SARAJEVO JOURNAL OF MATHEMATICS
Vol.15 (28), No.2, (2019), 265 – 281
DOI: 10.5644/SJM.15.02.10

GENERALIZED HEAT EQUATION UNDER CONFORM DERIVATIVE

M'HAMED ELOMARI, SAID MELLIANI, AND LALLA SAADIA CHADLI

ABSTRACT. In the present work, we establish the existence and uniqueness result of the linear heat equation with Conform derivative in Colombeau generalized algebra. We using for the first time the notion of a generalized conformable semigroup and the purpose of introducing Conform derivative is regularizing it in Colombeau.

1. INTRODUCTION

In this paper the heat equation is considered with the first order time derivative changed to a conform derivative. for each type of data we always ask what is the optimal corresponding nonlocal model to be applied.Moreover, many authors studied Conform operators with local, nonlocal, singular and non-singular kernels [2]. The Riemann and Caputo fractional calculus may not provide us the required kernel in order to extract important information from these types of systems. At this stage, we ask the following question. Can we generalize the standard fractional Riemann-Liouville integrals in a way such that we obtain unification to Riemann-Liouville, Hadamard and other fractional derivatives. The importance of this procedure is to decide which differentiation operator should be used as a starting point for the iteration procedure. For the standard fractional calculus, we iterate the usual integral of a function and using the Cauchy formula we obtain the integral of higher integer orders and then replace this integer by any real number. In [1] the autor presented and developped the definitions of Conform derivative and set the basic concepts in this new simple interesting fractional calculus. The fractional versions of chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions, Laplace transforms and linear differential systems are proposed and discussed. The present paper concerns the study of existence and uniqueness to equation with Conform differentiation in extended Colombeau algebra. We consider Conform differentiation for indicating to existence and uniqueness heat equation in extended Colombeau algebra. The

2010 *Mathematics Subject Classification.* 34xx.
Key words and phrases. Conform derivative, Colombeau algebra, generalized conformable semigroup, generalized solution.

reason for introducing conform derivatives into algebra of generalized functions was the possibility of solving nonlinear problems with singularities and derivatives of arbitrary real order in it. We use specific space of Colombeau algebra type in order to give a sense of our problem. Colombeau algebras is a differential algebra, commutative, associative in which we can inject D' the set of distributions so that the product of smooth functions and the usual derivative of distributions are respected [5], [6], [7]. From the previous ideas we will discuss the existence and uniqueness of such equation in a specific spaces coincide with the usual spaces when $\alpha \to 1$.
The paper is organized as follows: After this introduction, we present some concepts concerning the Colombeau's algebra. In section 3 we give the definitions and we prove some properties concerning the Conform derivative. The embedding of this derivation in Colombeau algebra takes place in section 4. In the last section we discuss the existence and uniqueness of our problem.

2. PRELIMINAIRIES

We shall fix the notation and introduce a number of known as well as new classes of generalized functions here. For more details, see [7].
Let Ω be an open subset of $\mathbb{R}^n$. The basic objects of the theory as we use it are families $(u_\varepsilon)_{\varepsilon\in(0;1]}$ of smooth functions $u_\varepsilon \in \mathcal{C}^\infty(\Omega)$ for $0 < \varepsilon \leq 1$. We single out the following subalgebra.
Moderate families, denoted by $\mathcal{E}_M(\Omega)$, are defined by the property :

$$\forall K \subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \exists p \geq 0 : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O_{\varepsilon\to 0}(\varepsilon^{-p})$$

Null families, denoted by $\mathcal{E}_M(\Omega)$, are defined by the property :

$$\forall K \subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \forall q \geq 0 : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O_{\varepsilon\to 0}(\varepsilon^{q})$$

Thus moderate families satisfy a locally uniform polynomial estimate as $\varepsilon \mapsto 0$, together with all derivatives, while null functions vanish faster than any power of ε in the same situation. The null families from a differential ideal in the collection of moderate families.
The Colombeau algebra is the factor algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$$

The algebra $\mathcal{G}(\Omega)$ just defined coincides with the special Colombeau algebra in [5], where the notation $\mathcal{G}^s(\Omega)$ has been employed. It was called the simplified Colombeau algebra in [5].
The Colombeau algebra on a closed half space $\mathbb{R}^n \times [0,1)$ is defined in a similary way. The restriction of an element $u \in \mathcal{G}(\mathbb{R} \times [0,1))$ to the line $\{t = 0\}$ is defined on representatives by

$$u/\{t = 0\} = \textit{Class of } \left(u_\varepsilon(\cdot,0)\right)_{\varepsilon\in(0,1]}$$

Similarly, restrictions of the elements of $\mathcal{G}(\Omega)$ to open subsets of Ω are defined on representatives. One can see that $\Omega \longrightarrow \mathcal{G}(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$. The space of compactly supported distributions is imbedded in $\mathcal{G}(\Omega)$ by convolution:

$$i : \begin{cases} \mathcal{E}^{'}(\Omega) \longrightarrow \mathcal{G}(\Omega) \\ \omega \to i(\omega) = \ \textit{class of} \ \Big(\omega * (\phi_\varepsilon)/\Omega\Big)_{\varepsilon\in(0,1]} \end{cases}$$

where

$$\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$$

is obtained by scaling a fixed test function $\mathcal{S}(\mathbb{R}^n)$ of integral one with all moments vanishing. By the sheaf property, this can be extended in a unique way to an imbedding of the space of distributions $\mathcal{D}(\Omega)$.

One of the main features of the Colombeau construction is the fact that this imbedding renders $\mathcal{C}^\infty(\Omega)$ a faithful subalgebra. In fact, given $f \in \mathcal{C}^\infty(\Omega)$, one can define a corresponding element of $\mathcal{G}(\Omega)$ by the constant imbedding $\sigma(f) =$ *class of* $[(\varepsilon, x) \longrightarrow f(x)]$. Then the important equality $i(f) = \sigma(f)$ holds in $\mathcal{G}(\Omega)$.

If $u \in \mathcal{G}(\Omega)$ and f is a smooth function which is of at most polynomial growth at infinity, together with all its derivatives, the superposition $f(u)$ is a well-defined element of $\mathcal{G}(\Omega)$.

We need a couple of further notions from the theory of Colombeau generalized functions. An element u of $\mathcal{G}(\Omega)$ is called of local L^p-type $(1 \leq p \leq \infty)$, if it has a representative with the property $\lim\limits_{\varepsilon\longrightarrow 0} \| u_\varepsilon \|_{L^p(K)}$ exists for every $K \subset \Omega$.

Regularity theory is based on the subalgebra $\mathcal{G}^\infty(\Omega)$ of regular generalized functions in $\mathcal{G}(\Omega)$. It is defined by those elements which have a representative satisfying

$$\forall K \subset \Omega, \forall \alpha \in \mathbb{N}_0^n, \exists p \geq 0 : \sup_{x\in K} \left|\partial^\alpha u_\varepsilon(x)\right| = O_{\varepsilon\to 0}(\varepsilon^{-p})$$

Observe the change of quantifiers with respect to formula 2; locally, all derivatives of a regular generalized function have the same order of growth in $\varepsilon > 0$. One has that (see [6]).

$$\mathcal{G}^\infty(\Omega) \cap \mathcal{D}^{'}(\Omega) = \mathcal{C}^\infty(\Omega)$$

For the purpose of describing the regularity of Colombeau generalized functions, $\mathcal{G}^\infty(\Omega)$ plays the same role as $\mathcal{C}^\infty(\Omega)$ does in the setting of distributions.

