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STABILITY AND SOLVABILITY OF DISCRETE GENERALIZED PROPORTIONAL CAPUTO FRACTIONAL NEURAL NETWORK MODELS

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Dedicated to Professor Mehmed Nurkanović on the occasion of his 65th birthday

ABSTRACT. This study examines stability in variable-order fractional discrete neural networks modeled via the generalized proportional Caputo fractional difference operator. By employing the Krasnoselskii fixed-point theorem, we establish solution existence under Lipschitz continuity, and we prove Ulam–Hyers stability. Numerical simulations validate that balancing network parameters and fractional orders ensure robustness.

1. INTRODUCTION

Fractional calculus has secured considerable attention due to its ability to capture memory and hereditary characteristics inherent in numerous physical, biological, and computational systems. Traditional integer-order differential models often fall short in representing systems with long-term memory and nonlocal interactions. In this context, fractional-order neural networks have become a focal point in both theoretical and applied mathematics. The pioneering work of Podlubny [26] introduced a robust framework for fractional differential equations, while Kilbas et al. [19] developed a comprehensive theory encompassing various definitions of fractional operators and their applications. Within the discrete proportional framework, an important step was made in [16], while our recent contribution [9] introduced the theory of discrete generalized proportional derivatives. These two works constitute the base of the present article, where we extend the stability and solvability analysis to DGPCFD (discrete generalized proportional Caputo fractional derivative) neural networks.

In addition to the above, several authors have contributed to the development. For instance, Machado et al. [22] analyzed memory-dependent systems and emphasized the relevance of fractional models in dynamic networks. Li and Chen [20] explored synchronization and control in fractional-order chaotic neural networks, showing improved convergence and dynamic behavior. Yang et al. [29] further applied fractional neural models to emulate brain dynamics and signal propagation, highlighting their biological relevance.

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A notable recent development is the generalized proportional Caputo (GPC) fractional derivative proposed by Atangana and Baleanu [6], introducing a flexible proportionality factor to model weighted memory. This operator has since been adapted to various fields, including control theory [13], epidemiological models [17], and neural modeling [27]. Its discrete counterpart enables digital simulation and analysis, crucial for computational neuroscience and artificial intelligence applications.

Discrete fractional calculus provides tools for modeling systems in digital or sampled-data environments. Abdeljawad [1] formalized definitions of discrete fractional differences, laying the groundwork for discrete-time analysis. Goodrich and Peterson [15] advanced the theoretical framework, while Baleanu and Fernández [7] provided a survey of the latest developments. Applications in digital control systems were demonstrated by Lin et al. [21, 24] and Yu et al. [30], including discrete modeling of neural learning and synaptic dynamics.

The solvability of discrete-time fractional neural networks is essential for ensuring model predictability and consistency. Various fixed-point techniques have been used to establish existence and uniqueness, particularly in Banach space settings. Agarwal et al. [2] used the Schauder and Banach fixed-point theorems to solve boundary value problems, while Zhou et al. [32] extended these results to impulsive and multi-delay fractional systems. Zhang et al. [31] and Ali et al. [4] provided solvability conditions of discrete neural networks described with fractional operator of time-varying delays.

A vital concern in modeling is stability, which guarantees the system's robustness under small perturbations. Ulam–Hyers stability, in particular, has found prominence in the analysis of fractional systems. Jung [18] introduced the classical concept, while Ahmad et al. [3, 12] and Araci et al. [5] extended it to fractional difference equations. Wang et al. [28] applied it in neural networks with impulsive and delay effects. Recent contributions by Mandal and Singh [23] and Cai et al. [11] provide generalized Ulam-type stability results for discrete and hybrid systems.

Despite the advances in both discrete-time fractional models and neural dynamics, there remains a gap in the literature regarding a unified analysis of solvability and Ulam-type stability in discrete generalized proportional Caputo fractional neural networks. While several studies have separately explored discrete fractional operators or stability analysis [8, 14, 25], the integration of these concepts under the DGPCFD framework in neural network dynamics is relatively uncharted.

Motivated by this, the present study aims to formulate a DGPCFD neural network model that captures memory effects and nonlocality in a discrete-time setting. It establishes solvability through fixed-point theory under appropriate Lipschitz and compactness conditions. Furthermore, the study analyzes Ulam–Hyers stability and its variants to ensure solution robustness and convergence. The theoretical outcomes are validated through numerical simulations, providing practical insights into time-delay systems and adaptive neural processes.

This research contributes to advancing the theoretical framework of fractional neural systems and offers practical utility in modeling memory-dependent processes in AI, biology, and computational engineering.

