell and Mersenne numbers with negative indices

n	1	2	3	4	5	6	7	8	9
J_{-n}	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$	$-\frac{5}{16}$	$\frac{11}{32}$	$-\frac{21}{64}$	$\frac{43}{128}$	$-\frac{85}{256}$	$\frac{171}{512}$
P_{-n}	1	-2	5	-12	29	-70	169	-408	985
M_{-n}	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$

In [3], the author derived the following formulas for the powers of Fibonacci numbers:

$$
\begin{aligned}
&F_{n+1} = F_n + F_{n-1},\\
&F^2_{n+1} = 2F^2_n + 2F^2_{n-1} - F^2_{n-2},\\
&F^3_{n+1} = 3F^3_n + 6F^3_{n-1} - 3F^3_{n-2} - F^3_{n-3},\\
&F^4_{n+1} = 5F^4_n + 15F^4_{n-1} - 15F^4_{n-2} - 5F^4_{n-3} - F^4_{n-4},\\
&F^5_{n+1} = 8F^5_n + 40F^5_{n-1} - 60F^5_{n-2} - 40F^5_{n-3} + 8F^5_{n-4} + F^5_{n-5},\\
&F^6_{n+1} = 13F^6_n + 104F^6_{n-1} - 260F^6_{n-2} - 260F^6_{n-3} + 104F^6_{n-4} + 13F^6_{n-5} - F^6_{n-6},
\end{aligned}
$$

and some more higher powers. Motivated by this, in the present paper, we aim to find similar formulas for the powers of Jacobsthal, Pell, and Mersenne numbers expressed as the sum of the powers of lower terms.
These formulas help us to calculate the higher powers of these number sequences as shown in the few examples at the end of this paper. Also by using these formulas we aim to find recurrence relations and generating functions for a few important sequences, such as squares of Jacobsthal numbers, cubes of Jacobsthal numbers, squares of Pell numbers, cubes of Pell numbers, squares of Mersenne numbers and cubes of Mersenne numbers which are listed in [19] as A139818, A253103, A079291, A110272, A060867 and A128831, respectively.

2. POWERS OF JACOBSTHAL NUMBERS

In this section, we discuss the powers of Jacobsthal numbers, that is, the m^{th} power of J_n by using the m^{th} power of the lower Jacobsthal numbers $J_{n-1}, J_{n-2}, \ldots, J_{n-m-1}$. We begin with the following lemma.

Lemma 2.1. *For any integers n and m, the following identity holds*

$$2^{m-1} J_{n-m-1} = (-1)^m (J_{m-1} J_{n-1} - J_m J_{n-2}). \tag{2.1}$$

Proof. The proof can be done by induction on m. □

Theorem 2.1. *The first few powers of* J_n *are*

$$\begin{aligned}
&\mathrm{J}_n^2 = 3\mathrm{J}_{n-1}^2 + 6\mathrm{J}_{n-2}^2 - 8\mathrm{J}_{n-3}^2,\\
&\mathrm{J}_n^3 = 5\mathrm{J}_{n-1}^3 + 30\mathrm{J}_{n-2}^3 - 40\mathrm{J}_{n-3}^3 - 64\mathrm{J}_{n-4}^3,\\
&\mathrm{J}_n^4 = 11\mathrm{J}_{n-1}^4 + 110\mathrm{J}_{n-2}^4 - 440\mathrm{J}_{n-3}^4 - 704\mathrm{J}_{n-4}^4 + 1024\mathrm{J}_{n-5}^4,\\
&\mathrm{J}_n^5 = 21\mathrm{J}_{n-1}^5 + 462\mathrm{J}_{n-2}^5 - 3080\mathrm{J}_{n-3}^5 - 14784\mathrm{J}_{n-4}^5 + 21504\mathrm{J}_{n-5}^5 + 32768\mathrm{J}_{n-6}^5,\\
&\mathrm{J}_n^6 = 43\mathrm{J}_{n-1}^6 + 1806\mathrm{J}_{n-2}^6 - 26488\mathrm{J}_{n-3}^6 - 211904\mathrm{J}_{n-4}^6 + 924672\mathrm{J}_{n-5}^6 + 1409024\mathrm{J}_{n-6}^6\\
&\qquad - 2097152\mathrm{J}_{n-7}^6.
\end{aligned}$$

Proof. From Equality (1.2) we have that

$$\mathrm{J}_n = \mathrm{J}_{n-1} + 2\mathrm{J}_{n-2}, \tag{2.2}$$

and by using $m = 2$ in Equality (2.1) we obtain

$$2\mathrm{J}_{n-3} = \mathrm{J}_{n-1} - \mathrm{J}_{n-2}. \tag{2.3}$$

If we square both sides of Equalities (2.2) and (2.3) and add them, we obtain the second power of Jacobsthal numbers as follows,

$$\mathrm{J}_n^2 = 3\mathrm{J}_{n-1}^2 + 6\mathrm{J}_{n-2}^2 - 8\mathrm{J}_{n-3}^2. \tag{2.4}$$

Now we calculate the third power of J_n, from Equalities (1.2) and (2.1) we have

$$\begin{aligned}
\mathrm{J}_n^3 &= \mathrm{J}_{n-1}^3 + 6\mathrm{J}_{n-1}^2\mathrm{J}_{n-2} + 12\mathrm{J}_{n-1}\mathrm{J}_{n-2}^2 + 8\mathrm{J}_{n-2}^3,\\
8\mathrm{J}_{n-3}^3 &= \mathrm{J}_{n-1}^3 - 3\mathrm{J}_{n-1}^2\mathrm{J}_{n-2} + 3\mathrm{J}_{n-1}\mathrm{J}_{n-2}^2 - \mathrm{J}_{n-2}^3,\\
64\mathrm{J}_{n-4}^3 &= -\mathrm{J}_{n-1}^3 + 9\mathrm{J}_{n-1}^2\mathrm{J}_{n-2} - 27\mathrm{J}_{n-1}\mathrm{J}_{n-2}^2 + 27\mathrm{J}_{n-2}^3,
\end{aligned}$$

and to eliminate non-simple powers, we multiply the first equation by a, the second equation by b, where a and b are non-zero numbers and add them to the third equation, thus obtaining the following system

$$\begin{aligned}
&6a - 3b + 9 = 0\\
&12a + 3b - 27 = 0,
\end{aligned}$$

which has the solution $a = 1$ and $b = 5$. Then we get the third power of J_n as follows

$$\mathrm{J}_n^3 = 5\mathrm{J}_{n-1}^3 + 30\mathrm{J}_{n-2}^3 - 40\mathrm{J}_{n-3}^3 - 64\mathrm{J}_{n-4}^3. \tag{2.5}$$

For the fourth power of J_n, using Equalities (1.2) and (2.1), we obtain the following

$$\begin{aligned}
\mathrm{J}_n^4 &= \mathrm{J}_{n-1}^4 + 8\mathrm{J}_{n-2}^3\mathrm{J}_{n-3} + 24\mathrm{J}_{n-1}^2\mathrm{J}_{n-2}^2 + 32\mathrm{J}_{n-1}\mathrm{J}_{n-2}^3 + 16\mathrm{J}_{n-2}^4,\\
16\mathrm{J}_{n-3}^4 &= \mathrm{J}_{n-1}^4 - 4\mathrm{J}_{n-1}^3\mathrm{J}_{n-2} + 6\mathrm{J}_{n-1}^2\mathrm{J}_{n-2}^2 - 4\mathrm{J}_{n-1}\mathrm{J}_{n-2}^3 + \mathrm{J}_{n-2}^4,\\
256\mathrm{J}_{n-4}^4 &= \mathrm{J}_{n-1}^4 - 12\mathrm{J}_{n-1}^3\mathrm{J}_{n-2} + 54\mathrm{J}_{n-1}^2\mathrm{J}_{n-2}^2 - 108\mathrm{J}_{n-1}\mathrm{J}_{n-2}^3 + 81\mathrm{J}_{n-2}^4,\\
4096\mathrm{J}_{n-5}^4 &= 81\mathrm{J}_{n-1}^4 - 540\mathrm{J}_{n-1}^3\mathrm{J}_{n-2} + 1350\mathrm{J}_{n-1}^2\mathrm{J}_{n-2}^2 - 1500\mathrm{J}_{n-1}\mathrm{J}_{n-2}^3 + 625\mathrm{J}_{n-2}^4.
\end{aligned}$$

