GENERALIZED HEAT EQUATION UNDER CONFORM DERIVATIVE

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ABSTRACT. In the present work, we establish the existence and uniqueness result of the linear heat equation with Conform derivative in Colombeau generalized algebra. We using for the first time the notion of a generalized conformable semigroup and the purpose of introducing Conform derivative is regularizing it in Colombeau.

1. INTRODUCTION

In this paper the heat equation is considered with the first order time derivative changed to a conform derivative. for each type of data we always ask what is the optimal corresponding nonlocal model to be applied. Moreover, many authors studied Conform operators with local, nonlocal, singular and non-singular kernels [2]. The Riemann and Caputo fractional calculus may not provide us the required kernel in order to extract important information from these types of systems. At this stage, we ask the following question. Can we generalize the standard fractional Riemann-Liouville integrals in a way such that we obtain unification to Riemann-Liouville, Hadamard and other fractional derivatives. The importance of this procedure is to decide which differentiation operator should be used as a starting point for the iteration procedure. For the standard fractional calculus, we iterate the usual integral of a function and using the Cauchy formula we obtain the integral of higher integer orders and then replace this integer by any real number. In [1] the author presented and developed the definitions of Conform derivative and set the basic concepts in this new simple interesting fractional calculus. The fractional versions of chain rule, exponential functions, Gronwall’s inequality, integration by parts, Taylor power series expansions, Laplace transforms and linear differential systems are proposed and discussed. The present paper concerns the study of existence and uniqueness to equation with Conform differentiation in extended Colombeau algebra. We consider Conform differentiation for indicating to existence and uniqueness heat equation in extended Colombeau algebra. The

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reason for introducing conform derivatives into algebra of generalized functions was the possibility of solving nonlinear problems with singularities and derivatives of arbitrary real order in it. We use specific space of Colombeau algebra type in order to give a sense of our problem. Colombeau algebras is a differential algebra, commutative, associative in which we can inject $D'$ the set of distributions so that the product of smooth functions and the usual derivative of distributions are respected [5], [6], [7]. From the previous ideas we will discuss the existence and uniqueness of such equation in a specific spaces coincide with the usual spaces when $\alpha \to 1$.

The paper is organized as follows: After this introduction, we present some concepts concerning the Colombeau’s algebra. In section 3 we give the definitions and we prove some properties concerning the Conform derivative. The embedding of this derivation in Colombeau algebra takes place in section 4. In the last section we discuss the existence and uniqueness of our problem.

2. PRELIMINARIES

We shall fix the notation and introduce a number of known as well as new classes of generalized functions here. For more details, see [7].

Let $\Omega$ be an open subset of $\mathbb{R}^n$. The basic objects of the theory as we use it are families $(u_\varepsilon)_{\varepsilon \in (0,1]}$ of smooth functions $u_\varepsilon \in C^\infty(\Omega)$ for $0 < \varepsilon \leq 1$. We single out the following subalgebra.

Moderate families, denoted by $E_M(\Omega)$, are defined by the property :
\[
\forall K \subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \exists p \geq 0 : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O_{\varepsilon \to 0}(\varepsilon^{-p})
\]

Null families, denoted by $E_M(\Omega)$, are defined by the property :
\[
\forall K \subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \forall q \geq 0 : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O_{\varepsilon \to 0}(\varepsilon^q)
\]

Thus moderate families satisfy a locally uniform polynomial estimate as $\varepsilon \to 0$, together with all derivatives, while null functions vanish faster than any power of $\varepsilon$ in the same situation. The null families form a differential ideal in the collection of moderate families.

The Colombeau algebra is the factor algebra
\[G(\Omega) = E_M(\Omega) / N(\Omega)\]

The algebra $G(\Omega)$ just defined coincides with the special Colombeau algebra in [5], where the notation $G^s(\Omega)$ has been employed. It was called the simplified Colombeau algebra in [5].

The Colombeau algebra on a closed half space $\mathbb{R}^n \times [0,1)$ is defined in a similar way. The restriction of an element $u \in G(\mathbb{R} \times [0,1))$ to the line $\{ t = 0 \}$ is defined on representatives by
\[u/\{ t = 0 \} = \text{Class of } (u_\varepsilon(\cdot,0))_{\varepsilon \in (0,1]}\]
Similarly, restrictions of the elements of $G(\Omega)$ to open subsets of $\Omega$ are defined on representatives. One can see that $\Omega \rightarrow G(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$. The space of compactly supported distributions is imbedded in $G(\Omega)$ by convolution:

$$i : \mathcal{E}'(\Omega) \rightarrow G(\Omega)$$

$$\omega \mapsto i(\omega) = \text{class of } (\omega * (\phi_\varepsilon)/\Omega)_{\varepsilon \in (0,1]}$$

where

$$\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$$

is obtained by scaling a fixed test function $\mathcal{S}(\mathbb{R}^n)$ of integral one with all moments vanishing. By the sheaf property, this can be extended in a unique way to an imbedding of the space of distributions $\mathcal{D}(\Omega)$.

One of the main features of the Colombeau construction is the fact that this imbedding renders $C^\infty(\Omega)$ a faithful subalgebra. In fact, given $f \in C^\infty(\Omega)$, one can define a corresponding element of $G(\Omega)$ by the constant imbedding $\sigma(f) = \text{class of } [(\varepsilon, x) \mapsto f(x)]$. Then the important equality $i(f) = \sigma(f)$ holds in $G(\Omega)$.

If $u \in G(\Omega)$ and $f$ is a smooth function which is of at most polynomial growth at infinity, together with all its derivatives, the superposition $f(u)$ is a well-defined element of $G(\Omega)$.

We need a couple of further notions from the theory of Colombeau generalized functions. An element $u$ of $G(\Omega)$ is called of local $L^p$-type ($1 \leq p \leq \infty$), if it has a representative with the property $\lim_{\varepsilon \to 0} \| u_\varepsilon \|_{L^p(K)}$ exists for every $K \subset \Omega$.

Regularity theory is based on the subalgebra $G^m(\Omega)$ of regular generalized functions in $G(\Omega)$. It is defined by those elements which have a representative satisfying

$$\forall K \subset \Omega, \forall \alpha \in \mathbb{N}^n_0, \exists p \geq 0 : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O_{\varepsilon \to 0}(\varepsilon^{-p})$$

Observe the change of quantifiers with respect to formula 2; locally, all derivatives of a regular generalized function have the same order of growth in $\varepsilon > 0$. One has that (see [6]).

$$G^m(\Omega) \cap \mathcal{D}'(\Omega) = C^m(\Omega)$$

For the purpose of describing the regularity of Colombeau generalized functions, $G^m(\Omega)$ plays the same role as $C^m(\Omega)$ does in the setting of distributions. A net $(r_\varepsilon)_{\varepsilon \in (0,1]}$ of complex numbers is called a slow scale net if

$$|r_\varepsilon| = O_{\varepsilon \to 0}(\varepsilon^{-p})$$

for every $p \geq 0$. We refer to [6] for a detailed discussion of slow scale nets. Finally, an element $u \in G(\Omega)$ is called of total slow scale type, if for some representative, $\| \partial^\alpha u_\varepsilon \|_{L^p(K)}$ forms a slow scale net for every $K \subset \Omega$ and $\alpha \in \mathbb{N}^n_0$.