A net$(r_\varepsilon)_{\varepsilon\in(0,1]}$ of complex numbers is called a slow scale net if

$$|r_\varepsilon| = O_{\varepsilon\to 0}(\varepsilon^{-p})$$

for every $p \geq 0$. We refer to [6] for a detailed discussion of slow scale nets. Finally, an element $u \in \mathcal{G}(\Omega)$ is called of total slow scale type, if for some representative, $\| \partial^\alpha u_\varepsilon \|_{L^\infty(K)}$ forms a slow scale net for every $K \subset \Omega$ and $\alpha \in \mathbb{N}_0^n$.

We end this section by recalling the association relation on the Colombeau algebra $\mathcal{G}(\Omega)$. It identifies elements of $u \in \mathcal{G}(\Omega)$ if they coincide in the weak limit. That is, $u,v \in \mathcal{G}(\Omega)$ are called associated,

$$u \approx v, \text{ if } \lim_{\varepsilon \longrightarrow 0} \int (u_\varepsilon(x) - v_\varepsilon(x))\psi(x)dx = 0$$

3. Conformable derivative

In this section we will give some definition and properties concerning the new derivative important in the following.

Definition 3.1. *[11] Let $\alpha \in (n,n+1]$ and $f : [0,\infty) \to \mathbb{R}$ be n-differentiable at $t > 0$, then the conformable derivative of f of order α is defined by*

$$f^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f^{(n)}(t + \varepsilon t^{n+1-\alpha}) - f^{(n)}(t)}{\varepsilon}$$
$$f^{(\alpha)}(0) = \lim_{t \to 0} f^{(\alpha)}(t)$$

Remark 3.1. [11] As consequence of the previous definition, one can easily show that

$$f^{(\alpha)}(t) = t^{n+1-\alpha} f^{(n+1)}(t)$$

where $\alpha \in (n,n+1]$, and f is $(n+1)$-differentiable at $t > 0$.

In [2] we find the following proposition.

Proposition 3.1. *[2] Let f,g two function α-derivatives and $a,b \in \mathbb{R}$. We have the following properties.*

(1) $(af + bg)^{(\alpha)} = af^{(\alpha)} + bg^{(\alpha)}$,
(2) $(fg)^{(\alpha)} = f^{(\alpha)}g + fg^{(\alpha)}$,
(3) $(t^p)^{(\alpha)} = pt^{p-\alpha}$,
(4) $\left(\frac{f}{g}\right)^{(\alpha)} = \frac{f^{(\alpha)}g - fg^{(\alpha)}}{g^2}$,
(5) *If* $c \in \mathbb{R}$, $c^{(\alpha)} = 0$.

Proposition 3.2. *If x is a continuous map, then $t \to x^{(\alpha)}(t)$ is a continuous map.*

Proof. Since x is a continuous map $t \in \mathbb{R}_+^* \to x\left(t + \varepsilon t^{1-\alpha}\right)$ is a continuous, thus $\forall\beta > 0$, $\exists\alpha > 0$,

$$\left|\frac{x\left(t + \varepsilon t^{1-\alpha}\right) - x\left(t_0 + \varepsilon t_0^{1-\alpha}\right)}{\varepsilon}\right| \leq \beta$$

whenever $|t - t_0| \leq \alpha$, by passing to limite $\varepsilon \to 0$ we get $\left|x^{(\alpha)}(t) - x^{(\alpha)}(t_0)\right| \leq \beta$ as desired. □

Proposition 3.3. *Let $f : X \to X$, be a Lipschitz map. i.e.*
$|f(x) - f(y)| \leq k|x - y|$, $\forall x,y \in X$ *and* $k \in]0,1[$. *The problem of Cauchy*

$$\begin{cases} x^{(\alpha)}(t) = f(x(t)), & t > 0 \\ x(0) = x_0 \end{cases}$$

has a unique solution.

Proof. By 5.3 x is continuous, the sequence $x_{n+1} = f(x_n)$ is a Cauchy's sequence, since $\mathbb{R}$ is a complet space then x_n converge to the unique solution of (3.3) $\square$

Definition 3.2. *[11] Let $\alpha \in (n, n+1]$. We define the α-integral by:*

$$(I^\alpha f)(t) = \int_0^t s^{\alpha - n} f(s) ds.$$

According to the previous writing we set

$$dx_\alpha = \frac{dx}{x^{\alpha-1}}$$

Theorem 3.1. *[11] We have the following inequality*

$$(I^\alpha f)^{(\alpha)}(t) = f(t).$$

for $t \geq 0$

Example 3.1. *It is easy to prove that:*

$$I^\alpha(\sin(t)) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+\alpha}}{(2n+\alpha)(2n+1)!}$$

where $\alpha \in (1,2)$

Definition 3.3. *[9] Let $\alpha \in (0,1)$. For a Banach space X.*
A family $\{T(t)\}_{t \geq 0} \subset \mathcal{L}(X,X)$ is called a conform α-semigroup if:

(1) $T(0) = I$
(2) $T\left((s+t)^{\frac{1}{\alpha}}\right) = T\left(s^{\frac{1}{\alpha}}\right) T\left(t^{\frac{1}{\alpha}}\right)$*, for all $s, t \in [0, \infty)$*

Example 3.2. *Let A be a bounded linear operator on X. Define $T(t) = e^{2\sqrt{t}A}$. Then $\left\{T(t)\right\}_{t\geq 0}$ is a $\frac{1}{2}$-semigroup. Indeed:*

(1) $T(0) = e^{0A} = I$
(2) $\forall s, t \in [0, \infty)$, $T\left((s+t)^2\right) = e^{2(t+s)A} = e^{2tA} e^{2sA} = T(s)T(t)$

Definition 3.4. *[9] An α-semigroup $T(t)$ is called a C_0-semigroup if, for each fixed $x \in X$, $T(t)x \to x$ as $t \to 0^+$*

The conformable α-derivative of $T(t)$ at $t = 0$ is called the α-infinitesimal generator of the conform α-semigroup $T(t)$, with domain equals

$$\left\{ x \in X,\ \lim_{t \to 0} T(t)x \text{ exist } \right\}$$

In the sequel $\alpha \in (0,1)$.

4. Imbedding of the conformable differentiation into extended Colombeau algebra of generalized functions

Let $u_\varepsilon(x)$ represents a Colombeau generalized function $u \in \mathcal{G}^e(_R)$: The conformable derivative for $0 < \alpha < 1$ is defined by:

$$\mathcal{D}^\alpha u_\varepsilon(x) = x^{1-\alpha} u_\varepsilon'(x)$$

we have

$$| \mathcal{D}^{\alpha} u_{\varepsilon}(x) | = | x^{1-\alpha} u_{\varepsilon}^{'}(x) |$$

$$| \mathcal{D}^{\alpha} u_{\varepsilon}(x) | \leq | x^{1-\alpha} | | u_{\varepsilon}^{'}(x) |$$

$$\sup_{x \in K} | \mathcal{D}^{\alpha} u_{\varepsilon}(x) | \leq | a^{1-\alpha} | \sup_{x \in K} | u_{\varepsilon}^{'}(x) |$$

so,

$$\sup_{x \in K} | \mathcal{D}^{\alpha} u_{\varepsilon}(x) | \leq C_1 \, \varepsilon^{p}$$

we use the regularization for $1 < \alpha < 1$

$$\tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) = \int_{\mathbb{R}} \mathcal{D}^{\alpha} u_{\varepsilon}(s) \phi_{\varepsilon}(x-s) ds$$

The convolution form is given by:

$$\tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) = \mathcal{D}^{\alpha} u_{\varepsilon} * \phi_{\varepsilon}(x)$$

we indicate that $| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) - \mathcal{D}^{\alpha} u_{\varepsilon}(x) | \approx 0$