To eliminate non-simple powers, we produce the following system

$$\begin{aligned}
8a - 4b - 12c &= 540\\
24a + 6b + 54c &= -1350\\
32a - 4b - 108c &= 1500,
\end{aligned}$$

where a, b and c are non-zero numbers. This system has the solution $a = -4$, $b = -110$ and $c = -11$. Therefore, if we multiply the first three equations by -4, -100, and -11, respectively, and add them to the fourth equation, we obtain the fourth power of J_n as follows

$$\mathrm{J}_n^4 = 11\mathrm{J}_{n-1}^4 + 110\mathrm{J}_{n-2}^4 - 440\mathrm{J}_{n-3}^4 - 704\mathrm{J}_{n-4}^4 + 1024\mathrm{J}_{n-5}^4. \tag{2.6}$$

By the same method, we can calculate the fifth and sixth powers of J_{n+1} as follows

$$\begin{aligned}
\mathrm{J}_n^5 &= 21\mathrm{J}_{n-1}^5 + 462\mathrm{J}_{n-2}^5 - 3080\mathrm{J}_{n-3}^5 - 14784\mathrm{J}_{n-4}^5 + 21504\mathrm{J}_{n-5}^5 + 32768\mathrm{J}_{n-6}^5,\\
\mathrm{J}_n^6 &= 43\mathrm{J}_{n-1}^6 + 1806\mathrm{J}_{n-2}^6 - 26488\mathrm{J}_{n-3}^6 - 211904\mathrm{J}_{n-4}^6 + 924672\mathrm{J}_{n-5}^6 + 1409024\mathrm{J}_{n-6}^6\\
&\quad - 2097152\mathrm{J}_{n-7}^6.
\end{aligned}$$

□

Let $[\mathrm{J}_n]$ be the coefficient of J_n, then by writing only the coefficients of the lower powers of J_n, we construct the following table

TABLE 4. First few powers of J_n.

	$[\mathrm{J}_{n-1}]$	$[\mathrm{J}_{n-2}]$	$[\mathrm{J}_{n-3}]$	$[\mathrm{J}_{n-4}]$	$[\mathrm{J}_{n-5}]$	$[\mathrm{J}_{n-6}]$	$[\mathrm{J}_{n-7}]$
J_n^0	1						
J_n^1	1	2					
J_n^2	3	6	-8				
J_n^3	5	30	-40	-64			
J_n^4	11	110	-440	-704	1024		
J_n^5	21	462	-3080	-14784	21504	32768	
J_n^6	43	1806	-26488	-211904	924672	1409024	-2097152

From Table 6, we can observe the following pattern:

For $r \geq 1$, the first element of the r^{th} row is J_r. If we multiply each number in the r^{th} row by J_{r+1} and divide it by the lower Jacobsthal numbers J_r, J_{r-1}, $\ldots$, J_2, J_1, we will obtain the $(r+1)^{st}$ row, except for the last element; the last elements of the rows are powers of 2. Moreover, the first two numbers in each row are positive, and the next two alternate in sign.

Therefore we can write the p^{th} power of J_n as the sum of p^{th} powers of $\mathrm{J}_{n-1}, \mathrm{J}_{n-2}, ..., \mathrm{J}_{n-p-1}$. This result is stated in the following theorem.

Theorem 2.2. *For a fixed non-negative integer p, the p^{th} power of J_n is as follows*

$$\mathrm{J}_n^p = \sum_{i=1}^{p+1} q_{i-1}\mathrm{J}_{n-i}^p,$$

the coefficients of q_i are

$$q_i = (-1)^{\lfloor \frac{i}{2} \rfloor} a_i \frac{\mathrm{J}_{p+1}\mathrm{J}_p ... \mathrm{J}_{p-i+1}}{\mathrm{J}_1\mathrm{J}_2 ... \mathrm{J}_{i+1}} \ ; \quad i = 0, 1, ..., p, \tag{2.7}$$

where a_n satisfies the recurrence relation of $a_n = a_{n-1}2^n$, $a_0 = 1$, for $n \geq 1$.

Proof. We use induction on i,

$$\begin{aligned}
&p = 1;\ q_0 = 1,\\
&p = 2;\ q_0 = a_0\frac{\mathrm{J}_2}{\mathrm{J}_1} = 1\ ; q_1 = a_1\frac{\mathrm{J}_2\mathrm{J}_1}{\mathrm{J}_1\mathrm{J}_2} = 2,\\
&p = 3\ ; q_0 = a_0\frac{\mathrm{J}_3}{\mathrm{J}_1} = 3\ ; q_1 = a_1\frac{\mathrm{J}_3\mathrm{J}_2}{\mathrm{J}_1\mathrm{J}_2} = 6\ ; q_2 = -a_2\frac{\mathrm{J}_3\mathrm{J}_2\mathrm{J}_1}{\mathrm{J}_1\mathrm{J}_2\mathrm{J}_3} = -8.
\end{aligned}$$

Then the first three powers are as follows

$$\begin{aligned}
&\mathrm{J}_n^0 = 1,\\
&\mathrm{J}_n^1 = \mathrm{J}_{n-1} + 2\mathrm{J}_{n-2},\\
&\mathrm{J}_n^2 = 3\mathrm{J}_{n-1}^2 + 6\mathrm{J}_{n-2}^2 - 8\mathrm{J}_{n-3}^2.
\end{aligned}$$

Now let the coefficients of p^{th} power be

$$\begin{aligned}
&q_0 = a_0\frac{\mathrm{J}_{p+1}}{J_1},\ q_1 = a_1\frac{\mathrm{J}_{p+1}\mathrm{J}_p}{\mathrm{J}_1\mathrm{J}_2},\ q_2 = -a_2\frac{\mathrm{J}_{p+1}\mathrm{J}_p\mathrm{J}_{p-1}}{\mathrm{J}_1\mathrm{J}_2\mathrm{J}_3},\ ...,\\
&q_{p-1} = (-1)^{\lfloor \frac{p-1}{2} \rfloor} a_{p-1}\frac{\mathrm{J}_{p+1}\mathrm{J}_p\mathrm{J}_{p-1}...\mathrm{J}_2}{\mathrm{J}_1\mathrm{J}_2\mathrm{J}_3...\mathrm{J}_\mathrm{p}} = (-1)^{\lfloor \frac{p-1}{2} \rfloor} a_{p-1}\mathrm{J}_{p+1},\\
&q_p = (-1)^{\lfloor \frac{p}{2} \rfloor} a_p\frac{\mathrm{J}_{p+1}\mathrm{J}_p\mathrm{J}_{p-1}...\mathrm{J}_2\mathrm{J}_1}{\mathrm{J}_1\mathrm{J}_2\mathrm{J}_3...\mathrm{J}_p\mathrm{J}_{p+1}} = (-1)^{\lfloor \frac{p}{2} \rfloor} a_p.
\end{aligned}$$

If we multiply these coefficients by J_{p+2} and divide it by $\mathrm{J}_{p+1}, J_p, ..., \mathrm{J}_2, \mathrm{J}_1$, respectively, then we will get the following coefficients

$$q_0 = a_0 \frac{\mathrm{J}_{p+2}}{J_1},\ q_1 = a_1 \frac{\mathrm{J}_{p+2}\mathrm{J}_{p+1}}{\mathrm{J}_1\mathrm{J}_2},\ q_2 = -a_2 \frac{\mathrm{J}_{p+2}\mathrm{J}_{p+1}\mathrm{J}_p}{\mathrm{J}_1\mathrm{J}_2\mathrm{J}_3},\ ...,$$

$$q_{p-1} = (-1)^{\lfloor \frac{p-1}{2} \rfloor} a_{p-1} \frac{\mathrm{J}_{p+2}\mathrm{J}_{p+1}\mathrm{J}_p\mathrm{J}_{p-1}...\mathrm{J}_3}{\mathrm{J}_1\mathrm{J}_2\mathrm{J}_3...\mathrm{J_p}},$$

$$q_p = (-1)^{\lfloor \frac{p}{2} \rfloor} a_p \frac{\mathrm{J}_{p+1}\mathrm{J}_p\mathrm{J}_{p-1}...\mathrm{J}_2\mathrm{J}_1}{\mathrm{J}_1\mathrm{J}_2\mathrm{J}_3...\mathrm{J}_p\mathrm{J}_{p+1}} = (-1)^{\lfloor \frac{p}{2} \rfloor} a_p \mathrm{J}_{p+2}.$$

These are the coefficients of $(p+1)^{st}$ power of J_n, except for the last coefficient, which can be obtained from the recurrence relation for a_n, that is

$$q_{p+1} = (-1)^{\lfloor \frac{p}{2} \rfloor} a_p 2^{p+1}.$$

Therefore the result is true for every non-negative integer i. □

Using the above Theorem, we can extend the table to include larger powers. For example, to obtain the coefficients of the 7^{th} power from the 6^{th} power, we multiply each coefficient by $\mathrm{J}_8 = 85$ and divide by $43, 21, 11, 5, 3, 1, 1$, respectively. The last coefficient of the seventh power is $-2097152 \cdot 2^8 = -268435456$. See Table 5 for some larger powers of J_n, (the coefficients of terms are listed vertically).