We end this section by recalling the association relation on the Colombeau algebra $G(\Omega)$. It identifies elements of $u \in G(\Omega)$ if they coincide in the weak limit. That
is, \( u, v \in G(\Omega) \) are called associated,

\[
u \approx v, \text{ if } \lim_{\varepsilon \to 0} \int (u_\varepsilon(x) - v_\varepsilon(x))\psi(x)dx = 0\]

### 3. Conformable Derivative

In this section we will give some definition and properties concerning the new derivative important in the following.

**Definition 3.1.** [11] Let \( \alpha \in (n, n + 1) \) and \( f : [0, \infty) \to \mathbb{R} \) be \( n \)-differentiable at \( t > 0 \), then the conformable derivative of \( f \) of order \( \alpha \) is defined by

\[
f^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f^{(n)}(t + \varepsilon t^{n+1-\alpha}) - f^{(n)}(t)}{\varepsilon}
\]

\[
f^{(\alpha)}(0) = \lim_{t \to 0} f^{(\alpha)}(t)
\]

**Remark 3.1.** [11] As consequence of the previous definition, one can easily show that

\[
f^{(\alpha)}(t) = t^{n+1-\alpha}f^{(n+1)}(t)
\]

where \( \alpha \in (n, n + 1) \), and \( f \) is \((n + 1)\)-differentiable at \( t > 0 \).

In [2] we find the following proposition.

**Proposition 3.1.** [2] Let \( f, g \) two function \( \alpha \)-derivatives and \( a, b \in \mathbb{R} \). We have the following properties.

1. \((af + bg)^{(\alpha)} = af^{(\alpha)} + bg^{(\alpha)}\),
2. \((fg)^{(\alpha)} = f^{(\alpha)}g + fg^{(\alpha)}\),
3. \((t^p)^{(\alpha)} = pt^{p-\alpha}\),
4. \(( \frac{f}{g} )^{(\alpha)} = \frac{f^{(\alpha)}g - fg^{(\alpha)}}{g^2}\),
5. If \( c \in \mathbb{R} \), \( c^{(\alpha)} = 0 \).

**Proposition 3.2.** If \( x \) is a continuous map, then \( t \to x^{(\alpha)}(t) \) is a continuous map.

**Proof.** Since \( x \) is a continuous map \( t \in \mathbb{R}^+ \to x(t + \varepsilon t^{1-\alpha}) \) is a continuous, thus \( \forall \beta > 0, \exists \alpha > 0, \) \( \frac{|x(t + \alpha t^{1-\alpha}) - x(t_0 + \alpha t_0^{1-\alpha})|}{\varepsilon} \leq \beta \)

whenever \(|t - t_0| \leq \alpha\), by passing to limite \( \varepsilon \to 0 \) we get \( |x^{(\alpha)}(t) - x^{(\alpha)}(t_0)| \leq \beta \) as desired. \( \square \)

**Proposition 3.3.** Let \( f : X \to X \), be a Lipschitz map. i.e.

\[
|f(x) - f(y)| \leq k|x - y|, \forall x, y \in X \text{ and } k \in [0, 1].
\]

The problem of Cauchy

\[
\begin{align*}
x^{(\alpha)}(t) &= f(x(t)), & t > 0 \\
x(0) &= x_0
\end{align*}
\]

has a unique solution.
Proof. By 5.3 \( x \) is continuous, the sequence \( x_{n+1} = f(x_n) \) is a Cauchy’s sequence, since \( \mathbb{R} \) is a complet space then \( x_n \) converge to the unique solution of (3.3) \( \square \)

**Definition 3.2.** [11] Let \( \alpha \in (n, n+1] \). We define the \( \alpha \)-integral by:

\[
(I^\alpha f)(t) = \int_0^t s^{\alpha - n} f(s)ds.
\]

According to the previous writing we set

\[
dx_\alpha = \frac{dx}{x^{\alpha - 1}}
\]

**Theorem 3.1.** [11] We have the following inequality

\[
(I^\alpha f)^{(\alpha)}(t) = f(t).
\]

for \( t \geq 0 \)

**Example 3.1.** It is easy to prove that:

\[
I^\alpha(\sin(t)) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+\alpha}}{(2n+\alpha)(2n+1)!}
\]

where \( \alpha \in (1, 2) \)

**Definition 3.3.** [9] Let \( \alpha \in (0, 1) \). For a Banach space \( X \).

A family \( \{T(t)\}_{t \geq 0} \subset L(X, X) \) is called a conform \( \alpha \)-semigroup if:

1. \( T(0) = I \)
2. \( T((s+t)^{\frac{1}{\alpha}}) = T(s^{\frac{1}{\alpha}})T(t^{\frac{1}{\alpha}}) \), for all \( s, t \in [0, \infty) \)

**Example 3.2.** Let \( A \) be a bounded linear operator on \( X \). Define \( T(t) = e^{2\sqrt{t}A} \). Then \( \{T(t)\}_{t \geq 0} \) is a \( \frac{1}{2} \)-semigroup. Indeed:

1. \( T(0) = e^{0A} = I \)
2. \( \forall s, t \in [0, \infty), T((s+t)^{2}) = e^{2(t+s)A} = e^{2A}e^{2sA} = T(s)T(t) \)

**Definition 3.4.** [9] An \( \alpha \)-semigroup \( T(t) \) is called a \( C_0 \)-semigroup if, for each fixed \( x \in X \), \( T(t)x \to x \) as \( t \to 0^+ \)

The conformable \( \alpha \)-derivative of \( T(t) \) at \( t = 0 \) is called the \( \alpha \)-infinitesimal generator of the conform \( \alpha \)-semigroup \( T(t) \), with domain equals

\[
\{x \in X, \lim_{t \to 0} T(t)x \text{ exist}\}
\]

In the sequel \( \alpha \in (0, 1) \).