$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) - \mathcal{D}^{\alpha} u_{\varepsilon}(x) | = | \mathcal{D}^{\alpha} u_{\varepsilon} * \phi_{\varepsilon}(x) - \mathcal{D}^{\alpha} u_{\varepsilon}(x) |$$

$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) - \mathcal{D}^{\alpha} u_{\varepsilon}(x) | = | \mathcal{D}^{\alpha} u_{\varepsilon} * \phi_{\varepsilon}(x) - \mathcal{D}^{\alpha} u_{\varepsilon} * \delta(x) |$$

$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) - \mathcal{D}^{\alpha} u_{\varepsilon}(x) | = | \mathcal{D}^{\alpha} u_{\varepsilon} * (\phi_{\varepsilon}(x) - \delta(x)) |$$

$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) - \mathcal{D}^{\alpha} u_{\varepsilon}(x) | = | \int_{\mathbb{R}} \mathcal{D}^{\alpha} u_{\varepsilon}(x-s) \, (\phi_{\varepsilon}(s) - \delta(s)) ds |$$

$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) - \mathcal{D}^{\alpha} u_{\varepsilon}(x) | = \int_{\mathbb{R}} | \mathcal{D}^{\alpha} u_{\varepsilon}(x-s) \, | \, | \, \phi_{\varepsilon}(s) - \delta(s) \, | \, ds \longrightarrow 0$$

as $\varepsilon \longrightarrow 0$. Since $\lim | \phi_{\varepsilon}(s) - \delta(s) |$, then

$$\tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) \approx \mathcal{D}^{\alpha} u_{\varepsilon}(x)$$

Using the fact that $\phi_{\varepsilon}(x)$ has the compact support on K_0, so by Holder inequalities, have the following calculations:

we have

$$\tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) = \mathcal{D}^{\alpha} u_{\varepsilon} * \phi_{\varepsilon}(x) = \int_{\mathbb{R}} \mathcal{D}^{\alpha} u_{\varepsilon}(x-s) \phi_{\varepsilon}(s) ds$$

$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) | = | \int_{\mathbb{R}} \mathcal{D}^{\alpha} u_{\varepsilon}(x-s) \phi_{\varepsilon}(s) ds | = | \int_{K_0} \mathcal{D}^{\alpha} u_{\varepsilon}(x-s) \phi_{\varepsilon}(s) ds |$$

$$| \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) | = \int_{K_0} | \mathcal{D}^{\alpha} u_{\varepsilon}(x-s) \, | \, | \, \phi_{\varepsilon}(s) \, | \, ds$$

$$\sup_{x \in K} | \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) | = \sup_{x \in K} \{ \int_{K_0} | \mathcal{D}^{\alpha} u_{\varepsilon}(x-s) \, | \, | \, \phi_{\varepsilon}(s) \, | \, ds \}$$

so,

$$\sup_{x \in K} | \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) | \leq \sup_{x \in K_1} | \mathcal{D}^{\alpha} u_{\varepsilon}(x) | \int_{K_0} | \phi_{\varepsilon}(s) | \, ds$$

$$\sup_{x \in K} | \tilde{\mathcal{D}}^{\alpha} u_{\varepsilon}(x) | \leq C_1 \, \varepsilon^{p}$$

and

$$\frac{d}{dx}(\tilde{\mathcal{D}}^{\alpha}u_{\varepsilon}(x)) = \frac{d}{dx}(\mathcal{D}^{\alpha}u_{\varepsilon}) * \phi_{\varepsilon}(x) = \mathcal{D}^{\alpha}u_{\varepsilon} * \frac{d}{dx}(\phi_{\varepsilon}(x))$$

then,

$$\sup_{x\in K} \mid \frac{d}{dx}(\tilde{\mathcal{D}}^{\alpha}u_{\varepsilon}(x)) \mid \leq \sup_{x\in K_1} \mid \mathcal{D}^{\alpha}u_{\varepsilon}(x) \mid \int_{K_0} \mid \frac{d}{dx}(\phi_{\varepsilon}(s)) \mid ds \leq C_2\, \varepsilon^{p}$$

In order to prove moderateness for higher derivatives a similar calculation is applied.

$$\sup_{x\in K} \mid \partial^{n}\tilde{\mathcal{D}}^{\alpha}u_{\varepsilon}(x) \mid \leq C_3\, \varepsilon^{p}$$

5. Generalized conformable semigroup

We define

$$\mathcal{E}_M(\mathbb{R}) := \left\{ (x_\varepsilon)_\varepsilon \in (\mathbb{R})^{(0,1)} / \exists m \in \mathbb{N}, |x_\varepsilon| = O_{\varepsilon\to 0}(\varepsilon^{-m}) \right\}$$

and

$$\mathcal{N}(\mathbb{R}) := \left\{ (x_\varepsilon)_\varepsilon \in (\mathbb{R})^{(0,1)} / \forall m \in \mathbb{N}, |x_\varepsilon| = O_{\varepsilon\to 0}(\varepsilon^{m}) \right\}$$

It is easy to prove that

Proposition 5.1. *The space $\mathcal{E}_M(\mathbb{R})$ is an algebra and $\mathcal{N}(\mathbb{R})$ ideal in $\mathcal{E}_M(\mathbb{R})$*

Definition 5.1. *We define the Colombeau algebra type by:*

$$\tilde{\mathbb{R}} = \mathcal{E}_M(\mathbb{R})/\mathcal{N}(\mathbb{R})$$

5.1. Locally convex and complete spaces

Definition 5.2. *: Let X be a vector space with a seminorms familly $(p_i)_{i\in I}$. If τ_i is the topology defined by the only semi-norm p_i. If τ is the super bound of topology τ_i. The spece provided with this topology τ is called a locally convex space*

A basis of 0-neighbourhood is the set of all "balls" of the seminorms $(p_i)_{i\in I}$

$$B(i,r) = \left\{ x \in X / \quad p_i(x) < r \right\}, \quad \forall i \in I \text{ and } r > 0.$$

Then, $(x_n)n \in \mathbb{N}$ is a Cauchy sequence iff

$$\big(\forall \varepsilon > 0\big)\big(\forall i \in I\big)\big(\exists n_0 \in \mathbb{N}\big)\big(\forall n, p \in \mathbb{N} \text{ if } n \geq n_0 \Rightarrow p_i(x_{n+p} - x_n) < \varepsilon\big)$$

Definition 5.3. *We said that $\mathcal{D}$ is dense in locally convex space X iff*

$$\big(\forall x \in X\big) \quad \big(\exists y \in \mathcal{D}\big) \quad \big(\forall \varepsilon > 0\big) \quad \big(\forall i \in I\big) \quad \textit{we have} \quad p_i(x-y) < \varepsilon$$

and X is sequentially complete if any Cauchy sequence converges to an element e in X.

5.2. **Generalized semigroup**

Definition 5.4. *Let X be a locally convex space with a seminorm familly $(p_i)_{i\in I}$. We define*

$$\mathcal{E}_M(X) := \left\{(x_\varepsilon)_\varepsilon \in (X)^{(0,1)} / \exists m \in \mathbb{N}, \forall i \in I, p_i(x_\varepsilon) = O_{\varepsilon\to 0}(\varepsilon^{-m})\right\}$$

and

$$\mathcal{N}(X) := \left\{(x_\varepsilon)_\varepsilon \in (X)^{(0,1)} / \forall m \in \mathbb{N}, \forall i \in I, \quad p_i(x_\varepsilon) = O_{\varepsilon\to 0}(\varepsilon^{m})\right\}$$

We define the Colombeau algebra type by:

$$\tilde{X} = \mathcal{E}_M(X)/\mathcal{N}^s(X)$$

First, we are looking if it is possible to define a map $A : \tilde{X} \longrightarrow \tilde{X}$ by means of a given family $(A_\varepsilon)_{\varepsilon\in(0,1)}$ of maps $A_\varepsilon : X \longrightarrow X$ where A_ε is a linear and continuous operator. The general requirement is given in the following

Lemma 5.1. *: Let $(A_\varepsilon)_{\varepsilon\in(0,1)}$ be a given family of maps $A_\varepsilon : X \longrightarrow X$. For each $(x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$ and $(y_\varepsilon)_\varepsilon \in \mathcal{N}(X)$, suppose that*

(1) $\left(A_\varepsilon x_\varepsilon\right)_\varepsilon \in \mathcal{E}_M(X)$

(2) $\left(A_\varepsilon(x_\varepsilon + y_\varepsilon)\right)_\varepsilon - \left(A_\varepsilon x_\varepsilon\right)_\varepsilon \in \mathcal{N}(X)$.