TABLE 5. Some larger powers of J_n.

J_n^7	J_n^8	J_n^9	J_n^{10}	J_n^{11}
85	171	341	683	1365
7310	29070	116622	465806	1864590
−204680	−1666680	−132217160	−1062403768	−847766920
−3602368	−56000448	−909340608	−14443712448	−231949029312
26199040	896007168	27776222208	903388560384	28677334533120
119767040	6826721280	465582391296	28908433932288	1879048205598720
−17825790	−30482104320	−3464799191040	−473291569496064	−58731181123829760
−268435456	−45902462976	−15652739874816	−3563607111499776	−972864741439438848
	68719476736	23433341566976	16004972290244608	7282262392061296640
		35184372088832	24030926136672256	32802214176557629440
			−36028797018963968	−49179307930885816320
				−100719222642454151823360

3. POWERS OF PELL NUMBERS

In this section, we aim to compute powers of Pell numbers in a manner similar to Jacobsthal numbers, beginning with the following lemma.

Lemma 3.1. *For any integers n and m the following identity holds*

$$\mathrm{P}_{n-m-1} = (-1)^m (\mathrm{P}_{m-1}\mathrm{P}_{n-1} - \mathrm{P}_m\mathrm{P}_{n-2}). \tag{3.1}$$

Proof. The proof can be done by induction on m. □

Theorem 3.1. *The first few powers of* P_n *are as follows*

$$\begin{aligned}
&\mathrm{P}_n^2 = 5\mathrm{P}_{n-1}^2 + 5\mathrm{P}_{n-2}^2 - \mathrm{P}_{n-3}^2,\\
&\mathrm{P}_n^3 = 12\mathrm{P}_{n-1}^3 + 30\mathrm{P}_{n-2}^3 - 12\mathrm{P}_{n-3}^3 - \mathrm{P}_{n-4}^3,\\
&\mathrm{P}_n^4 = 29\mathrm{P}_{n-1}^4 + 174\mathrm{P}_{n-2}^4 - 174\mathrm{P}_{n-3}^4 - 29\mathrm{P}_{n-4}^4 + \mathrm{P}_{n-5}^4,\\
&\mathrm{P}_n^5 = 70\mathrm{P}_{n-1}^5 + 1015\mathrm{P}_{n-2}^5 - 2436\mathrm{P}_{n-3}^5 - 1015\mathrm{P}_{n-4}^5 + 70\mathrm{P}_{n-5}^5 + \mathrm{P}_{n-6}^5,\\
&\mathrm{P}_n^6 = 169\mathrm{P}_{n-1}^6 + 5915\mathrm{P}_{n-2}^6 - 34307\mathrm{P}_{n-3}^6 - 34307\mathrm{P}_{n-4}^6 + 5915\mathrm{P}_{n-5}^6 + 169\mathrm{P}_{n-6}^6\\
&\qquad - \mathrm{P}_{n-7}^6.
\end{aligned}$$

Proof. From Equality (1.3) we have

$$\mathrm{P}_n = 2\mathrm{P}_{n-1} + \mathrm{P}_{n-2},$$

then by using Equality (3.1) with $m = 2$ and squaring both sides we get

$$\begin{aligned}
\mathrm{P}_n^2 &= 4\mathrm{P}_{n-1}^2 + 4\mathrm{P}_{n-1}\mathrm{P}_{n-2} + \mathrm{P}_{n-2}^2,\\
\mathrm{P}_{n-3}^2 &= \mathrm{P}_{n-1}^2 - 4\mathrm{P}_{n-1}\mathrm{P}_{n-2} + 4\mathrm{P}_{n-2}^2,
\end{aligned}$$

and by adding these equations we obtain the second power of P_n as follows

$$\mathrm{P}_n^2 = 5\mathrm{P}_{n-1}^2 + 5\mathrm{P}_{n-2}^2 - \mathrm{P}_{n-3}^2. \tag{3.2}$$

For the third power by using Equalities (1.3) and (3.1) with $m = 2$ and $m = 3$, we have

$$\begin{aligned}
\mathrm{P}_n^3 &= 8\mathrm{P}_{n-1}^3 + 12\mathrm{P}_{n-1}^2\mathrm{P}_{n-2} + 6\mathrm{P}_{n-1}\mathrm{P}_{n-2}^2 + \mathrm{P}_{n-2}^3,\\
\mathrm{P}_{n-3}^3 &= \mathrm{P}_{n-1}^3 - 6\mathrm{P}_{n-1}^2\mathrm{P}_{n-2} + 12\mathrm{P}_{n-1}\mathrm{P}_{n-2}^2 - 8\mathrm{P}_{n-2}^3,\\
\mathrm{P}_{n-4}^3 &= -8\mathrm{P}_{n-1}^3 + 60\mathrm{P}_{n-1}^2\mathrm{P}_{n-2} - 150\mathrm{P}_{n-1}\mathrm{P}_{n-2}^2 + 125\mathrm{P}_{n-2}^3.
\end{aligned}$$

To eliminate the non-simple powers, we multiply the first equation by a, the second by b, where a and b are non-zero numbers and then add with the third equation, obtaining the following system,

$$\begin{aligned}
&12a - 6b + 60 = 0\\
&6a + 12b - 150 = 0,
\end{aligned}$$

which has a solution $a = 1$ and $b = 12$. Therefore the third power can be written as follows

$$\mathrm{P}_n^3 = 12\mathrm{P}_{n-1}^3 + 30\mathrm{P}_{n-2}^3 - 12\mathrm{P}_{n-3}^3 - \mathrm{P}_{n-4}^3. \tag{3.3}$$

Now, by using Equalities (1.3) and (3.1) with $m = 2,3,4$, we have

$$\begin{aligned}
\mathrm{P}_n^4 &= 16\mathrm{P}_{n-1}^4 + 32\mathrm{P}_{n-1}^3\mathrm{P}_{n-2} + 24\mathrm{P}_{n-1}^2\mathrm{P}_{n-2}^2 + 8\mathrm{P}_{n-1}\mathrm{P}_{n-2}^3 + \mathrm{P}_{n-2}^4,\\
\mathrm{P}_{n-3}^4 &= \mathrm{P}_{n-1}^4 - 8\mathrm{P}_{n-1}^3\mathrm{P}_{n-2} + 24\mathrm{P}_{n-1}^2\mathrm{P}_{n-2}^2 - 32\mathrm{P}_{n-1}\mathrm{P}_{n-2}^3 + 16\mathrm{P}_{n-2}^4,\\
\mathrm{P}_{n-4}^4 &= 16\mathrm{P}_{n-1}^4 - 160\mathrm{P}_{n-1}^3\mathrm{P}_{n-2} + 600\mathrm{P}_{n-1}^2\mathrm{P}_{n-2}^2 - 1000\mathrm{P}_{n-1}\mathrm{P}_{n-2}^3 + 625\mathrm{P}_{n-2}^4,\\
\mathrm{P}_{n-5}^4 &= 625\mathrm{P}_{n-1}^4 - 6000\mathrm{P}_{n-1}^3\mathrm{P}_{n-2} + 21600\mathrm{P}_{n-1}^2\mathrm{P}_{n-2}^2 - 34560\mathrm{P}_{n-1}\mathrm{P}_{n-2}^3 + 20736\mathrm{P}_{n-2}^4,
\end{aligned}$$

if we multiply the first equation by a, second by b, the third by c, where a, b and c are non-zero numbers and add with the fourth, we get the following system to eliminate the non-simple powers

$$\begin{aligned}
32a - 8b - 160c - 6000 &= 0\\
24a + 24b + 600c + 21600 &= 0\\
8a - 32b - 1000c - 34560 &= 0.
\end{aligned}$$