4. IMBEDDING OF THE CONFORMABLE DIFFERENTIATION INTO EXTENDED
   COLOMBEAU ALGEBRA OF GENERALIZED FUNCTIONS

Let \( u_\varepsilon(x) \) represents a Colombeau generalized function \( u \in \mathcal{G}^e(\mathbb{R}) \): The conformable derivative for \( 0 < \alpha < 1 \) is defined by:

\[
D^\alpha u_\varepsilon(x) = x^{1-\alpha}u'_\varepsilon(x)
\]
we have
\[ | \mathcal{D}^\alpha u_e(x) | = | x^{1-\alpha} u'_e(x) | \\
| \mathcal{D}^\alpha u_e(x) | \leq | x^{1-\alpha} | \sup_{x \in K} | u'_e(x) | \]
so,
\[ \sup_{x \in K} | \mathcal{D}^\alpha u_e(x) | \leq C_1 e^p \]

we use the regularization for $1 < \alpha < 1$
\[ \mathcal{D}^\alpha u_e(x) = \int_{\mathbb{R}} \mathcal{D}^\alpha u_e(s) \phi_e(x - s) ds \]

The convolution form is given by:
\[ \mathcal{D}^\alpha u_e(x) = \mathcal{D}^\alpha u_e \ast \phi_e(x) \]
we indicate that $| \mathcal{D}^\alpha u_e(x) - \mathcal{D}^\alpha u_e(x) | \approx 0$
\[ | \mathcal{D}^\alpha u_e(x) - \mathcal{D}^\alpha u_e(x) | = | \mathcal{D}^\alpha u_e \ast \phi_e(x) - \mathcal{D}^\alpha u_e(x) | \\
| \mathcal{D}^\alpha u_e(x) - \mathcal{D}^\alpha u_e(x) | = | \mathcal{D}^\alpha u_e \ast \phi_e(x) - \mathcal{D}^\alpha u_e \ast \delta(x) | \\
| \mathcal{D}^\alpha u_e(x) - \mathcal{D}^\alpha u_e(x) | = | \mathcal{D}^\alpha u_e \ast (\phi_e(x) - \delta(x)) | \\
| \mathcal{D}^\alpha u_e(x) - \mathcal{D}^\alpha u_e(x) | = | \int_{\mathbb{R}} \mathcal{D}^\alpha u_e(x - s) (\phi_e(s) - \delta(s)) ds | \\
| \mathcal{D}^\alpha u_e(x) - \mathcal{D}^\alpha u_e(x) | = \int_{\mathbb{R}} | \mathcal{D}^\alpha u_e(x - s) | | \phi_e(s) - \delta(s) | ds \rightarrow 0 \]
as $\varepsilon \rightarrow 0$. Since $\lim | \phi_e(x) - \delta(s) |$, then
\[ \mathcal{D}^\alpha u_e(x) \approx \mathcal{D}^\alpha u_e(x) \]

Using the fact that $\phi_e(x)$ has the compact support on $K_0$, so by Holder inequalities, have the following calculations:
we have
\[ \mathcal{D}^\alpha u_e(x) = \mathcal{D}^\alpha u_e \ast \phi_e(x) = \int_{\mathbb{R}} \mathcal{D}^\alpha u_e(x - s) \phi_e(s) ds \]
\[ | \mathcal{D}^\alpha u_e(x) | = | \int_{\mathbb{R}} \mathcal{D}^\alpha u_e(x - s) \phi_e(s) ds | = | \int_{K_0} \mathcal{D}^\alpha u_e(x - s) \phi_e(s) ds | \\
| \mathcal{D}^\alpha u_e(x) | = \int_{K_0} | \mathcal{D}^\alpha u_e(x - s) | | \phi_e(s) | ds \]
\[ \sup_{x \in K} | \mathcal{D}^\alpha u_e(x) | = \sup_{x \in K_1} \int_{K_0} | \mathcal{D}^\alpha u_e(x - s) | | \phi_e(s) | ds \]
so,
\[ \sup_{x \in K} | \mathcal{D}^\alpha u_e(x) | \leq \sup_{x \in K_1} | \mathcal{D}^\alpha u_e(x) | \int_{K_0} | \phi_e(s) | ds \]
\[ \sup_{x \in K} | \mathcal{D}^\alpha u_e(x) | \leq C_1 e^p \]
and
\[
\frac{d}{dx}(\tilde{D}_\alpha u_\epsilon(x)) = \frac{d}{dx}(D_\alpha u_\epsilon) \ast \phi_\epsilon(x) = D_\alpha u_\epsilon \ast \frac{d}{dx}(\phi_\epsilon(x))
\]
then,
\[
\sup_{x \in K} \left| \frac{d}{dx}(\tilde{D}_\alpha u_\epsilon(x)) \right| \leq \sup_{x \in K_1} \left| D_\alpha u_\epsilon(x) \right| \int_{K_0} \left| \frac{d}{dx}(\phi_\epsilon(s)) \right| ds \leq C_2 \epsilon^p
\]
In order to prove moderateness for higher derivatives a similar calculation is applied.
\[
\sup_{x \in K} \left| \partial^n \tilde{D}_\alpha u_\epsilon(x) \right| \leq C_3 \epsilon^p
\]

5. **Generalized Conformable Semigroup**

We define
\[
\mathcal{E}_M(\mathbb{R}) := \left\{ (x_\epsilon)_\epsilon \in (\mathbb{R}^{(0,1)} \cap \mathbb{N}) : x_\epsilon = O_{\epsilon \to 0}(\epsilon^{-m}) \right\}
\]
and
\[
\mathcal{N}(\mathbb{R}) := \left\{ (x_\epsilon)_\epsilon \in (\mathbb{R}^{(0,1)} \cap \mathbb{N}) : \forall m \in \mathbb{N}, x_\epsilon = O_{\epsilon \to 0}(\epsilon^m) \right\}
\]
It is easy to prove that

**Proposition 5.1.** The space \( \mathcal{E}_M(\mathbb{R}) \) is an algebra and \( \mathcal{N}(\mathbb{R}) \) ideal in \( \mathcal{E}_M(\mathbb{R}) \)

**Definition 5.1.** We define the Colombeau algebra type by:
\[
\tilde{\mathbb{R}} = \mathcal{E}_M(\mathbb{R}) / \mathcal{N}(\mathbb{R})
\]

**5.1. Locally convex and complete spaces**

**Definition 5.2.** : Let \( X \) be a vector space with a seminorms family \( (p_i)_{i \in I} \). If \( \tau_i \) is the topology defined by the only semi-norm \( p_i \). If \( \tau \) is the super bound of topology \( \tau_i \). The space provided with this topology \( \tau \) is called a locally convex space

A basis of 0-neighbourhood is the set of all “balls” of the seminorms \( (p_i)_{i \in I} \)
\[
B(i, r) = \left\{ x \in X : p_i(x) < r \right\}, \quad \forall i \in I \text{ and } r > 0.
\]
Then, \((x_n)_{n \in \mathbb{N}}\) is a Cauchy sequence iff
\[
(\forall \epsilon > 0) \left( \forall i \in I \right) \left( \exists n_0 \in \mathbb{N} \right) \left( \forall n, p \in \mathbb{N} \text{ if } n \geq n_0 \Rightarrow p_i(x_{n+p} - x_n) < \epsilon \right)
\]

**Definition 5.3.** We said that \( \mathcal{D} \) is dense in locally convex space \( X \) iff
\[
(\forall x \in X) \left( \exists y \in \mathcal{D} \right) \left( \forall \epsilon > 0 \right) \left( \forall i \in I \right) \text{ we have } p_i(x - y) < \epsilon
\]
and \( X \) is sequentially complete if any Cauchy sequence converges to an element \( e \) in \( X \).
5.2. Generalized semigroup