Then

$$A : \begin{cases} \tilde{X} \longrightarrow \tilde{X} \\ x = \left[x_\varepsilon\right] \longmapsto Ax = \left[A_\varepsilon x_\varepsilon\right] \end{cases}$$

is well defined.

Proof. From the first property we see that the class $\left[(A_\varepsilon x_\varepsilon)_\varepsilon\right] \in \tilde{X}$
Let $x_\varepsilon + x_\varepsilon$ be another representative of $x = [x_\varepsilon]$
From the second property we have

$$\left(A_\varepsilon(x_\varepsilon + y_\varepsilon)\right)_\varepsilon - \left(A_\varepsilon x_\varepsilon\right)_\varepsilon \in \mathcal{N}(X)$$

and

$$\left[\left(A_\varepsilon(x_\varepsilon + y_\varepsilon)\right)_\varepsilon\right] = \left[\left(A_\varepsilon x_\varepsilon\right)\right)_\varepsilon\right] \textit{ in } \tilde{X}$$

Then A is well defined. □

Definition 5.5. *Let $\mathcal{SE}_M\left(\mathbb{R}_+ : \mathcal{L}_c(X)\right)$ is the space of nets $\left(S_\varepsilon\right)_\varepsilon$ of strongly continuous mappings $S_\varepsilon : \mathbb{R}_+ \longrightarrow \mathcal{L}_c(X), \quad \varepsilon \in (0,1)$ with the property that for every $T > 0$ there exists $a \in \mathbb{R}$ such that*

$$\sup_{t\in[0,T)} \|S_\varepsilon(t^{\frac{1}{\alpha}})\| = O_{\varepsilon\to 0}(\varepsilon^a), \tag{5.1}$$

and $\mathcal{SN}(\mathbb{R}_+ : \mathcal{L}_c(X))$ is the space of nets $(N_\varepsilon)_\varepsilon$ of strongly continuous mappings $N_\varepsilon : \mathbb{R}_+ \longrightarrow \mathcal{L}_c(X)$, $\varepsilon \in (0,1)$ with the properties:

For every $b \in \mathbb{R}$ and $T > 0$

$$\sup_{t\in[0,T)} \|N_\varepsilon(t^{\frac{1}{\alpha}})\| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^b), \tag{5.2}$$

There exist $t_0 > 0$ and $a \in \mathbb{R}$ such that

$$\sup_{t<t_0} \|\frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}\| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^a), \tag{5.3}$$

There exists a net $\left(H_\varepsilon\right)_\varepsilon$ in $\mathcal{L}_c(X)$ and $\varepsilon_0 \in (0,1)$ such that

$$\lim_{t\longrightarrow 0} \frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t} H_\varepsilon x,\ x \in X, \tag{5.4}$$

For every $b > 0$,

$$\|H_\varepsilon\| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^b), \tag{5.5}$$

Proposition 5.2. *$\mathcal{SE}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$ is algebra with respect to composition and $\mathcal{SN}\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$ is an ideal of $\mathcal{SE}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$*

Proof. Let $\left(S_\varepsilon(t^{\frac{1}{\alpha}})\right)_\varepsilon \in \mathcal{SE}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$ and $\left(N_\varepsilon(t^{\frac{1}{\alpha}})\right)_\varepsilon \in \mathcal{SN}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$ We will prove only the second assertion, i.e., That

$$\left(S_\varepsilon(t^{\frac{1}{\alpha}})N_\varepsilon(t^{\frac{1}{\alpha}})\right)_\varepsilon, \left(N_\varepsilon(t^{\frac{1}{\alpha}})S_\varepsilon(t^{\frac{1}{\alpha}})\right)_\varepsilon \in \mathcal{SN}_M\big(\mathbb{R}_+ : \mathcal{L}_c(X)\big)$$

where $S_\varepsilon(t^{\frac{1}{\alpha}})N_\varepsilon(t^{\frac{1}{\alpha}})$ denotes the composition.
Let $\varepsilon \in (0,1)$. By (2) and (3), for some $a \in \mathbb{R}$ and every $b \in \mathbb{R}$,

$$\|S_\varepsilon(t^{\frac{1}{\alpha}})N_\varepsilon(t^{\frac{1}{\alpha}})\| \le \|S_\varepsilon(t^{\frac{1}{\alpha}})\|\|N_\varepsilon(t^{\frac{1}{\alpha}})\| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{a+b}),$$

The same holds for $\|N_\varepsilon(t^{\frac{1}{\alpha}})S_\varepsilon(t^{\frac{1}{\alpha}})\|$. Further, (2) and (5) yield

$$\begin{aligned}\sup_{t<t_0}\|\frac{S_\varepsilon(t^{\frac{1}{\alpha}})N_\varepsilon(t^{\frac{1}{\alpha}})}{t}\| &\le \sup_{t<t_0}\|S_\varepsilon(t^{\frac{1}{\alpha}})\| \sup_{t<t_0}\|N_\varepsilon(t^{\frac{1}{\alpha}})\| \\ &= \mathcal{O}_{\varepsilon\to 0}(\varepsilon^a),\end{aligned}$$

for some $t_0 > 0$ and $a \in \mathbb{R}$. Also,

$$\sup_{t<t_0}\|\frac{S_\varepsilon(t^{\frac{1}{\alpha}})N_\varepsilon(t^{\frac{1}{\alpha}})}{t}\| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^a),$$

for some $t_0 > 0$ and $a \in \mathbb{R}$. Let now $\varepsilon \in (0,1)$ be fixed. We have

$$\begin{aligned}\|\frac{S_\varepsilon(t^{\frac{1}{\alpha}})N_\varepsilon(t^{\frac{1}{\alpha}})}{t}x - S_\varepsilon(0)H_\varepsilon x\| &= \|S_\varepsilon(t^{\frac{1}{\alpha}})\frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}x - S_\varepsilon(t^{\frac{1}{\alpha}})H_\varepsilon x + S_\varepsilon(t^{\frac{1}{\alpha}})H_\varepsilon x - S_\varepsilon(0)H_\varepsilon x\| \\ &\le \|S_\varepsilon(t^{\frac{1}{\alpha}})\|\|\frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}x - S_\varepsilon(t^{\frac{1}{\alpha}})H_\varepsilon x\| + \|S_\varepsilon(t^{\frac{1}{\alpha}})H_\varepsilon x - S_\varepsilon(0)H_\varepsilon x\|\end{aligned}$$