This system has the solution $a = -1, b = -174$ and $c = -29$, thus the fourth power of P_n is given as follows

$$\mathrm{P}_n^4 = 29\mathrm{P}_{n-1}^4 + 174\mathrm{P}_{n-2}^4 - 174\mathrm{P}_{n-3}^4 - 29\mathrm{P}_{n-4}^4 + \mathrm{P}_{n-5}^4. \tag{3.4}$$

By a similar method we can find the fifth and sixth powers of P_n as follows

$$\begin{aligned}
\mathrm{P}_n^5 &= 70\mathrm{P}_{n-1}^5 + 1015\mathrm{P}_{n-2}^5 - 2436\mathrm{P}_{n-3}^5 - 1015\mathrm{P}_{n-4}^5 + 70\mathrm{P}_{n-5}^5 + \mathrm{P}_{n-6}^5,\\
\mathrm{P}_n^6 &= 169\mathrm{P}_{n-1}^6 + 5915\mathrm{P}_{n-2}^6 - 34307\mathrm{P}_{n-3}^6 - 34307\mathrm{P}_{n-4}^6 + 5915\mathrm{P}_{n-5}^6 + 169\mathrm{P}_{n-6}^6 - \mathrm{P}_{n-7}^6.
\end{aligned}$$

□

Let $[\mathrm{P}_n]$ denote the coefficient of P_n, then we can create the following table:

TABLE 6. First few powers of P_n.

	$[\mathrm{P}_{n-1}]$	$[\mathrm{P}_{n-2}]$	$[\mathrm{P}_{n-3}]$	$[\mathrm{P}_{n-4}]$	$[\mathrm{P}_{n-5}]$	$[\mathrm{P}_{n-6}]$	$[\mathrm{P}_{n-7}]$
P_n^0	1						
P_n^1	2	1					
P_n^2	5	5	-1				
P_n^3	12	30	-12	-1			
P_n^4	29	174	-174	-29	1		
P_n^5	70	1015	-2436	-1015	70	1	
P_n^6	169	5915	-34307	-34307	5915	169	-1

From the above table, we can observe the following pattern: If we multiply the coefficients of the m^{th} power by P_{m+2} and divide by the lower Pell numbers $\mathrm{P}_{m+1}, \mathrm{P}_m, \ldots, \mathrm{P}_2, \mathrm{P}_1$, respectively, we obtain the coefficients of the next power, except for the last coefficient, which is either 1 or -1. The first two elements are positive, while the next two alternate as negative.

Theorem 3.2. *For a fixed non-negative integer m, the m^{th} power of* P_n *is as follows*

$$\mathrm{P}_n^m = \sum_{i=1}^{m+1} q_{i-1}\mathrm{P}_{n-i}, \tag{3.5}$$

where

$$q_i = (-1)^{\lfloor \frac{i}{2} \rfloor} \frac{\mathrm{P}_{m+1}\mathrm{P}_m \ldots \mathrm{P}_{m-i+1}}{\mathrm{P}_1\mathrm{P}_2\ldots\mathrm{P}_{i+1}}; \quad i = 0, 1, \ldots, m. \tag{3.6}$$

Proof. We use induction on i, we have

$$\begin{aligned}
&m = 0;\ q_0 = 1,\\
&m = 1;\ q_0 = \frac{\mathrm{P}_2}{\mathrm{P}_1} = 2\ ; q_1 = \frac{\mathrm{P}_2\mathrm{P}_1}{\mathrm{P}_1\mathrm{P}_2} = 1,\\
&m = 2\ ; q_0 = \frac{\mathrm{P}_3}{\mathrm{P}_1} = 5\ ; q_1 = \frac{\mathrm{P}_3\mathrm{P}_2}{\mathrm{P}_1\mathrm{P}_2} = 5\ ; q_2 = -\frac{\mathrm{P}_3\mathrm{P}_2\mathrm{P}_1}{\mathrm{P}_1\mathrm{P}_2\mathrm{P}_3} = -1.
\end{aligned}$$

Then the first three powers are as follows

$$\begin{aligned}
&\mathrm{P}_n^0 = 1,\\
&\mathrm{P}_n^1 = 2\mathrm{P}_{n-1} + \mathrm{P}_{n-2},\\
&\mathrm{P}_n^2 = 5\mathrm{P}_{n-1}^2 + 5\mathrm{P}_{n-2}^2 - \mathrm{P}_{n-3}^2.
\end{aligned}$$

Suppose the coefficients of P_n^m are as follows

$$\begin{aligned}
q_0 &= \frac{\mathrm{P}_{m+1}}{\mathrm{P}_1} = \mathrm{P}_{m+1},\\
q_1 &= \frac{\mathrm{P}_{m+1}\mathrm{P}_m}{\mathrm{P}_1\mathrm{P}_2},\\
&\vdots\\
q_{m-1} &= (-1)^{\lfloor \frac{m-1}{2} \rfloor} \frac{\mathrm{P}_{m+1}\mathrm{P}_m\ldots\mathrm{P}_2}{\mathrm{P}_1\mathrm{P}_2\ldots\mathrm{P}_m} = (-1)^{\lfloor \frac{m-1}{2} \rfloor}\mathrm{P}_{m+1},\\
q_m &= (-1)^{\lfloor \frac{m}{2} \rfloor} \frac{\mathrm{P}_{m+1}\mathrm{P}_m\ldots\mathrm{P}_1}{\mathrm{P}_1\mathrm{P}_2\ldots\mathrm{P}_{m+1}} = (-1)^{\lfloor \frac{m}{2} \rfloor}.
\end{aligned}$$

Now, if we multiply each coefficient by P_{m+2} and divide it by $\mathrm{P}_{m+1}, \mathrm{P}_m, \ldots, \mathrm{P}_2, \mathrm{P}_1$, then we will have

$$\begin{aligned}
q_0 &= \frac{\mathrm{P}_{m+1}\mathrm{P}_{m+2}}{\mathrm{P}_{m+1}} = \mathrm{P}_{m+2},\\
q_1 &= \frac{\mathrm{P}_{m+1}\mathrm{P}_m\mathrm{P}_{m+2}}{\mathrm{P}_1\mathrm{P}_2\mathrm{P}_m} = \frac{\mathrm{P}_{m+2}\mathrm{P}_{m+1}}{\mathrm{P}_1\mathrm{P}_2},\\
&\vdots\\
q_{m-1} &= (-1)^{\lfloor \frac{m-1}{2} \rfloor} \frac{\mathrm{P}_{m+2}\mathrm{P}_{m+1}\mathrm{P}_m\ldots\mathrm{P}_2}{\mathrm{P}_1\mathrm{P}_2\ldots\mathrm{P}_m\mathrm{P}_2} = (-1)^{\lfloor \frac{m-1}{2} \rfloor} \frac{\mathrm{P}_{m+2}\mathrm{P}_{m+1}}{\mathrm{P}_1\mathrm{P}_2},
\end{aligned}$$

$$q_m = (-1)^{\lfloor \frac{m}{2} \rfloor} \frac{\mathrm{P}_{m+2}\mathrm{P}_{m+1}\mathrm{P}_m \ldots \mathrm{P}_1}{\mathrm{P}_1\mathrm{P}_2 \ldots \mathrm{P}_{m+1}\mathrm{P}_1} = (-1)^{\lfloor \frac{m}{2} \rfloor} \frac{\mathrm{P}_{m+2}}{\mathrm{P}_1}.$$

These are the coefficients of $(m+1)^{st}$ power, the last coefficient may be either 1 or -1. Thus the theorem holds for every non-negative integer i. □

By using this theorem, Table 7 includes larger powers of Pell numbers, all coefficients are listed vertically.

TABLE 7. Some larger powers of P_n.