**Definition 5.4.** Let $X$ be a locally convex space with a seminorm family $(p_i)_{i \in I}$. We define

$$E_M(X) := \{(x_\varepsilon) \in (X)^{(0,1)} / \exists m \in \mathbb{N}, \forall i \in I, p_i(x_\varepsilon) = O_{\varepsilon \to 0}(\varepsilon^{-m})\}$$

and

$$\mathcal{N}(X) := \{(x_\varepsilon) \in (X)^{(0,1)} / \forall m \in \mathbb{N}, \forall i \in I, p_i(x_\varepsilon) = O_{\varepsilon \to 0}(\varepsilon^m)\}$$

We define the Colombeau algebra type by:

$$\hat{X} = E_M(X) / \mathcal{N}(X)$$

First, we are looking if it is possible to define a map $A : \hat{X} \to \hat{X}$ by means of a given family $(A_\varepsilon)_{\varepsilon \in (0,1)}$ of maps $A_\varepsilon : X \to X$ where $A_\varepsilon$ is a linear and continuous operator. The general requirement is given in the following

**Lemma 5.1.** Let $(A_\varepsilon)_{\varepsilon \in (0,1)}$ be a given family of maps $A_\varepsilon : X \to X$. For each $(x_\varepsilon) \in E_M(X)$ and $(y_\varepsilon) \in \mathcal{N}(X)$, suppose that

1. $(A_\varepsilon x_\varepsilon)_\varepsilon \in E_M(X)$
2. $(A_\varepsilon (x_\varepsilon + y_\varepsilon))_\varepsilon - (A_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{N}(X)$.

Then

$$A : \left\{ \hat{X} \to \hat{X} \right\}$$

is well defined.

**Proof.** From the first property we see that the class $[(A_\varepsilon x_\varepsilon)_\varepsilon] \in \hat{X}$

Let $x_\varepsilon + x_\varepsilon$ be another representative of $x = [x_\varepsilon]$

From the second property we have

$$(A_\varepsilon (x_\varepsilon + y_\varepsilon))_\varepsilon - (A_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{N}(X)$$

and

$$[(A_\varepsilon (x_\varepsilon + y_\varepsilon))_\varepsilon] = [(A_\varepsilon x_\varepsilon)_\varepsilon]$$

in $\hat{X}$

Then $A$ is well defined. \qed

**Definition 5.5.** Let $S E_M(\mathbb{R}_+ : L_c(X))$ is the space of nets $(S_\varepsilon)_\varepsilon$ of strongly continuous mappings $S_\varepsilon : \mathbb{R}_+ \to L_c(X), \varepsilon \in (0,1)$ with the property that for every $T > 0$ there exists $a \in \mathbb{R}$ such that

$$\sup_{t \in [0,T]} \|S_\varepsilon(t^{\frac{1}{2}})\| = O_{\varepsilon \to 0}(\varepsilon^a),$$

(5.1)

and $S \mathcal{N}(\mathbb{R}_+ : L_c(X))$ is the space of nets $(N_\varepsilon)_\varepsilon$ of strongly continuous mappings $N_\varepsilon : \mathbb{R}_+ \to L_c(X), \varepsilon \in (0,1)$ with the properties:
For every $b \in \mathbb{R}$ and $T > 0$

$$\sup_{t \in [0,T]} \|N_\varepsilon(t^{\frac{1}{\varepsilon}})\| = O_{\varepsilon \to 0}(\varepsilon^b), \quad (5.2)$$

There exist $t_0 > 0$ and $a \in \mathbb{R}$ such that

$$\sup_{t < t_0} \left\| \frac{N_\varepsilon(t^{\frac{1}{\varepsilon}})}{t} \right\| = O_{\varepsilon \to 0}(\varepsilon^a), \quad (5.3)$$

There exists a net $(H_\varepsilon)_\varepsilon$ in $L_c(X)$ and $\varepsilon_0 \in (0,1)$ such that

$$\lim_{t \to \varepsilon_0} \frac{N_\varepsilon(t^{\frac{1}{\varepsilon}})}{t}H_\varepsilon, \quad x \in X, \quad (5.4)$$

For every $b > 0$,

$$\|H_\varepsilon\| = O_{\varepsilon \to 0}(\varepsilon^b), \quad (5.5)$$

**Proposition 5.2.** $S^\varepsilon M(\mathbb{R}_+: L_c(X))$ is algebra with respect to composition and $S^\varepsilon N(\mathbb{R}_+: L_c(X))$ is an ideal of $S^\varepsilon M(\mathbb{R}_+: L_c(X))$

*Proof.* Let $(S_\varepsilon(t^{\frac{1}{\varepsilon}}))_\varepsilon \in S^\varepsilon M(\mathbb{R}_+: L_c(X))$ and $(N_\varepsilon(t^{\frac{1}{\varepsilon}}))_\varepsilon \in S^\varepsilon N(\mathbb{R}_+: L_c(X))$

We will prove only the second assertion, i.e., That

$$(S_\varepsilon(t^{\frac{1}{\varepsilon}}))_\varepsilon \cdot (N_\varepsilon(t^{\frac{1}{\varepsilon}}))_\varepsilon \in S^\varepsilon N(\mathbb{R}_+: L_c(X))$$

where $S_\varepsilon(t^{\frac{1}{\varepsilon}})N_\varepsilon(t^{\frac{1}{\varepsilon}})$ denotes the composition.

Let $\varepsilon \in (0,1)$. By (2) and (3), for some $a \in \mathbb{R}$ and every $b \in \mathbb{R}$,

$$\|S_\varepsilon(t^{\frac{1}{\varepsilon}})N_\varepsilon(t^{\frac{1}{\varepsilon}})\| \leq \|S_\varepsilon(t^{\frac{1}{\varepsilon}})\|\|N_\varepsilon(t^{\frac{1}{\varepsilon}})\| = O_{\varepsilon \to 0}(\varepsilon^{a+b}),$$