By the first and the fifty properties in 5.5 as well as by the continuity of $t \longrightarrow S_\varepsilon(t)(H_\varepsilon x)$ as zero, it follows that the last expression tend to zero as $t \mapsto 0$. Similarly, we have

$$\begin{aligned}
\|\frac{N_\varepsilon(t^{\frac{1}{\alpha}})S_\varepsilon(t^{\frac{1}{\alpha}})}{t}x - H_\varepsilon S_\varepsilon(0)x\| &= \|\frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}S_\varepsilon(t^{\frac{1}{\alpha}})x - \frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}S_\varepsilon(0)x \\
&\quad + \frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}S_\varepsilon(0)x - H_\varepsilon S_\varepsilon(0)x\| \\
&\leq \|\frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}\|\|S_\varepsilon(t^{\frac{1}{\alpha}})x - S_\varepsilon(0)x\| \\
&\quad + \|\frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}(S_\varepsilon(0)x) - H_\varepsilon(S_\varepsilon(0)x)\|
\end{aligned}$$

Assumptions (4), (5) and (2) imply that the last expression tends to zero as $t \mapsto 0$. Thus (5) is proved in both cases. □

Now we define Colombeau type algebra as the factor algebra:

$$\mathcal{SG}\big(\mathbb{R}_+ : \mathcal{L}(X)\big) = \mathcal{SE}_M\big(\mathbb{R}_+ : \mathcal{L}(X)\big)/\mathcal{SN}\big(\mathbb{R}_+ : \mathcal{L}(X)\big)$$

Elements of $\mathcal{SG}\big(\mathbb{R}_+ : \mathcal{L}(X)\big)$ will be denoted by $S = [S_\varepsilon]$, where $(S_\varepsilon)_\varepsilon$ is a representative of the above class.

Definition 5.6. *$S \in \mathcal{SG}\big(\mathbb{R}_+ : \mathcal{L}(X)\big)$ is a called a Colombeau C_0-Semigroup if it has a representative $(S_\varepsilon)_\varepsilon$ such that, for some $\varepsilon_0 > 0$, S_ε is a C_0-Semigroup, for every $\varepsilon < \varepsilon_0$.*

Example 5.1. *We take $\mathcal{G} = \mathcal{G}\big(\mathbb{R}^+\big)$, and we we define $T_\varepsilon(t)u_\varepsilon(x) = u_\varepsilon\big(x + \frac{t^\alpha}{\alpha}\big)$. Then*

$$T(t) = \overline{\big(T_\varepsilon(t)\big)_\varepsilon}$$

define a conformable semigroup on $\mathcal{G}$

In the sequel we will use only representatives $\big(S_\varepsilon\big)_\varepsilon$ of a Colombeau C_0-semigroup S which are C_0-semigroups, for ε small enough.

Proposition 5.3. *Let $\big(S_\varepsilon\big)_\varepsilon$ and $\big(\tilde{S}_\varepsilon\big)_\varepsilon$ be representatives of a Colombeau C_0-semigroup S, with the infinitesimal generators A_ε, $\quad\varepsilon < \varepsilon_0$, and $\tilde{A}_\varepsilon$, $\quad\varepsilon < \tilde{\varepsilon}_0$, respectively, where ε_0 and $\tilde{\varepsilon}_0$ correspond **(**in the sense of definition 5.6 to $\big(S_\varepsilon\big)_\varepsilon$ and $\big(\tilde{S}_\varepsilon\big)_\varepsilon$, respectively.*
Then, $D\big(A_\varepsilon\big) = D\big(\tilde{A}_\varepsilon\big)$, for every $\varepsilon < \overline{\varepsilon} = \min\big\{\varepsilon_0, \tilde{\varepsilon}_0\big\}$ and $A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to an element of $\mathcal{L}(X)$, denoted again by $A_\varepsilon - \tilde{A}_\varepsilon$.
Moreover, for every $a \in \mathbb{R}$,

$$\|A_\varepsilon - \tilde{A}_\varepsilon\| = O_{\varepsilon\to 0}(\varepsilon^a), \tag{5.6}$$

Proof. Denote $N_\varepsilon(S_\varepsilon - \tilde{S}_\varepsilon)_\varepsilon \in \mathcal{SN}(\mathbb{R}_+, \mathcal{L}(X))$.
Let $\varepsilon < \overline{\varepsilon}_0$ be fixed and $x \in X$. we have

$$\frac{S_\varepsilon(t^{\frac{1}{\alpha}})x - x}{t} - \frac{\tilde{S}_\varepsilon(t^{\frac{1}{\alpha}})x - x}{t} = \frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}x$$

This implies by letting $t \mapsto 0$, that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$. Now we have

$$\Big(A_\varepsilon - (\tilde{A})_\varepsilon\Big)x = \lim_{t\longrightarrow 0}\frac{S_\varepsilon(t^{\frac{1}{\alpha}})x - x}{t} \tag{5.7}$$

$$- \lim_{t\longrightarrow 0}\frac{\tilde{S}_\varepsilon(t^{\frac{1}{\alpha}})x - x}{t} \tag{5.8}$$

$$= \lim_{t\longrightarrow 0}\frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}x = H_\varepsilon x, \quad x \in D(A_\varepsilon) \tag{5.9}$$

since $D(A_\varepsilon)$ is dense in X, properties (4), (5) and (7) imply that for every $a \in \mathbb{R}$,

$$\|A_\varepsilon - \tilde{A}_\varepsilon\| = O_{\varepsilon\to 0}(\varepsilon^a). \qquad \square$$

Now we define the infinitesimal generator of a Colombeau C_0-semigroup S. Denote by $\mathcal{A}$ the set of pairs $((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon)$ where A_ε is a closed linear operator on X with the dense domain $D(A_\varepsilon) \subset X$, for every $\varepsilon \in (0,1)$. We introduce an equivalence relation in A

$$\Big(\big(A_\varepsilon\big)_\varepsilon, \big(D(A_\varepsilon)\big)_\varepsilon\Big) \sim \Big(\big(\tilde{A}_\varepsilon\big)_\varepsilon, \big(D(\tilde{A}_\varepsilon)\big)_\varepsilon\Big)$$

if there exist $\varepsilon_0 \in (0,1)$ such that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \varepsilon_0$, and for every $a \in \mathbb{R}$ ther exist $C > 0$ and $\varepsilon_a \leq \varepsilon_0$ such that, for $x \in A(A_\varepsilon)$, $\|(A_\varepsilon - \tilde{A}_\varepsilon)x\| \leq C\varepsilon^a\|x\|$, $x \in D(A_\varepsilon), \varepsilon \leq \varepsilon_a$.

Since A_ε has a dense domain in X, $R_\varepsilon := A_\varepsilon - \tilde{A}_\varepsilon$ ca, be extended to be an operator in $\mathcal{L}_c(X)$ satisfying $\|(A_\varepsilon - \tilde{A}_\varepsilon)x\| = O_{\varepsilon\to 0}(\varepsilon^a)$, for every $a \in \mathbb{R}$. such an operator R_ε is called the zero operator.