P_n^7	P_n^8	P_n^9	P_n^{10}	P_n^{11}	P_n^{12}
408	985	2378	5741	13860	33461
34476	200940	1171165	6826049	39785130	231884730
−482664	−6791772	−95567064	−1344731653	−18921807828	−266250046986
−1166438	−39618670	−1345902818	−45720876202	−1553165059215	−52761884311059
482664	396186670	3248730940	266442347522	21851425660680	1792084691254935
34476	6791772	1345902818	266442347522	52755584809356	10445293629028764
−408	−200940	−95567064	−45720876202	−21851425660680	−10445293629028764
−1	−985	−1171165	−1344731653	−1553165059215	−1792084691254935
	1	2378	6826049	18921807828	52761884311059
		1	5741	39785130	266250046986
			−1	−13860	−231884730
				−1	−33461
					1

4. POWERS OF MERSENNE NUMBERS

In this section we will discuss the powers of Mersenne numbers, begining with the following lemma.

Lemma 4.1. *For any integers n and m the following identity holds for n^{th} Mersenne number* M_n*,*

$$-2^{m-1}\mathrm{M}_{n-m-1} = \mathrm{M}_{m-1}\mathrm{M}_{n-1} - \mathrm{M}_m\mathrm{M}_{n-2}. \tag{4.1}$$

Theorem 4.1. *The first few powers of* M_n *are*

$$\mathrm{M}_n^2 = 7\mathrm{M}_{n-1}^2 - 14\mathrm{M}_{n-2}^2 + 8\mathrm{M}_{n-3}^2,$$
$$\mathrm{M}_n^3 = 15\mathrm{M}_{n-1}^3 - 70\mathrm{M}_{n-2}^3 + 120\mathrm{M}_{n-3}^3 - 64\mathrm{M}_{n-4}^3,$$
$$\mathrm{M}_n^4 = 31\mathrm{M}_{n-1}^4 - 310\mathrm{M}_{n-2}^4 + 1240\mathrm{M}_{n-3}^4 - 1984\mathrm{M}_{n-4}^4 + 1024\mathrm{M}_{n-5}^4,$$
$$\mathrm{M}_n^5 = 63\mathrm{M}_{n-1}^5 - 1302\mathrm{M}_{n-2}^5 + 11160\mathrm{M}_{n-3}^5 - 41664\mathrm{M}_{n-4}^5 + 64512\mathrm{M}_{n-5}^5 - 32768\mathrm{M}_{n-6}^5,$$
$$\mathrm{M}_n^6 = 127\mathrm{M}_{n-1}^6 - 5334\mathrm{M}_{n-2}^6 + 94488\mathrm{M}_{n-3}^6 - 755904\mathrm{M}_{n-4}^6 + 2731008\mathrm{M}_{n-5}^6 - 4161536\mathrm{M}_{n-6}^6 + 2097152\mathrm{M}_{n-7}^6.$$

Proof. From Equality (1.4) we have

$$\mathrm{M}_n = 3\mathrm{M}_{n-1} - 2\mathrm{M}_{n-2}, \tag{4.2}$$

by using Equality (4.1) with $m = 2$, and squaring both sides of the above equations we obtain

$$\begin{aligned} \mathrm{M}_n^2 &= 9\mathrm{M}_{n-1}^2 - 12\mathrm{M}_{n-1}\mathrm{M}_{n-2} + 4\mathrm{M}_{n-2}^2, \\ 4\mathrm{M}_{n-3}^2 &= \mathrm{M}_{n-1}^2 - 6\mathrm{M}_{n-1}\mathrm{M}_{n-2} + 9\mathrm{M}_{n-2}^2. \end{aligned}$$

By adding these equations we obtain the second power of M_n as

$$\mathrm{M}_n^2 = 7\mathrm{M}_{n-1}^2 - 14\mathrm{M}_{n-2}^2 + 8\mathrm{M}_{n-3}^2. \tag{4.3}$$

For the third power by using Equalities (1.4) and (4.1) with $m = 2,3$ we will obtain

$$\begin{aligned} \mathrm{M}_n^3 &= 27\mathrm{M}_{n-1}^3 - 54\mathrm{M}_{n-1}^2\mathrm{M}_{n-2} + 36\mathrm{M}_{n-1}\mathrm{M}_{n-2}^2 - 8\mathrm{M}_{n-2}^3, \\ (-2)^3\mathrm{M}_{n-3}^3 &= \mathrm{M}_{n-1}^3 - 9\mathrm{M}_{n-1}^2\mathrm{M}_{n-2} + 27\mathrm{M}_{n-1}\mathrm{M}_{n-2}^2 - 27\mathrm{M}_{n-2}^3, \\ (-4)^3\mathrm{M}_{n-4}^3 &= 27\mathrm{M}_{n-1}^3 - 189\mathrm{M}_{n-1}^2\mathrm{M}_{n-2} + 441\mathrm{M}_{n-1}\mathrm{M}_{n-2}^2 - 343\mathrm{M}_{n-2}^3. \end{aligned}$$

To eliminate the non-simple powers we multiply the first equation by a, the second by b, where a and b are non-zero numbers and adding the third we obtain the following system

$$\begin{aligned} -6a - b - 21 &= 0 \\ 4a + 3b + 49 &= 0. \end{aligned}$$

This system has the solution $a = -1$ and $b = -15$, therefore the third power is as follows

$$\mathrm{M}_n^3 = 15\mathrm{M}_{n-1}^3 - 70\mathrm{M}_{n-2}^3 + 120\mathrm{M}_{n-3}^3 - 64\mathrm{M}_{n-4}^3. \tag{4.4}$$

Now, calculate the fourth power, by using Equalities (1.4) and (4.1) with $m = 2,3,4$. We obtain

$$\begin{aligned} \mathrm{M}_n^4 &= 81\mathrm{M}_{n-1}^4 - 216\mathrm{M}_{n-1}^3\mathrm{M}_{n-2} + 216\mathrm{M}_{n-1}^2\mathrm{M}_{n-2}^2 - 96\mathrm{M}_{n-1}\mathrm{M}_{n-2}^3 \\ &\quad + 16\mathrm{M}_{n-2}^4, \\ (-2)^4\mathrm{M}_{n-3}^4 &= \mathrm{M}_{n-1}^4 - 12\mathrm{M}_{n-1}^3\mathrm{M}_{n-2} + 54\mathrm{M}_{n-1}^2\mathrm{M}_{n-2}^2 - 108\mathrm{M}_{n-1}\mathrm{M}_{n-2}^3 + 81\mathrm{M}_{n-2}^4, \\ (-4)^4\mathrm{M}_{n-4}^4 &= 81\mathrm{M}_{n-1}^4 - 756\mathrm{M}_{n-1}^3\mathrm{M}_{n-2} + 2646\mathrm{M}_{n-1}^2\mathrm{M}_{n-2}^2 - 4116\mathrm{M}_{n-1}\mathrm{M}_{n-2}^3 \\ &\quad + 2401\mathrm{M}_{n-2}^4, \\ (-8)^4\mathrm{M}_{n-5}^4 &= 2401\mathrm{M}_{n-1}^4 - 20580\mathrm{M}_{n-1}^3\mathrm{M}_{n-2} + 66150\mathrm{M}_{n-1}^2\mathrm{M}_{n-2}^2 \\ &\quad - 94500\mathrm{M}_{n-1}\mathrm{M}_{n-2}^3 + 50625\mathrm{M}_{n-2}^4. \end{aligned}$$

If we multiply the first equation by a, the second by b, the third by c, where a, b and c are non-zero numbers and add the fourth, we get the following system to eliminate the non-simple powers

$$\begin{aligned}&-216a-12b-756c-20580=0\\&216a+54b+2646c+66150=0\\&-96a-108b-4116c-94500=0.\end{aligned}$$

This system has the solution $a=-4, b=310$ and $c=-31$, thus the fourth power of M_n is

$$\mathrm{M}_n^4=31\mathrm{M}_{n-1}^4-310\mathrm{M}_{n-2}^4+1240\mathrm{M}_{n-3}^4-1984\mathrm{M}_{n-4}^4+1024\mathrm{M}_{n-5}^4. \tag{4.5}$$

By a similar method we can obtain the fifth and sixth powers. □

By using the above theorem we can create Table 8.

TABLE 8. First few powers of M_n.