The same holds for $\|N_\varepsilon(t^{\frac{1}{\varepsilon}})S_\varepsilon(t^{\frac{1}{\varepsilon}})\|$. Further, (2) and (5) yield

$$\sup_{t < t_0} \left\| \frac{S_\varepsilon(t^{\frac{1}{\varepsilon}})N_\varepsilon(t^{\frac{1}{\varepsilon}})}{t} \right\| \leq \sup_{t < t_0} \left\| S_\varepsilon(t^{\frac{1}{\varepsilon}}) \right\| \sup_{t < t_0} \left\| N_\varepsilon(t^{\frac{1}{\varepsilon}}) \right\|$$

for some $t_0 > 0$ and $a \in \mathbb{R}$. Also,

$$\sup_{t < t_0} \left\| \frac{S_\varepsilon(t^{\frac{1}{\varepsilon}})N_\varepsilon(t^{\frac{1}{\varepsilon}})}{t} \right\| = O_{\varepsilon \to 0}(\varepsilon^a),$$

for some $t_0 > 0$ and $a \in \mathbb{R}$. Let now $\varepsilon \in (0,1)$ be fixed. We have

$$\left\| \frac{S_\varepsilon(t^{\frac{1}{\varepsilon}})N_\varepsilon(t^{\frac{1}{\varepsilon}})}{t}x - S_\varepsilon(0)H_\varepsilon x \right\| = \left\| S_\varepsilon(t^{\frac{1}{\varepsilon}}) \frac{N_\varepsilon(t^{\frac{1}{\varepsilon}})}{t}x - S_\varepsilon(t^{\frac{1}{\varepsilon}})H_\varepsilon x + S_\varepsilon(t^{\frac{1}{\varepsilon}})H_\varepsilon x - S_\varepsilon(0)H_\varepsilon x \right\|$$

$$\leq \left\| S_\varepsilon(t^{\frac{1}{\varepsilon}}) \right\| \left\| \frac{N_\varepsilon(t^{\frac{1}{\varepsilon}})}{t}x - S_\varepsilon(t^{\frac{1}{\varepsilon}})H_\varepsilon x \right\| + \left\| S_\varepsilon(t^{\frac{1}{\varepsilon}})H_\varepsilon x - S_\varepsilon(0)H_\varepsilon x \right\|$$

By the first and the fifty properties in 5.5 as well as by the continuity of $t \mapsto S_\varepsilon(t)(H_\varepsilon x)$ as zero, it follows that the last expression tend to zero as $t \to 0$.

Similarly, we have
\[ \| \frac{N_\varepsilon(t^{\frac{1}{\alpha}})S_\varepsilon(t^{\frac{1}{\alpha}})}{t} x - H_\varepsilon S_\varepsilon(0)x \| = \| \frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t} S_\varepsilon(t^{\frac{1}{\alpha}}) x - \frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t} S_\varepsilon(0)x \| \\
+ \frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t} S_\varepsilon(0)x - H_\varepsilon S_\varepsilon(0)x \|
\]

Thus

\[ \| S_\varepsilon - \tilde{S}_\varepsilon \| \leq \| S_\varepsilon(t^{\frac{1}{\alpha}})x - S_\varepsilon(0)x \| + \| S_\varepsilon(0)x - H_\varepsilon S_\varepsilon(0)x \| \]

Assumptions (4), (5) and (2) imply that the last expression tends to zero as \( t \to 0 \). Thus (5) is proved in both cases. \( \square \)

Now we define Colombeau type algebra as the factor algebra:

\[ SG(\mathbb{R}_+: L(X)) = S'EM(\mathbb{R}_+: L(X)) / S\mathcal{N}(\mathbb{R}_+: L(X)) \]

Elements of \( SG(\mathbb{R}_+: L(X)) \) will be denoted by \( S = [S_\varepsilon] \), where \( (S_\varepsilon)_\varepsilon \) is a representative of the above class.

**Definition 5.6.** \( S \in SG(\mathbb{R}_+: L(X)) \) is a called a Colombeau \( C_0 \)-Semigroup if it has a representative \( (S_\varepsilon)_\varepsilon \) such that, for some \( \varepsilon_0 > 0 \), \( S_\varepsilon \) is a \( C_0 \)-Semigroup, for every \( \varepsilon < \varepsilon_0 \).

**Example 5.1.** We take \( G = G(\mathbb{R}^+) \), and we define \( T_\varepsilon(t)u_\varepsilon(x) = u_\varepsilon(x + \varepsilon^\alpha) \). Then

\[ T(t) = [T_\varepsilon(t)]_\varepsilon \]

define a conformable semigroup on \( G \).

In the sequel we will use only representatives \( (S_\varepsilon)_\varepsilon \) of a Colombeau \( C_0 \)-semigroup \( S \) which are \( C_0 \)-semigroups, for \( \varepsilon \) small enough.

**Proposition 5.3.** Let \( (S_\varepsilon)_\varepsilon \) and \( (\tilde{S}_\varepsilon)_\varepsilon \) be representatives of a Colombeau \( C_0 \)-semigroup \( S \), with the infinitesimal generators \( A_\varepsilon \), \( \varepsilon < \varepsilon_0 \), and \( \tilde{A}_\varepsilon \), \( \varepsilon < \varepsilon_0 \), respectively, where \( \varepsilon_0 \) and \( \varepsilon_0 \) correspond (in the sense of definition 5.6) to \( (S_\varepsilon)_\varepsilon \) and \( (\tilde{S}_\varepsilon)_\varepsilon \) respectively.

Then, \( D(A_\varepsilon) = D(\tilde{A}_\varepsilon) \), for every \( \varepsilon < \varepsilon = \min \{ \varepsilon_0, \varepsilon_0 \} \) and \( A_\varepsilon - \tilde{A}_\varepsilon \) can be extended to an element of \( L(X) \), denoted again by \( A_\varepsilon - \tilde{A}_\varepsilon \). Moreover, for every \( a \in \mathbb{R} \),

\[ \| A_\varepsilon - \tilde{A}_\varepsilon \| = O_{\varepsilon \to 0}(\varepsilon^\alpha), \] (5.6)

**Proof.** Denote \( N_\varepsilon(S_\varepsilon - \tilde{S}_\varepsilon)_\varepsilon \in S\mathcal{N}(\mathbb{R}_+, L(X)) \).

Let \( \varepsilon < \varepsilon_0 \) be fixed and \( x \in X \). We have

\[ \frac{S_\varepsilon(t^{\frac{1}{\alpha}})x - x}{t} - \frac{\tilde{S}_\varepsilon(t^{\frac{1}{\alpha}})x - x}{t} = \frac{N_\varepsilon(t^{\frac{1}{\alpha}})x}{t} \]
This implies by letting $t \to 0$, that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$. Now we have
\[
\left( A_\varepsilon - (\tilde{A}_\varepsilon) \right) x = \lim_{t \to 0} \frac{S_\varepsilon(t^\frac{1}{\alpha})x - x}{t} \quad (5.7)
\]
\[
- \lim_{t \to 0} \frac{\tilde{S}_\varepsilon(t^\frac{1}{\alpha})x - x}{t} \quad (5.8)
\]
\[
= \lim_{t \to 0} \frac{N_\varepsilon(t^\frac{1}{\alpha})x}{t} = H_\varepsilon x, \quad x \in D(A_\varepsilon) \quad (5.9)
\]
since $D(A_\varepsilon)$ is dense in $X$, properties (4), (5) and (7) imply that for every $a \in \mathbb{R}$,
\[
\|A_\varepsilon - \tilde{A}_\varepsilon\| = O_{\varepsilon \to 0}(\varepsilon^\alpha). \quad \Box
\]

Now we define the infinitesimal generator of a Colombeau $C_0$-semigroup $S$. Denote by $\mathcal{A}$ the set of pairs $((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon)$ where $A_\varepsilon$ is a closed linear operator on $X$ with the dense domain $D(A_\varepsilon) \subset X$, for every $\varepsilon \in (0, 1)$. We introduce an equivalence relation in $A$
\[
\left( (A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon \right) \sim \left( (\tilde{A}_\varepsilon)_\varepsilon, (D(\tilde{A}_\varepsilon))_\varepsilon \right)
\]
if there exist $\varepsilon_0 \in (0, 1)$ such that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \varepsilon_0$, and for every $a \in \mathbb{R}$ there exist $C > 0$ and $\varepsilon_d \leq \varepsilon_0$ such that, for $x \in A(A_\varepsilon)$, $\|(A_\varepsilon - \tilde{A}_\varepsilon)x\| \leq C\varepsilon^\alpha \|x\|, x \in D(A_\varepsilon), \varepsilon \leq \varepsilon_d$.