We denote by A the corresponding element of the quotient space $\mathcal{A}/\sim$. Due to propostion 5.3, the following definition makes sense

Definition 5.7. *$A \in \mathcal{A}/\sim$ is the infinitesimal generator of a Colombeau C_0-semigroup S if there exists a representative $(A_\varepsilon)_\varepsilon$ of A such that A_ε is theinfinitesimal generator of S_ε, for ε small enough.*

By Pazy we have the following proposition

Proposition 5.4. *Let S be a Colombeau C_0-semigroup with the infinitesimal generator A. Then there exists $\varepsilon_0 \in (0,1)$ such that:*

- *Mapping $t \longmapsto S_\varepsilon(t^{\frac{1}{\alpha}})x : \mathbb{R}_+ \longrightarrow X$ is continuous for every $x \in X$ and $\varepsilon < \varepsilon_0$*
- $$\lim_{h\longrightarrow 0}\int_t^{t+h} S_\varepsilon(s^{\frac{1}{\alpha}})x ds_\alpha = S_\varepsilon(t^{\frac{1}{\alpha}})x, \quad \varepsilon < \varepsilon_0, \quad x \in X$$
- $$\int_0^t S_\varepsilon(s^{\frac{1}{\alpha}})x ds_\alpha \quad \in D(A_\varepsilon), \quad \varepsilon < \varepsilon_0, \quad x \in X$$
- *For every $x \in D(A_\varepsilon)$ and $t \geq 0$ $S_\varepsilon(t^{\frac{1}{\alpha}})x \in D(A_\varepsilon)$ and*

$$\frac{d^\alpha}{dt^\alpha}S_\varepsilon(t^{\frac{1}{\alpha}})x = A_\varepsilon S_\varepsilon(t^{\frac{1}{\alpha}})x = S_\varepsilon(t^{\frac{1}{\alpha}})A_\varepsilon x, \quad \varepsilon < \varepsilon_0 \tag{5.10}$$

- *Let $\big(S_\varepsilon\big)_\varepsilon$ and $\big(\tilde{S}_\varepsilon\big)_\varepsilon$ be representative of Colombeau C_0-semigroup S, with infinitesimal generators A_ε and $\tilde{A}_\varepsilon$, $\varepsilon < \varepsilon_0$, respectively. Then, for every $a \in \mathbb{R}$ and $t \geq 0$*
$$\|\frac{d^\alpha}{dt^\alpha} S_\varepsilon(t^{\frac{1}{\alpha}}) - \tilde{A}_\varepsilon S_\varepsilon(t^{\frac{1}{\alpha}})\| = O_{\varepsilon\to 0}(\varepsilon^a). \tag{5.11}$$
- *For every $x \in D\big(A_\varepsilon\big)$ and every $t, s \geq 0$,*
$$S_\varepsilon(t^{\frac{1}{\alpha}})x - S_\varepsilon(s^{\frac{1}{\alpha}})x = \int_s^t S_\varepsilon(\tau^{\frac{1}{\alpha}})A_\varepsilon x d\tau_\alpha = \int_s^t A_\varepsilon S_\varepsilon(\tau^{\frac{1}{\alpha}})x d\tau_\alpha$$

Theorem 5.2. *Let S and $\tilde{S}$ be Colombeau C_0-semigroups with infinitesimal generators A and $\tilde{A}$, respectively. If $A = \tilde{A}$ then $S = \tilde{S}$.*

Proof. Let ε be small enough and $x \in D\big(A_\varepsilon\big) = D\big(\tilde{A}_\varepsilon\big)$. Proposition 5.4 property 4 implies that for $t \geq 0$, the mapping $s \longmapsto \tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x, \quad t \geq s \geq 0$ is differentiable and
$$\begin{aligned}\frac{d^\alpha}{ds^\alpha}\big(\tilde{S}_\varepsilon((t-s)^{\frac{1}{\alpha}})S_\varepsilon(s^{\frac{1}{\alpha}})x\big) &= -\tilde{A}_\varepsilon\tilde{S}_\varepsilon((t-s)^{\frac{1}{\alpha}})S_\varepsilon(s^{\frac{1}{\alpha}})x \\ &\quad + \tilde{S}_\varepsilon((t-s)^{\frac{1}{\alpha}})A_\varepsilon S_\varepsilon(s^{\frac{1}{\alpha}})x, \quad t \geq s \geq 0.\end{aligned}$$
The assumption $A = \tilde{A}$ implies that $A_\varepsilon = \tilde{A}_\varepsilon + R_\varepsilon$, where R_ε is a zero operator. Since A_ε commutes with $\tilde{S}_\varepsilon$, for every $x \in D\big(A_\varepsilon\big)$
$$\frac{d^\alpha}{ds^\alpha}\big(\tilde{S}_\varepsilon((t-s)^{\frac{1}{\alpha}})S_\varepsilon(s^{\frac{1}{\alpha}})x\big) = S_\varepsilon\big((t-s)^{\frac{1}{\alpha}}\big)R_\varepsilon S_\varepsilon(s^{\frac{1}{\alpha}})x, \quad t \geq s \geq 0.$$
and this implies
$$\tilde{S}_\varepsilon\big((t-s)^{\frac{1}{\alpha}}\big)S_\varepsilon(s^{\frac{1}{\alpha}})x - \tilde{S}_\varepsilon(t^{\frac{1}{\alpha}})x = \int_0^s \tilde{S}_\varepsilon\big((t-u)^{\frac{1}{\alpha}}\big)R_\varepsilon S_\varepsilon(u^{\frac{1}{\alpha}})x du_\alpha,\ t \geq s \geq 0. \tag{5.12}$$
Putting $s = t$ in (11), we obtain
$$S_\varepsilon(t^{\frac{1}{\alpha}}) - \tilde{S}_\varepsilon(t^{\frac{1}{\alpha}})x) = \int_0^s \tilde{S}_\varepsilon\big((t-u)^{\frac{1}{\alpha}}\big)R_\varepsilon S_\varepsilon(u^{\frac{1}{\alpha}})x du_\alpha, \quad t \geq 0,\ x \in D(A_\varepsilon). \tag{5.13}$$
Since $D(A_\varepsilon)$ is dense in X, uniform boundedness of S and $\tilde{S}$ on $[0,t]$ implies that (11) holds for every $y \in X$. Let us prove that $(N_\varepsilon)_\varepsilon = (S_\varepsilon - \tilde{S}_\varepsilon)_\varepsilon \in \mathcal{SN}(\mathbb{R}_+ : \mathcal{L}_c(X))$. The formula (12) and definition 5.4 imply that for some $C > 0$ and $a, \tilde{a} \in \mathbb{R}$,
$$\begin{aligned}\sup_{t\in[0,T)} \|N_\varepsilon(t^{\frac{1}{\alpha}})x\| &\leq \sup_{t\in[0,T)} \int_0^t \|\tilde{S}_\varepsilon\big((t-u)^{\frac{1}{\alpha}}\big)\|\|R_\varepsilon\|\|S_\varepsilon(u^{\frac{1}{\alpha}})\|\|x\| du_\alpha \\ &\leq T\, C\, \varepsilon^{a+\tilde{a}}\|R_\varepsilon\|\|x\|, \quad x \in X\end{aligned}$$
Since $\|R_\varepsilon\| = O_{\varepsilon\to 0}(\varepsilon^b)$, for every $b \in \mathbb{R}, \big(N_\varepsilon(t)\big)_\varepsilon$ satisfies condition (3) in definition 5.4. Condition (3) follows from the boundedness of $(\tilde{S}_\varepsilon)_\varepsilon, (S_\varepsilon)_\varepsilon$ on dounded domain $[0,t)$, the properties of $(R_\varepsilon)_\varepsilon$ and the following expression:
$$\|\frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t}\| = \|\frac{1}{t}\int_0^t \tilde{S}_\varepsilon\big((t-u)^{\frac{1}{\alpha}}\big)R_\varepsilon S_\varepsilon(u^{\frac{1}{\alpha}})x du_\alpha\| \leq \|\tilde{S}_\varepsilon(t^{\frac{1}{\alpha}})\|\|R_\varepsilon\|\|S_\varepsilon\| \leq const,\ x \in X,\ t \leq t_0,$$

for some $t_0 > 0$. Also,

$$\lim_{t\longrightarrow 0}\frac{N_\varepsilon(t^{\frac{1}{\alpha}})}{t} = \lim_{t\longrightarrow 0}\frac{\tilde{S}_\varepsilon(t^{\frac{1}{\alpha}})x - x}{t} - \lim_{t\longrightarrow 0}\frac{S_\varepsilon(t^{\frac{1}{\alpha}})x - x}{t} = R_\varepsilon x, \quad \forall x \in D\big(A_\varepsilon\big).$$