	$[\mathrm{M}_{n-1}]$	$[\mathrm{M}_{n-2}]$	$[\mathrm{M}_{n-3}]$	$[\mathrm{M}_{n-4}]$	$[\mathrm{M}_{n-5}]$	$[\mathrm{M}_{n-6}]$	$[\mathrm{M}_{n-7}]$
M_n^0	1						
M_n^1	3	−2					
M_n^2	7	−14	8				
M_n^3	15	−70	120	−64			
M_n^4	31	−310	1240	−291984	1024		
M_n^5	63	−1302	11160	−41664	64512	−32768	
M_n^6	127	−5334	94488	−755904	2731008	−4161536	2097152

From the table, we can observe the following pattern:

If we multiply the coefficients of the p^{th} power by M_{p+2} and divide it by lower Mersenne numbers $\mathrm{M}_{p+1},\mathrm{M}_p,\ldots,\mathrm{M}_2,\mathrm{M}_1$ respectively, we will obtain the coefficients of the $(p+1)^{st}$ power. The signs alternate, and the last terms of each column are powers of 2.

Theorem 4.2. *For a fixed non-negative integer p, the p^{th} power of M_n is as follows*

$$\mathrm{M}_n^p=\sum_{i=1}^{p+1}q_{i-1}\mathrm{M}_{n-i}^p, \tag{4.6}$$

the coefficients of q_i are given by

$$q_i=(-1)^i a_i\frac{\mathrm{M}_{p+1}\mathrm{M}_p\ldots\mathrm{M}_{p-i+1}}{\mathrm{M}_1\mathrm{M}_2\ldots\mathrm{M}_{i+1}},\quad i=0,1,\ldots,p, \tag{4.7}$$

where a_n satisfy the recurrence relation $a_n=a_{n-1}2^n$, $a_0=1$, $n\geq 1$.

Proof. We use induction on i, we have that

$$\begin{aligned}
&p=0;\ q_0=1,\\
&p=1;\ q_0=a_0\frac{\mathrm{M}_2}{\mathrm{M}_1}=3\ ;q_1=-a_1\frac{\mathrm{M}_2\mathrm{M}_1}{\mathrm{M}_1\mathrm{M}_2}=-2,\\
&p=2\ ;q_0=a_0\frac{\mathrm{M}_3}{\mathrm{M}_1}=7\ ;q_1=-a_1\frac{\mathrm{M}_3\mathrm{M}_2}{\mathrm{M}_1\mathrm{M}_2}=-14\ ;q_2=-a_2\frac{\mathrm{M}_3\mathrm{M}_2\mathrm{M}_1}{\mathrm{M}_1\mathrm{M}_2\mathrm{M}_3}=8.
\end{aligned}$$

Thus the first three powers are

$$\begin{aligned}
&\mathrm{M}_n^0=1,\\
&\mathrm{M}_n^1=3\mathrm{M}_{n-1}-2\mathrm{M}_{n-2},\\
&\mathrm{M}_n^2=7\mathrm{M}_{n-1}^2-14\mathrm{M}_{n-2}^2+8\mathrm{M}_{n-3}^2.
\end{aligned}$$

Now suppose the coefficients of p^{th} power are as follows

$$\begin{aligned}
q_0&=a_0\frac{\mathrm{M}_{p+1}}{\mathrm{M}_1}=\mathrm{M}_{p+1},\\
q_1&=-a_1\frac{\mathrm{M}_{p+1}\mathrm{M}_p}{\mathrm{M}_1\mathrm{M}_2},\\
&\vdots\\
q_{p-1}&=(-1)^{p-1}a_{p-1}\frac{\mathrm{M}_{p+1}\mathrm{M}_p...\mathrm{M}_2}{\mathrm{M}_1\mathrm{M}_2...\mathrm{M}_p}=(-1)^{p-1}a_{p-1}\mathrm{M}_{p+1},\\
q_p&=(-1)^pa_p\frac{\mathrm{M}_{p+1}\mathrm{M}_p...\mathrm{M}_1}{\mathrm{M}_1\mathrm{M}_2...\mathrm{M}_{p+1}}=(-1)^pa_p.
\end{aligned}$$

If we multiply each coefficient in M_{p+2} and divide it $\mathrm{M}_{p+1},\mathrm{M}_p,...,\mathrm{M}_1$, we will get

$$\begin{aligned}
q_0&=\mathrm{M}_{p+2},\\
q_1&=-a_1\frac{\mathrm{M}_{p+2}\mathrm{M}_{p+1}}{\mathrm{M}_1\mathrm{M}_2},\\
&\vdots\\
q_{p-1}&=(-1)^{p-1}a_{p-1}\frac{\mathrm{M}_{p+1}\mathrm{M}_{p+2}\mathrm{M}_p...\mathrm{M}_2}{\mathrm{M}_1\mathrm{M}_2...\mathrm{M}_p\mathrm{M}_2}=(-1)^{p-1}a_{p-1}\frac{\mathrm{M}_{p+2}\mathrm{M}_{p+1}}{\mathrm{M}_1\mathrm{M}_2},\\
q_p&=(-1)^pa_p\frac{\mathrm{M}_{p+2}\mathrm{M}_{p+1}\mathrm{M}_p...\mathrm{M}_1}{\mathrm{M}_1\mathrm{M}_2...\mathrm{M}_{p+1}\mathrm{M}_1}=(-1)^pa_p\mathrm{M}_{p+2}.
\end{aligned}$$

These are the coefficients of the $(p+1)^{st}$ power, the last coefficient is $(-1)^{p+1}a_p2^{p+1}$, thus the theorem is true for every non-negative integer i. □

By using this theorem we can obtain higher powers as given in Table 9.

TABLE 9. Some larger powers of Mersenne numbers.

M_n^7	M_n^8	M_n^9	M_n^{10}	M_n^{11}
255	511	1023	2047	4095
−21590	−86870	−348502	−1396054	−5588310
777240	6304280	50781720	407647768	3266766360
−12850368	−211823808	−3439615168	−55440096448	−890302725312
99486720	3389180928	111842970624	3634008902656	117175326428160
−353730560	−25822330880	−1761082966016	−116288284884992	−7558738517524480
534773760	91089797120	13312123207680	1816661080408064	239975068524871680
−268435456	−137170518016	−46775146643456	−13678389311307776	−3734200281987022848
	68719476736	70300024700928	47968050187599872	28061309359745925120
		−35184372088832	−72022409665839104	−98310589193870376960
			36028797018963968	147537923792657448960
				73786976294838206464

5. SOME APPLICATIONS

In this section, we discuss applications of the previous theory. Our goal is to find recurrence relations and generating functions for certain integer sequences, including the squares and cubes of Jacobsthal, Pell, and Mersenne numbers.

Definition 5.1. *Let t_n be the sequence defined as*

$$1, 1, 9, 25, 121, 441, 1849, 7225, \ldots, \tag{5.1}$$

which is listed as A139818 *in [15]. This sequence consists of the squares of Jacobsthal numbers, where the n^{th} term is given by $t_n = \mathrm{J}_{n+1}^2$, $n \geq 0$.*

Theorem 5.1. *The recurrence relation and generating function for the sequence defined in equation* (5.1) *are given respectively as follows,*

$$t_n = 3t_{n-1} + 6t_{n-2} - 8t_{n-3}, \quad n \geq 3, \tag{5.2}$$

with initial values $t_0 = t_1 = 1, t_2 = 9$, and

$$t(x) = \sum_{k=0}^{\infty} t_n x^n = \frac{1-2x}{1-3x-6x^2+8x^3}. \tag{5.3}$$

Proof. From Equality (2.4) we have

$$\mathrm{J}_{n+1}^2 = 3\mathrm{J}_n^2 + 6\mathrm{J}_{n-1}^2 - 8\mathrm{J}_{n-2}^2,$$

let $t_n = \mathrm{J}_{n+1}^2$, then we obtain the following recurrence relation

$$t_n = 3t_{n-1} + 6t_{n-2} - 8t_{n-3},$$

with the initial values of $t_0 = \mathrm{J}_1^2 = 1$, $t_1 = \mathrm{J}_2^2 = 1$ and $t_2 = \mathrm{J}_3^2 = 9$. On the other hand, let $t(x) = \sum_{k=0}^{\infty} t_n x^n$, then

$$\begin{aligned} t(x) &= 1 + x + 9x^2 + 25x^3 + 121x^4 + \ldots, \\ -3xt(x) &= -3x - 3x^2 - 27x^3 - 75x^4 - \ldots, \\ -6x^2 t(x) &= -6x^2 - 6x^3 - 54x^4 - \ldots, \\ 8x^3 t(x) &= 8x^3 + 8x^4 + \ldots. \end{aligned}$$

By adding both sides of the above equations and applying the recurrence relation for t_n , we obtain the following result

$$t(x) = \frac{1-2x}{1-3x-6x^2+8x^3}.$$

□

Definition 5.2. *Let s_n be the sequence listed as* A253103 *in [15] and defined as follows*

$$1, 27, 125, 1331, 9261, 79507, 614125, 5000211, 39651821, \ldots, \tag{5.4}$$

the n^{th} term of this sequence is $s_n = \mathrm{J}_{n+2}^3$ for $n \geq 0$.