Since $A_\varepsilon$ has a dense domain in $X$, $R_\varepsilon := A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to be an operator in $L_c(X)$ satisfying $\|(A_\varepsilon - \tilde{A}_\varepsilon)x\| = O_{\varepsilon \to 0}(\varepsilon^\alpha)$, for every $a \in \mathbb{R}$, such an operator $R_\varepsilon$ is called the zero operator.

We denote by $\bar{A}$ the corresponding element of the quotient space $\mathcal{A}/\sim$. Due to proposition 5.3, the following definition makes sense

**Definition 5.7.** $\bar{A} \in \mathcal{A}/\sim$ is the infinitesimal generator of a Colombeau $C_0$-semigroup $S$ if there exists a representative $(A_\varepsilon)_\varepsilon$ of $\bar{A}$ such that $A_\varepsilon$ is the infinitesimal generator of $S_\varepsilon$ for $\varepsilon$ small enough.

By Pazy we have the following proposition

**Proposition 5.4.** Let $S$ be a Colombeau $C_0$-semigroup with the infinitesimal generator $A$. Then there exists $\varepsilon_0 \in (0, 1)$ such that:
\begin{itemize}
  \item Mapping $t \mapsto S_\varepsilon(t^\frac{1}{\alpha})x : \mathbb{R}_+ \to X$ is continuous for every $x \in X$ and $\varepsilon < \varepsilon_0$
  \item \[
  \lim_{h \to 0} \int_t^{t+h} S_\varepsilon(s^\frac{1}{\alpha})xds_\alpha = S_\varepsilon(t^\frac{1}{\alpha})x, \quad \varepsilon < \varepsilon_0, \quad x \in X
  \]
  \item \[
  \int_0^t S_\varepsilon(s^\frac{1}{\alpha})xds_\alpha \in D(A_\varepsilon), \quad \varepsilon < \varepsilon_0, \quad x \in X
  \]
  \item For every $x \in D(A_\varepsilon)$ and $t \geq 0 S_\varepsilon(t^\frac{1}{\alpha})x \in D(A_\varepsilon)$ and
  \[
  \frac{d^\alpha}{dt^\alpha} S_\varepsilon(t^\frac{1}{\alpha})x = A_\varepsilon S_\varepsilon(t^\frac{1}{\alpha})x = S_\varepsilon(t^\frac{1}{\alpha})A_\varepsilon x, \quad \varepsilon < \varepsilon_0
  \]
\end{itemize}
Let \((S_ε)_ε, \tilde{S}_ε\) be representative of Colombeau \(C_0\)-semigroup \(S\), with infinitesimal generators \(A_ε, \tilde{A}_ε\), \(ε < ε_0\), respectively. Then, for every \(a \in \mathbb{R}\) and \(t \geq 0\)
\[
\frac{d^α}{dt^α}S_ε(t^\frac{1}{α}) - \tilde{A}_εS_ε(t^\frac{1}{α}) = O_{ε \to 0}(ε^a).
\]  
(5.11)

For every \(x \in D(A_ε)\) and every \(t, s \geq 0\),
\[
S_ε(t^\frac{1}{α})x - S_ε(s^\frac{1}{α})x = \int_s^t S_ε(τ^\frac{1}{α})A_εxdτ_{α} = \int_s^t A_εS_ε(τ^\frac{1}{α})xdτ_{α}
\]

Theorem 5.2. Let \(S\) and \(\tilde{S}\) be Colombeau \(C_0\)-semigroups with infinitesimal generators \(A\) and \(\tilde{A}\), respectively. If \(A = \tilde{A}\) then \(S = \tilde{S}\).

Proof. Let \(ε\) be small enough and \(x \in D(A_ε) = D(\tilde{A}_ε)\). Proposition 5.4 property 4 implies that for \(t \geq 0\), the mapping \(s \mapsto \tilde{S}_ε(t - s)S_ε(s)x\), \(t \geq s \geq 0\) is differentiable and
\[
\frac{d^α}{ds^α}(\tilde{S}_ε((t - s)^\frac{1}{α})S_ε(s^\frac{1}{α})x) = -\tilde{A}_ε\tilde{S}_ε((t - s)^\frac{1}{α})S_ε(s^\frac{1}{α})x + \tilde{S}_ε((t - s)^\frac{1}{α})A_εS_ε(s^\frac{1}{α})x, \quad t \geq s \geq 0.
\]
The assumption \(A = \tilde{A}\) implies that \(A_ε = \tilde{A}_ε + R_ε\), where \(R_ε\) is a zero operator. Since \(A_ε\) commutes with \(\tilde{S}_ε\), for every \(x \in D(A_ε)\)
\[
\frac{d^α}{ds^α}(\tilde{S}_ε((t - s)^\frac{1}{α})S_ε(s^\frac{1}{α})x) = S_ε((t - s)^\frac{1}{α})R_εS_ε(s^\frac{1}{α})x, \quad t \geq s \geq 0.
\]
and this implies
\[
\tilde{S}_ε((t - s)^\frac{1}{α})S_ε(s^\frac{1}{α})x - \tilde{S}_ε(t^\frac{1}{α})x = \int_0^s \tilde{S}_ε((t - u)^\frac{1}{α})R_εS_ε(u^\frac{1}{α})xdμ_{α}, \quad t \geq s \geq 0.
\]  
(5.12)

Putting \(s = t\) in (11), we obtain
\[
S_ε(t^\frac{1}{α}) - \tilde{S}_ε(t^\frac{1}{α})x = \int_0^t \tilde{S}_ε((t - u)^\frac{1}{α})R_εS_ε(u^\frac{1}{α})xdμ_{α}, \quad t \geq 0, \quad x \in D(A_ε).
\]  
(5.13)

Since \(D(A_ε)\) is dense in \(X\), uniform boundedness of \(S\) and \(\tilde{S}\) on \([0, t]\) implies that (11) holds for every \(y \in X\). Let us prove that \((N_ε)_ε = (S_ε - \tilde{S}_ε)_ε \in \mathcal{SN}(\mathbb{R}_+ : L_c(X))\).