Since it is enough that (5) holds for a dense subset of X see the remark after definition 5.5 this concludes the proof. □

6. Main result

This section deals with applications of Colombeau α-C_0-semigroups in solving a class of heat equations with singular potentials and singular data. We consider the problem

$$\begin{cases} \partial_t^\alpha u(t,x) = (\Delta - v(x))u(t,x) \quad xin\mathbb{R},\ t\in\mathbb{R}^+ \\ u(0,t) = u_0(x) = \delta(x) \\ v(x) = \delta(x) \end{cases} \tag{6.1}$$

Before we discuss the problem we will defined some spaces on which we will work. First we set $\|.\|_{L^2(\mathbb{R})} = \|.\|_2$

Definition 6.1. *We define $H^{2,\alpha}$ by:*

$$H^{2,\alpha} = \big\{ f\in L^2(\mathbb{R}), \quad \widetilde{D}^\alpha f \in L^2(\mathbb{R})\big\}$$

with the norm

$$\|f\|_{H^{2,\alpha}} = \sqrt{\|f\|_2^2 + \|\widetilde{D}^\alpha f\|_2^2}$$

The Colmbeau algebra type defined as follows

$$\mathcal{G}_{H^{2,\alpha}} = \mathcal{E}_M\big(H^{2,\alpha}\big)/\mathcal{N}\big(H^{2,\alpha}\big)$$

where $\mathcal{E}_M\big(H^{2,\alpha}\big)$ the vector space of nets $\big(G_\varepsilon\big)_\varepsilon \in H^{2,\alpha}$ with the property: for every $T>0$ there exist $a\in\mathbb{R}$ such that

$$\|G_\varepsilon\|_{H^{2,\alpha}} = O(\varepsilon^a)$$

and $\mathcal{N}\big(H^{2,\alpha}\big)$ the vector space of nets $\big(G_\varepsilon\big)_\varepsilon \in H^{2,\alpha}$ with the property: for every $T>0$ for all $b\in\mathbb{R}$ such that

$$\|G_\varepsilon\|_{H^{2,\alpha}} = O(\varepsilon^b)$$

is a Colombeau type vector space.

Definition 6.2. *$\mathcal{E}_{C^1,H^{2,\alpha}}\big([0,T],\mathbb{R}\big)$ (respectively $\mathcal{N}_{C^1,H^{2,\alpha}}\big([0,T],\mathbb{R}\big)$), $T>0$, is the vector space of nets $\big(G_\varepsilon\big)_\varepsilon$ of functions*

$$G_\varepsilon \in \mathcal{C}\big([0,T],H^{2,\alpha}\big)\cap\mathcal{C}^1\big([0,T],L^2(\mathbb{R})\big)$$

with the property: for every $T_1\in(0,T)$ there exists $a\in\mathbb{R}$, (respectively, for every $a\in\mathbb{R}$) such that

$$\max\Big\{\sup_{t\in[0,T_1]}\|G_\varepsilon\|_{H^{2,\alpha}},\ \sup_{t\in[T_1,T]}\|\widetilde{D}^\alpha G_\varepsilon\|_{L^2(\mathbb{R})}\Big\} = O_{\varepsilon\to 0}(\varepsilon^a)$$

The quotient space

$$\mathcal{G}_{C^1,H^{2,\alpha}}\big([0,T],\mathbb{R}\big) = \mathcal{E}_{C^1,H^{2,\alpha}}\big([0,T],\mathbb{R}\big)/\mathcal{N}_{C^1,H^{2,\alpha}}\big([0,T],\mathbb{R}\big)$$

is a Colombeau type vector space.

Remark 6.1. The multiplication of potential $V \in \mathcal{G}_{H^{2,\alpha}}$ and $u \in \mathcal{G}_{C^1,H^{2,\alpha}}\big([0,T],\mathbb{R}\big)$ which is expected to be a solution to equation

$$\begin{cases} \partial_t^\alpha u(t,x) = (\Delta - v(x))u(t,x) \quad xin\mathbb{R},\ t \in \mathbb{R}^+ \\ u(0,t) = u_0(x) = \delta(x) \\ v(x) = \delta(x) \end{cases}$$

makes sense.

Definition 6.3. *Let A be represented by a net $(A_\varepsilon)_\varepsilon$, of linear operators with the common domain $H^{2,\alpha}(\mathbb{R})$ and with ranges in $L^2(\mathbb{R})$. A generalized function $G \in \mathcal{G}_{C^1,H^{2,\alpha}}$ $T > 0$, is said to be a solution to equation $\widetilde{D}^\alpha G = AG$ if*

$$\sup_{t\in[0,T]} \|\widetilde{D}^\alpha G_\varepsilon(t,.) - A_\varepsilon G_\varepsilon(t,.)\|_2 = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^a),\ \forall a \in \mathbb{R}.$$

Definition 6.4. *An element $V \in \mathcal{G}_{H^{2,\alpha}}$ is of logarithmic type if it has a representative $(V_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^{2,\alpha}}$ with the property*

$$\|V_\varepsilon\|_{H^{2,\alpha}} = \mathcal{O}_{\varepsilon\to 0}(\ln \varepsilon^{-1})$$

An element $V \in \mathcal{G}_{H^{2,\alpha}}$ is said to be of log-log type if it has a representative $(V_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^{2,\alpha}}$ with the property

$$\|V_\varepsilon\|_{H^{2,\alpha}} = \mathcal{O}_{\varepsilon\to 0}(\ln^a \ln \varepsilon^{-1})$$

We set

$$E(t,x) = \frac{1}{2\sqrt{\pi t}} \exp\frac{|x|^2}{4t}$$

Theorem 6.1. *Let $V \in \mathcal{G}_{H^{2,\alpha}}$ be of logarithmic type.*

(1) *Derivation of $A_\varepsilon u = \big(\Delta - V\big)u$, $u \in H^{2,\alpha}(\mathbb{R})$, are infinitesimal generators of conformable semigroups T_ε, for every $\varepsilon > 0$, and $(T_\varepsilon)_\varepsilon$ is a representative of a Colombeau conformable C_0-semigroup.*

$$T(t) \in \mathcal{SG}\big(\mathbb{R}^+, \mathcal{L}\left(L^2(\mathbb{R})\right)\big)$$

(2) *Let $V \in \mathcal{G}_{H^{2,\alpha}}$ and let T_ε be as in 1.*
Then, for every $T > 0$, the problem 6.1 has unique solution in $\mathcal{G}H^{2,\alpha}(\mathbb{R})$.

Proof.