Theorem 5.2. *The recurrence relation and generating function for the sequence given in equation* (5.4) *are respectively as follows*

$$s_n = 5s_{n-1} + 30s_{n-2} - 40s_{n-3} - 64s_{n-4}, \quad n \geq 4, \tag{5.5}$$

with the initial values $s_0 = 1$, $s_1 = 27$, $s_2 = 125$, $s_3 = 1331$, and

$$s(x) = \sum_{k=0}^{\infty} s_n x^n = \frac{1+22x-40x^2-64x^3}{1-5x-30x^2+40x^3+64x^4}. \tag{5.6}$$

Proof. From Equality (2.5) we have

$$\mathrm{J}_{n+1}^3 = 5\mathrm{J}_n^3 + 30\mathrm{J}_{n-1}^3 - 40\mathrm{J}_{n-2}^3 - 64\mathrm{J}_{n-3}^3,$$

let $s_n = J_{n+2}^3$, then we can obtain the recurrence relation as follows

$$s_n = 5s_{n-1} + 30s_{n-2} - 40s_{n-3} - 64s_{n-4}, \quad n \geq 4,$$

with the initial values $s_0 = \mathrm{J}_1^2 = 1$, $s_1 = \mathrm{J}_2^2 = 27$, $s_2 = \mathrm{J}_3^2 = 125$ and $s_3 = \mathrm{J}_4^2 = 1331$. On the other hand let $s(x) = \sum_{k=0}^{\infty} s_n x^n$ be generating function for (5.4), then

$$\begin{aligned} s(x) &= s_0 + s_1 x + s_2 x^2 + s_3 x^3 + s_4 x^4 + s_5 x^5 + \ldots, \\ -5xs(x) &= -5xs_0 - 5s_1 x^2 - 5s_2 x^3 - 5s_3 x^4 - 5s_4 x^5 - \ldots, \\ -30x^2 s(x) &= -30s_0 x^2 - 30s_1 x^3 - 30s_2 x^4 - 30s_3 x^5 - \ldots, \\ 40x^3 s(x) &= 40s_0 x^3 + 40s_1 x^4 + 40s_2 x^5 + \ldots, \\ 64x^4 s(x) &= 64s_0 x^4 + 64s_1 x^5 + \ldots, \end{aligned}$$

by adding these equations and using the recurrence relation we obtain

$$s(x) = \frac{1+22x-40x^2-64x^3}{1-5x-30x^2+40x^3+64x^4}.$$

□

Definition 5.3. *Let r_n be a sequence which is listed at* A079291 *in [15] defined as follows*

$$0,1,4,25,144,841,4900,28561,166464,970225,5654884,..., \tag{5.7}$$

this sequence consists of the squares of Pell numbers, that is the n^{th} term of it is $r_n = \mathrm{P}_n^2$ for $n \geq 0$.

Theorem 5.3. *The recurrence relation and generating function for the sequence given in* (5.7) *are given respectively as follows*

$$r_n = 5r_{n-1} + 5r_{n-2} - r_{n-3}, \quad n \geq 3, \tag{5.8}$$

with the initial values $r_0 = 0$, $r_1 = 1$ and $r_2 = 4$, and

$$r(x) = \frac{1-4x-6x^2}{1-5x-5x^2+x^3}. \tag{5.9}$$

Proof. From Equality (3.2) we have that

$$\mathrm{P}_{n+1}^2 = 5\mathrm{P}_n^2 + 5\mathrm{P}_{n-1}^2 - \mathrm{P}_{n-2}^2,$$

let $r_n = \mathrm{P}_n^2$, then the recurrence relation for r_n is

$$r_n = 5r_{n-1} + 5r_{n-2} - r_{n-3},$$

with the initial values $r_0 = \mathrm{P}_0^2 = 0$, $r_1 = \mathrm{P}_1^2 = 1$ and $r_2 = \mathrm{P}_2^2 = 4$. On the other hand, let $r(x) = \sum_{k=0}^{\infty} r_n x^n$, then

$$\begin{aligned}
r(x) &= r_0 + r_1x + r_2x^2 + r_3x^3 + r_4x^4 + ..., \\
-5xr(x) &= -5r_0x - 5r_1x^2 - 5r_2x^3 - 5r_3x^4 - ..., \\
-5x^2r(x) &= -5r_0x^2 - 5r_1x^3 - 5r_2x^4 - ..., \\
x^3r(x) &= r_0x^3 + r_1x^4 + ...,
\end{aligned}$$

by adding both sides of the above equations and using the recurrence relation we get the result as

$$r(x) = \frac{1-4x-6x^2}{1-5x-5x^2+x^3}.$$

□

Definition 5.4. *The sequence of u_n which is listed as* A110272 *in [15] is defined as follows*

$$0, 1, 8, 125, 1728, 24389, 343000, 4826809, 67917312, 955671625, \ldots, \tag{5.10}$$

the n^{th} term of this sequence is $u_n = \mathrm{P}_n^3$ for $n \geq 0$.

Theorem 5.4. *The recurrence relation and generating function for the sequence given in* (5.10) *are, respectively, as follows*

$$u_n = 12u_{n-1} + 30u_{n-2} - 12u_{n-3} - u_{n-4}, \quad n \geq 4, \tag{5.11}$$

with the initial values $u_0 = 0$, $u_1 = 1$, $u_2 = 8$, $u_3 = 125$, and

$$u(x) = \sum_{k=0}^{\infty} u_n x^n = \frac{x - 4x^2 - x^3}{1 - 12x - 30x^2 + 12x^3 + x^4}. \tag{5.12}$$

Proof. From Equality (3.3) we have that

$$\mathrm{P}_{n+1}^3 = 12\mathrm{P}_n^3 + 30\mathrm{P}_{n-1}^3 - 12\mathrm{P}_{n-2}^3 - \mathrm{P}_{n-3}^3,$$

let $u_n = \mathrm{P}_n^3$, then the recurrence relation for (5.10) can be obtained as

$$u_n = 12u_{n-1} + 30u_{n-2} - 12u_{n-3} - u_{n-4}, \quad n \geq 4,$$

with the initial values of $u_0 = \mathrm{P}_0^3 = 0$, $u_1 = \mathrm{P}_1^3 = 1$, $u_2 = \mathrm{P}_2^3 = 8$ and $u_3 = \mathrm{P}_3^3 = 125$. On the other hand let $u(x) = \sum_{k=0}^{\infty} u_n x^n$ be the generating function for (5.10), then

$$\begin{aligned}
u(x) &= u_0 + u_1 x + u_2 x^2 + u_3 x^3 + u_4 x^4 + u_5 x^5 + \ldots, \\
-12xu(x) &= -12u_0 x - 12u_1 x^2 - 12u_2 x^3 - 12u_3 x^4 - 12u_4 x^5 - \ldots, \\
-30x^2 u(x) &= -30u_0 x^2 - 30u_1 x^3 - 30u_2 x^4 - 30u_3 x^5 - \ldots, \\
12x^3 u(x) &= 12u_0 x^3 + 12u_1 x^4 + 12u_2 x^5 + \ldots, \\
x^4 u(x) &= u_0 x^4 + u_1 x^5 + \ldots,
\end{aligned}$$

by adding these equations and using the recurrence relation for u_n we obtain

$$u(x) = \frac{x - 4x^2 - x^3}{1 - 12x - 30x^2 + 12x^3 + x^4}.$$

$\square$

Definition 5.5. *The sequence v_n which is defined as follows*

$$1, 9, 49, 225, 961, 3969, 16129, 65025, 261121, 1046529, 4190209, \ldots, \tag{5.13}$$

is listed as A060867 *in [15], the n^{th} term of this sequence is $v_n = \mathrm{M}_{n+1}^2$ for $n \geq 0$.*