The formula (12) and definition 5.4 imply that for some \(C > 0\) and \(a, \tilde{a} \in \mathbb{R}\),
\[
\sup_{t \in [0, T]} ||N_ε(t^\frac{1}{α})x|| \leq \sup_{t \in [0, T]} \int_0^t ||\tilde{S}_ε((t - u)^\frac{1}{α})||||R_ε||||S_ε(u^\frac{1}{α})||||x||dμ_{α} \leq TC^α||R_ε|| ||x||, \quad x \in X
\]
Since \(||R_ε|| = O_{ε \to 0}(ε^α)\), for every \(b \in \mathbb{R}, (N_ε(t))_ε\) satisfies condition (3) in definition 5.4. Condition (3) follows from the boundedness of \((\tilde{S}_ε)_ε, (S_ε)_ε\) on bounded domain \([0, t]\), the properties of \((R_ε)_ε\) and the following expression:
\[
\frac{N_ε(t^\frac{1}{α})}{t} = \frac{1}{t} \int_0^t \tilde{S}_ε((t - u)^\frac{1}{α})R_εS_ε(u^\frac{1}{α})xdμ_{α} \leq ||\tilde{S}_ε(t^\frac{1}{α})|| ||R_ε|| ||S_ε|| \leq \text{const}, \quad x \in X, \quad t \leq t_0,
\]
for some \( t_0 > 0 \). Also,

\[
\lim_{t \to 0} \frac{N(t^\frac{1}{\alpha})}{t} = \lim_{t \to 0} \frac{\tilde{S}(t^\frac{1}{\alpha})x - x}{t} - \lim_{t \to 0} \frac{S(t^\frac{2}{\alpha})x - x}{t} = R_x, \quad \forall x \in D(A_\varepsilon).
\]

Since it is enough that (5) holds for a dense subset of \( X \) see the remark after definition 5.5 this concludes the proof. \( \square \)

6. MAIN RESULT

This section deals with applications of Colombeau \( \alpha \)-\( C_0 \)-semigroups in solving a class of heat equations with singular potentials and singular data. We consider the problem

\[
\begin{aligned}
\partial_t^\alpha u(t, x) &= (\Delta - v(x)) u(t, x) \quad \text{in } \mathbb{R}, \ t \in \mathbb{R}^+
\end{aligned}
\]

\[
\begin{aligned}
u(0, t) &= u_0(x) = \delta(x) \\
v(x) &= \delta(x)
\end{aligned}
\]

Before we discuss the problem we will defined some spaces on which we will work. First we set \( \| \cdot \|_{L^2(\mathbb{R})} = \| \cdot \|_2 \)

Definition 6.1. We define \( H^{2, \alpha} \) by:

\[
H^{2, \alpha} = \{ f \in L^2(\mathbb{R}), \quad \tilde{D}^\alpha f \in L^2(\mathbb{R}) \}
\]

with the norm

\[
\| f \|_{H^{2, \alpha}} = \sqrt{\| f \|^2_2 + \| \tilde{D}^\alpha f \|^2_2}
\]

The Colombeau algebra type defined as follows

\[
G_{H^{2, \alpha}} = \mathcal{E}_M(H^{2, \alpha}) / \mathcal{N}(H^{2, \alpha})
\]

where \( \mathcal{E}_M(H^{2, \alpha}) \) the vector space of nets \( (G_\varepsilon)_\varepsilon \in H^{2, \alpha} \) with the property: for every \( T > 0 \) there exist \( a \in \mathbb{R} \) such that

\[
\| G_\varepsilon \|_{H^{2, \alpha}} = O(\varepsilon^a)
\]

and \( \mathcal{N}(H^{2, \alpha}) \) the vector space of nets \( (G_\varepsilon)_\varepsilon \in H^{2, \alpha} \) with the property: for every \( T > 0 \) for all \( b \in \mathbb{R} \) such that

\[
\| G_\varepsilon \|_{H^{2, \alpha}} = O(\varepsilon^b)
\]

is a Colombeau type vector space.

Definition 6.2. \( \mathcal{E}^{1, H^{2, \alpha}}([0, T], \mathbb{R}) \) (respectively \( \mathcal{N}^{1, H^{2, \alpha}}([0, T], \mathbb{R}) \), \( T > 0 \), is the vector space of nets \( (G_\varepsilon)_\varepsilon \) of functions

\[
G_\varepsilon \in C([0, T], H^{2, \alpha}) \cap C^1([0, T], L^2(\mathbb{R}))
\]

with the property: for every \( T_1 \in (0, T) \) there exists \( a \in \mathbb{R} \), (respectively, for every \( a \in \mathbb{R} \)) such that

\[
\max \left\{ \sup_{t \in [0, T_1]} \| G_\varepsilon \|_{H^{2, \alpha}}, \sup_{t \in [T_1, T]} \| \tilde{D}^\alpha G_\varepsilon \|_{L^2(\mathbb{R})} \right\} = O_{\varepsilon \to 0}(\varepsilon^a)
\]
The quotient space
$$\mathcal{G}_{C^1, \mathbb{H}^2, a}(0, T, \mathbb{R}) = \mathcal{E}_{C^1, \mathbb{H}^2, a}(0, T, \mathbb{R}) / \mathcal{N}_{C^1, \mathbb{H}^2, a}(0, T, \mathbb{R})$$
is a Colombeau type vector space.

Remark 6.1. The multiplication of potential \( V \in \mathcal{G}_{H^2,a} \) and \( u \in \mathcal{G}_{C^1, \mathbb{H}^2, a}(0, T, \mathbb{R}) \)
which is expected to be a solution to equation
$$\begin{aligned}
\partial_t^a u(t, x) &= (\Delta - V(x)) u(t, x) \quad \forall t \in \mathbb{R}^+, \ t \in \mathbb{R}^+
\quad u(0, t) = u_0(x) = \delta(x) \\
v(x) &= \delta(x)
\end{aligned}$$
makes sense.