(1) Let $\varepsilon > 0$ small enough. By the Feynman-Kac formula the operator A_ε is the infinitesimal generator of the corresponding semigroup

$$T_\varepsilon(t)\psi(x) = \int_\Omega \Big(\exp\big(-\int_0^{t^\alpha} V_\varepsilon(\omega(s))ds\big)\Big)\psi(\omega(t^\alpha))d\mu_x(\omega)$$

for $\psi \in L^2(\mathbb{R})$, where $\Omega = \prod_{t\geq 0} \overline{\mathbb{R}}$ and μ_x is is the Wiener measure concentrated at $x \in \mathbb{R}$.
Since V is of logarithmic type, there exist $C > 0$ and $\eta \in (0,1)$ such that

$$\begin{aligned}\left|T_\varepsilon(t)\psi(x)\right| &\leq \exp\left(t^\alpha \sup_{s\in\mathbb{R}} |V_\varepsilon(s)|\right) \int_\Omega |\psi(\omega(t))| d\mu_x(\omega) \\ &\leq \varepsilon^{Ct^\alpha} \frac{1}{2\sqrt{\pi t^\alpha}} \int_\mathbb{R} \exp - \frac{|x-y|^2}{4t^\alpha} |\psi(y)| dy\end{aligned}$$

Therefore, there exist $C_0 > 0$ such that

$$\sup_{t\in(0,T]} \|T_\varepsilon(t)\psi(x)\|_2 \leq C_0 \varepsilon^{CT^\alpha} \|\psi\|_2$$

Thus $\overline{(T_\varepsilon)_\varepsilon} \in \mathcal{SG}\Big(\mathbb{R}^+, \mathcal{L}\big(L^2(\mathbb{R})\big)\Big)$

(2) (a) By the Duhamel principle, solution $u_\varepsilon(t,x)$ to equation (6.1) satisfies

$$u_\varepsilon(t,x) = \int_\mathbb{R} E(t^\alpha, x-y) b_\varepsilon(y) dy \tag{6.2}$$

$$+ \int_0^t \int_\mathbb{R} E((t-s)^\alpha, x-y) V_\varepsilon(y) u_\varepsilon(s,y) dy \tag{6.3}$$

Young's inequality implies

$$\|u_\varepsilon(t,.)\|_2 \leq \|b_\varepsilon\|_2 + \int_0^t \|V_\varepsilon(.)\|_L^\infty \|u_\varepsilon(s,.)\|_2 ds$$

Gronwall's inequality gives

$$\|u_\varepsilon(t,.)\|_2 \leq \|b_\varepsilon\|_2 \exp \int_0^t \|V_\varepsilon(.)\|_L^\infty, \ \forall t \in (0,T]$$

Since $V \in \mathcal{G}H^{2,\alpha}(\mathbb{R})$ is of logarithmic type and $(u_{0\varepsilon})_\varepsilon \in \mathcal{E}H^{2,\alpha}(\mathbb{R})$, it follows that $\sup_{t\in(0;T]} \|u_\varepsilon(t,.)\|_2$ has a moderate bound. Differentiation of equation 6.2 with respect to some spatial variable x gives

$$\begin{aligned}\frac{d}{dx} u_\varepsilon(t,x) = &\int_\mathbb{R} E(t^\alpha, y) \frac{d}{dx} b_{\varepsilon(x-y)} dy \\ &+ \int_0^t \int_\mathbb{R} E((t-s)^\alpha, y) \frac{d}{dx} V(x-y) u_\varepsilon(s, x-y) \\ &+ V(x-y) \frac{d}{dx} u_\varepsilon(s, x-y) dy ds\end{aligned}$$

then

$$\|\frac{d}{dx} u_\varepsilon(t,.)\|_2 \leq \|\frac{d}{dx} b\|_2 + \int_0^t \|\frac{d}{dx} V_\varepsilon(.)\|_{L^\infty \|u_\varepsilon(t,.)\|_2 + \|V_\varepsilon(.)\|_{L^\infty} \|\frac{d}{dx} u_\varepsilon(t,.)\|_2}$$

Also Gronwall's inequality implies that $\sup_{t\in(0,T]} \|u_\varepsilon(t,.)\|_2$ is moderate.
So $\overline{(u_\varepsilon)_\varepsilon} \in \mathcal{E}_{C^1,H^{2,\alpha}}(\mathbb{R})$.

(b) For uniqueness:
Let u_ε and v_ε two solution of (6.1).
e set $G_\varepsilon = u_\varepsilon - v_\varepsilon$, we get

$$\begin{aligned} G_\varepsilon(t,x) &= \int_{\mathbb{R}} E(t^\alpha, x-y) N_\varepsilon(y)dy \\ &+ \int_0^t \int_{\mathbb{R}} E((t-s)^\alpha, x-y) V_\varepsilon(y) G_\varepsilon(s,y)dy \\ &+ \int_0^t \int_{\mathbb{R}} E((t-s)^\alpha, x-y) N_\varepsilon(y)dy \end{aligned}$$

where $N_\varepsilon(x) = G(0,x)$, and $N_\varepsilon = \frac{d^\alpha}{dt^\alpha} G_\varepsilon - (\Delta - V)G_\varepsilon$
Then Young's and Gronwall's inequalities imply

$$\|G_\varepsilon(t,.)\|_2 \leq \|N\|_2 + \int_0^t \|V_\varepsilon(s,.)\|_{L^\infty} \|G_\varepsilon(s,.)\|_2 ds + \int_0^t \|N_\varepsilon(s,.)\|_2 ds_\alpha$$

thus $G_\varepsilon \in \mathcal{N}H^{2,\alpha}$. □

References

[1] T.Abdeljawad, *On Conformable Fractional Calculus*. Journal of Computational and Applied Mathematics, Vol. **279**, 57–66, (2015).

[2] A. Atangana, D. Baleanu, A. Alsaedi, *New properties of conformable derivative*. Open Mathematics, Open Math. **13** 889–898, (2015).

[3] A. V. Balakrishnan, *Fractional Powers Of Closed Operators And The Semigroups Generated By Them*, Pacific Journal of Mathematics **10**, pp. 419-439, 1960.

[4] H.A. Biagioni, *A nonlinear theory of generalized functions*, Lecture Notes in Mathematics, vol. **1421**,Springer, Berlin, 1990.

[5] J. F. Colombeau, *New generalized functions and Multiplication of distributions*, North-Holland, Amsterdam, (1984).

[6] J. F. Colombeau and A. Y. L. Roux, *Multiplications of distributions in elasticity and hydrodynamics*, J. Math. Phys., 29 (1988), 315–319.

[7] J. F. Colombeau, *Elementary Introduction to New Generalized Functions*, North-Holland Math. Studies Vol. 113, North-Holland, Amsterdam (1985).

[8] I. M. Gel'fand and G. E. Shilov, *Generalized functions*, Academic press, New York, Vol. I, (1964).

[9] M. Horani, R. Khalil, T. Abdeljawad, *Conformable Fractional Semigroups of Operators*. arXiv:1502.06014v1 [math.FA] 21 Nov (2014).

[10] G. Jumarie, *Modified Riemann-Liouville derivative and fractional Taylor series of non-differentiable functions further results*, Computers and Mathematics with Applications, vol. **51**, no. 9–10, pp. 1367–1376, (2006).

[11] R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new Definition of Fractional Derivative*, J. Comput. Appl. Math. **264**. pp. 65–0, (2014).

[12] A. A. Kilbas, H. M. Srivastava, JJ. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam (2006)

[13] S.Mirjana, *Nonlinear Schrödinger equation with singular potential and initial data*, Nonlinear Analysis 64 (2006) 1460–1474.

[14] A. Pazy (1983), Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York Inc.

[15] D. Rajter-Ciric, *Fractional derivatives of Colombeau Generalized stochastic processes defined on* $\mathbb{R}^+$, Appl. Anal. Discrete Math. 5 (2011), 283–297.

[16] M. Stojanovi'c, *Extension of Colombeau algebra to derivatives of arbitrary order* $\mathcal{D}^\alpha$, $\alpha \in \mathbb{R}_+ \cup \{0\}$. *Application to ODEs and PDEs with entire and fractional derivatives*, Nonlinear Analysis 5 (2009), 5458–5475.

Abbreviated titles of journal are available at AMS and zbMATH web sites.

(Received: July 27, 2018)
(Revised: August 21, 2019)

M'hamed Elomari, Said Melliani, and Lalla Saadia Chadli
Sultan Moulay Sliman University
Department of Mathematics
LAMSC, Sultan Moulay Slimane University
BP 523, 23000, Beni Mellal, Morocco
e-mail: *m.elomari@usms.ma*
e-mail: *said.melliani@gmail.com*
e-mail: *sa.chadli@yahoo.fr*