Theorem 5.5. *The recurrence relation and generating function for the sequence given in* (5.13) *are respectively as follows*

$$v_n = 7v_{n-1} - 14v_{n-2} + 8v_{n-3}, \tag{5.14}$$

with the initial values $v_0 = 1$, $v_1 = 9$, $v_2 = 49$, *and*

$$v(x) = \sum_{k=0}^{\infty} v_n x^n = \frac{1+2x}{1-7x+14x^2-8x^3}. \tag{5.15}$$

Proof. From Equality (4.3) we have

$$\mathrm{M}_{n+1}^2 = 7\mathrm{M}_n^2 - 14\mathrm{M}_{n-1}^2 + 8\mathrm{M}_{n-2}^2,$$

let $v_n = \mathrm{M}_{n+1}^2$, then the recurrence relation for (5.13) is

$$v_n = 7v_{n-1} - 14v_{n-2} + 8v_{n-3},$$

with the initial values of $v_0 = \mathrm{M}_1^2 = 1$, $v_1 = \mathrm{M}_2^2 = 9$ and $v_2 = \mathrm{M}_3^2 = 49$. On the other hand, let $v(x) = \sum_{k=0}^{\infty} v_n x^n$, then

$$\begin{aligned} v(x) &= v_0 + v_1 x + v_2 x^2 + v_3 x^3 + v_4 x^4 + \ldots, \\ -7xv(x) &= -7v_0 x - 7v_1 x^2 - 7v_2 x^3 - 7v_3 x^4 - \ldots, \\ 14x^2 v(x) &= 14v_0 x^2 + 14v_1 x^3 + 14v_2 x^4 - \ldots, \\ -8x^3 v(x) &= -8v_0 x^3 - 8v_1 x^4 + \ldots, \end{aligned}$$

by adding both sides of the above equations and using the recurrence relation for (5.13), we get the result

$$v(x) = \frac{1+2x}{1-7x+14x^2-8x^3}.$$

□

Definition 5.6. *Let* w_n *be the sequence*

$$1, 27, 343, 3375, 29791, 250047, 2048383, 16581375, 133432831, 1070599167, \ldots, \tag{5.16}$$

which is listed as A128831 *in [15], then the* n^{th} *term is* $w_n = \mathrm{M}_{n+1}^3$ *for* $n \geq 0$.

Theorem 5.6. *The recurrence relation and generating function for the sequence given in* (5.16) *are respectively as follows*

$$w_n = 15w_{n-1} - 70w_{n-2} + 120w_{n-3} - 64w_{n-4}, \quad n \geq 4, \tag{5.17}$$

with the initial values $u_0 = 0$, $u_1 = 1$, $u_2 = 8$, $u_3 = 125$, *and*

$$w(x) = \sum_{k=0}^{\infty} w_n x^n = \frac{1+12x+8x^2}{1-15x+70x^2-120x^3+64x^4}. \tag{5.18}$$

Proof. From Equality (4.4) we have that

$$\mathrm{M}_{n+1}^3 = 15\mathrm{M}_n^3 - 70\mathrm{M}_{n-1}^3 + 120\mathrm{M}_{n-2}^3 - 64\mathrm{M}_{n-3}^3,$$

let $w_n = \mathrm{M}_{n+1}^3$, then the recurrence relation of (5.16) is

$$w_n = 15w_{n-1} - 70w_{n-2} + 120w_{n-3} - 64w_{n-4}, \quad n \geq 4,$$

with the initial values $w_0 = \mathrm{M}_1^3 = 1$, $w_1 = \mathrm{M}_2^3 = 27$, $w_2 = \mathrm{M}_3^3 = 343$ and $w_3 = \mathrm{M}_4^3 = 3375$. On the other hand let $w(x) = \sum_{k=0}^{\infty} w_n x^n$ be the generating function for (5.16), then

$$\begin{aligned} w(x) &= w_0 + w_1 x + w_2 x^2 + w_3 x^3 + w_4 x^4 + w_5 x^5 + \dots, \\ -15xw(x) &= -15w_0 x - 15w_1 x^2 - 15w_2 x^3 - 15w_3 x^4 - 15w_4 x^5 - \dots, \\ 70x^2 w(x) &= 70w_0 x^2 + 70w_1 x^3 + 70w_2 x^4 + 70w_3 x^5 + \dots, \\ -120x^3 w(x) &= -120w_0 x^3 - 120w_1 x^4 - 120w_2 x^5 - \dots, \\ 64x^4 w(x) &= 64w_0 x^4 + 64w_1 x^5 + \dots,. \end{aligned}$$

By adding these equations and using the recurrence relation for w_n we obtain

$$w(x) = \frac{1 + 12x + 8x^2}{1 - 15x + 70x^2 - 120x^3 + 64x^4}.$$

□

6. Numerical Examples

In this section, we compute various powers of Jacobsthal, Pell, and Mersenne numbers and verify our results.

Example 6.1. *The 6^{th} power of J_7 is as follows*

$$\begin{aligned} \mathrm{J}_7^6 &= 43\mathrm{J}_6^6 + 1806\mathrm{J}_5^6 - 26488\mathrm{J}_4^6 - 211904\mathrm{J}_3^6 + 924672\mathrm{J}_2^6 + 1409024\mathrm{J}_1^6 - 2097152\mathrm{J}_0^6 \\ &= 43(21)^6 + 1806(11)^6 - 26488(5)^6 - 211904(3)^6 + 924672(1)^6 + 1409024(1)^6 \\ &\quad - 2097152(0)^6 \\ &= 6321363049. \end{aligned}$$

On the other hand

$$J_7^6 = 43^6 = 6321363049.$$

Example 6.2. *The 7^{th} power of P_8 is*

$$\begin{aligned} \mathrm{P}_8^7 &= 408\mathrm{P}_7^7 + 34476\mathrm{P}_6^7 - 482664\mathrm{P}_5^7 - 1166438\mathrm{P}_4^7 + 482664\mathrm{P}_3^7 + 34476\mathrm{P}_2^7 - 408\mathrm{P}_1^7 - P_0^7 \\ &= 408(169)^7 + 34476(70)^7 - 482664(29)^7 - 1166438(12)^7 + 482664(5)^7 + 34476(2)^7 \\ &\quad - 408(1)^7 - (0)^7 \\ &= 1882006597876580352. \end{aligned}$$

On the other hand

$$\mathrm{P}_8^7 = 408^7 = 1882006597876580352.$$

Example 6.3. *The* 8^{th} *power of* M_6 *is*

$$\begin{aligned}
\mathrm{M}_6^8 &= 511\mathrm{M}_5^8 - 86870\mathrm{M}_4^8 + 6304280\mathrm{M}_3^8 - 211823808\mathrm{M}_2^8 + 3389180928\mathrm{M}_1^8 - 25822330880\mathrm{M}_0^8 \\
&+ 91089797120\mathrm{M}_{-1}^8 - 137170518016\mathrm{M}_{-2}^8 + 68719476736\mathrm{M}_{-3}^8 \\
&= 511(31)^8 - 86870(15)^8 + 6304280(7)^8 - 211823808(3)^8 + 3389180928(1)^8 \\
&- 25822330880(0)^8 + 91089797120(\frac{1}{2})^8 - 137170518016(-\frac{3}{4})^8 + 68719476736(-\frac{7}{8})^8 \\
&= 248155780267521.
\end{aligned}$$

On the other hand

$$\mathrm{M}_6^8 = 63^8 = 248155780267521.$$

7. Conclusions

We found the formulas for the powers of Jacobsthal, Pell and Mersenne numbers. These formulas are given by the sum of the powers of successive lower terms. Furthermore we construct tables for these powers and set a pattern to reach a higher power easily. Some applications of these power formulas to find the recurrence relation for some integer sequences are given. All powers were checked by applying the formula to a particular value of n.

Conflict of interest

The authors declare no conflict of interest.

Data availability

Not applicable.

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