Definition 6.3. Let \( A \) be represented by a net \((A_\varepsilon)\varepsilon\) of linear operators with the
common domain \( H^{2, \alpha}(\mathbb{R}) \) and with ranges in \( L^2(\mathbb{R}) \). A generalized function \( G \in \mathcal{G}_{C^1, \mathbb{H}^2, a} \) \( T > 0 \), is said to be a solution to equation \( D^a G = AG \) if
$$\sup_{t \in [0, T]} ||D^a G_\varepsilon(t, \cdot) - A_\varepsilon G_\varepsilon(t, \cdot)||_2 = O_{\varepsilon \to 0}(\varepsilon\varepsilon), \ \forall \varepsilon \in \mathbb{R}.$$  

Definition 6.4. An element \( V \in \mathcal{G}_{H^2,a} \) is of logarithmic type if it has a representative
\((V_\varepsilon)\varepsilon \in \mathcal{E}_{C^1, \mathbb{H}^2, a}\) with the property
$$||V_\varepsilon||_{H^2,a} = O_{\varepsilon \to 0}(\ln \varepsilon^{-1})$$

An element \( V \in \mathcal{G}_{H^2,a} \) is said to be of log-log type if it has a representative \((V_\varepsilon)\varepsilon \in \mathcal{E}_{C^1, \mathbb{H}^2, a}\) with the property
$$||V_\varepsilon||_{H^2,a} = O_{\varepsilon \to 0}(\ln a \ln \varepsilon^{-1})$$

We set
$$E(t, x) = \frac{1}{2\sqrt{\pi t}} \exp \frac{|x|^2}{4t}$$

Theorem 6.1. Let \( V \in \mathcal{G}_{H^2,a} \) be of logarithmic type.

1. Derivation of \( A_\varepsilon u = (\Delta - V) u, u \in H^{2, \alpha}(\mathbb{R}) \), are infinitesimal generators of
conformable semigroups \( T_\varepsilon \) for every \( \varepsilon > 0 \), and \((T_\varepsilon)\varepsilon \) is a representative of a
Colombeau conformable \( C_0 \)-semigroup
$$T(t) = S G(\mathbb{R}^+, L (L^2(\mathbb{R})))$$

2. Let \( V \in \mathcal{G}_{H^2,a} \) and let \( T_\varepsilon \) be as in 1.
Then, for every \( T > 0 \), the problem 6.1 has unique solution in \( \mathcal{G}_{H^2,a}(\mathbb{R}) \).

Proof.

1. Let \( \varepsilon > 0 \) small enough. By the Feynman-Kac formula the operator \( A_\varepsilon \) is the
infinitesimal generator of the corresponding semigroup
$$T_\varepsilon(t) \psi(x) = \int_\Omega \left( \exp \left( - \int_0^\varepsilon V_\varepsilon(\omega(s)) ds \right) \right) \psi(\omega(t^\varepsilon)) d\mu_\varepsilon(\omega)$$
for $\psi \in L^2(\mathbb{R})$, where $\Omega = \prod_{t \geq 0} \mathbb{R}$ and $\mu_\alpha$ is is the Wiener measure concentrated at $x \in \mathbb{R}$.

Since $V$ is of logarithmic type, there exist $C > 0$ and $\eta \in (0, 1)$ such that

$$|T_\epsilon(t)\psi(x)| \leq \exp\left( t^\alpha \sup_{s \in \mathbb{R}} |V_\epsilon(s)| \right) \int_\mathbb{R} |\psi(\omega(t))| d\mu_\alpha(\omega)$$

$$\leq e^{Ct^\alpha} \frac{1}{2\sqrt{\pi}t^\alpha} \int_\mathbb{R} \exp -\frac{|x-y|^2}{4t^\alpha} |\psi(y)| dy$$

Therefore, there exist $C_0 > 0$ such that

$$\sup_{t \in (0,T]} ||T_\epsilon(t)\psi(x)||_2 \leq C_0 e^{Ct^\alpha} ||\psi||_2$$

Thus $\{T_\epsilon\}_\epsilon \in SG\left( \mathbb{R}^+, \mathcal{L}(L^2(\mathbb{R})) \right)$

(2) (a) By the Duhamel principle, solution $u_\epsilon(t, x)$ to equation (6.1) satisfies

$$u_\epsilon(t, x) = \int_\mathbb{R} E(t^\alpha, x-y)b_\epsilon(y)dy$$

$$+ \int_0^t \int_\mathbb{R} E((t-s)^\alpha, x-y)V_\epsilon(y)u_\epsilon(s, y)dy$$

(6.2)

Young’s inequality implies

$$||u_\epsilon(t, .)||_2 \leq ||b_\epsilon||_2 + \int_0^t ||V_\epsilon(.)||_2^2 ||u_\epsilon(s, .)||_2^2 ds$$

Gronwall’s inequality gives

$$||u_\epsilon(t, .)||_2 \leq ||b_\epsilon||_2 \exp \int_0^t ||V_\epsilon(.)||_2^2, \forall t \in (0,T]$$

Since $V \in G H^{2,\alpha}(\mathbb{R})$ is of logarithmic type and $(u_{0\epsilon})_\epsilon \in \mathcal{E} H^{2,\alpha}(\mathbb{R})$, it follows that $\sup_{t \in (0,T]} ||u_\epsilon(t, .)||_2$ has a moderate bound. Differentiation of equation 6.2 with respect to some spatial variable $x$ gives

$$\frac{d}{dx}u_\epsilon(t, x) = \int_\mathbb{R} E(t^\alpha, y)\frac{d}{dx}b_\epsilon(x-y)dy$$

$$+ \int_0^t \int_\mathbb{R} E((t-s)^\alpha, y)\frac{d}{dx}V(y)u_\epsilon(s, x-y)$$

$$+ V(x-y)\frac{d}{dx}u_\epsilon(s, x-y)dy ds$$

then

$$||\frac{d}{dx}u_\epsilon(t, .)||_2 \leq ||\frac{d}{dx}b||_2 + \int_0^t ||\frac{d}{dx}V(.)||_{L^\infty} ||u_\epsilon(t, .)||_2 + ||V(.)||_{L^\infty} ||u_\epsilon(t, .)||_2$$

Also Gronwall’s inequality implies that $\sup_{t \in (0,T]} ||u_\epsilon(t, .)||_2$ is moderate.

So $\{u_\epsilon\}_\epsilon \in \mathcal{E} C^{1, H^{2,\alpha}}(\mathbb{R})$. 
For uniqueness: Let $u_\varepsilon$ and $v_\varepsilon$ two solutions of (6.1). We set $G_\varepsilon = u_\varepsilon - v_\varepsilon$, we get

$$G_\varepsilon(t,x) = \int_\mathbb{R} E(t^\alpha, x-y) N_\varepsilon(y) dy$$

$$+ \int_0^t \int_\mathbb{R} E((t-s)^\alpha, x-y) V_\varepsilon(y) G_\varepsilon(s,y) dy$$

$$+ \int_0^t \int_\mathbb{R} E((t-s)^\alpha, x-y) N_\varepsilon(y) dy$$

where $N_\varepsilon(x) = G(0, x)$, and $N_\varepsilon = \frac{d^\alpha}{dt^\alpha} G_\varepsilon - (\Delta - V) G_\varepsilon$

Then Young's and Gronwall's inequalities imply

$$\|G_\varepsilon(t, .)\|_2 \leq \|N\|_2 + \int_0^t \|V_\varepsilon(s, .)\|_{L^\infty} \|G_\varepsilon(s, .)\|_2 ds + \int_0^t \|N_\varepsilon(s, .)\|_2 ds$$

thus $G_\varepsilon \in \mathcal{N}H^{2, \alpha}$. □

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