We organize the paper as follows. In Section 2, preliminaries on discrete fractional calculus are given. In Section 3, a governing equation of discrete fractional neural networks with variable order is introduced. In Section 4, using Krasnosel′skiĭ's fixed-point theorem, we establish the existence of solutions under Lipschitz continuity. In Section 5, we derive Ulam–Hyers stability criteria, proving that perturbations decay exponentially when $M < 1$. In Section 6, numerical simulations of two-dimensional and three-dimensional networks validate the theoretical results. Finally, in the last section, some conculsions are given.

2. Preliminaries

The following definitions of discrete generalized proportional fractional sum-difference are recalled. For $n \in \mathbb{N}_0$, we use the notation $\mathbb{N}_n = \mathbb{N} \cap [n, \infty)$.

Definition 2.1 (See [9, Definition 3.9]). *For $\rho \in (0,1]$ and $\alpha \in (0,1)$, the left generalized proportional fractional sum of $f : \mathbb{N}_a \to \mathbb{R}$ is defined as*

$$({}_aI^{\alpha,\rho} f)(t) = \sum_{s=a}^{t-\alpha} \rho^{t-s-\alpha} h_{\alpha-1}(t, s+1) f(s) \quad \text{for} \quad t \in \mathbb{N}_{a+\alpha}, \tag{2.1}$$

where the αth discrete Taylor monomials are defined as

$$h_\alpha(t,s) = \frac{\Gamma(t-s+1)}{\Gamma(t-s+1-\alpha)\Gamma(\alpha+1)} \tag{2.2}$$

whenever the right-hand side is well defined.

Lemma 2.1 (See [15, Theorem 2.27 (ii)]). *Let $t \in \mathbb{N}_a$. Then*

$$\Delta h_\alpha(t,a) = h_{\alpha-1}(t,a), \tag{2.3}$$

whenever these expressions make sense.

Definition 2.2 (See [9, Definition 4.1]). *For $\rho \in (0,1]$ and $\alpha \in (0,1)$, we define the left generalized proportional fractional difference of Caputo type starting at a as*

$$\left({}_a^{\mathrm{c}}D^{\alpha,\rho} f\right)(t) = {}_aI^{1-\alpha,\rho} \left(D^{1,\rho} f\right)(t) \quad \text{for} \quad t \in \mathbb{N}_{a+1-\alpha}, \tag{2.4}$$

where

$$(D^{1,\rho} f)(t) = f(t+1) - \rho f(t).$$

Remark 2.1. If $\rho = 1$, then the generalized proportional Caputo fractional difference is reduced to the classical Caputo fractional difference.

Lemma 2.2 (See [9, Theorem 4.3]). *For any $\rho \in (0,1]$, $\alpha \in (0,1)$, we have*

$${}_{a+1-\alpha}I^{\alpha,\rho} \left({}_a^{\mathrm{c}}D^{\alpha,\rho} f\right)(t) = f(t) - \rho^{t-a} f(a) \quad \text{for} \quad t \in \mathbb{N}_{a+1}. \tag{2.5}$$

Lemma 2.3 (See [10, Theorem 1, Krasnosel′skiĭ's Fixed-Point Theorem])**.** *Let E be a Banach space, let $\Omega \subset E$ be closed, convex, and nonempty, and let $S, T : \Omega \to E$ be two operators satisfying*

1. *S is a contraction,*
2. *T is continuous and $T(\Omega)$ is contained in a compact set,*
3. *$Sx + Ty \in \Omega$ for all $x, y \in \Omega$.*

Then, there exists $x \in \Omega$ with $Sx + Tx = x$.

3. Variable-Order DGPCFD Neural Network

Discrete-time fractional-order neural networks provide a powerful framework for modeling the dynamics of discrete nonlinear systems characterized by fractional-order behavior. These networks are highly effective due to their ability to capture the intricate dynamics of complex systems with high precision. As a result, they present a promising approach for developing generic, parametric, and nonlinear models applicable to a broad class of discrete nonlinear systems exhibiting fractional orders. In this study, we investigate a specific variant of discrete fractional neural networks described with a variable-order mechanism. The model is defined on $\mathcal{T} := \mathbb{N}_0 \cap [0, m\ell]$, where $m, \ell \in \mathbb{N}$, so

$$0 \le t \le m\ell \quad \text{for all} \quad t \in \mathcal{T}. \tag{3.1}$$

On each $\mathcal{T}_k := \mathcal{T} \cap [k\ell, (k+1)\ell - 1]$, $k \in \mathbb{N}_0 \cap [0, m-1]$, the model is governed by a DGPCFD of order $\alpha_k \in (0,1)$. The matrix $A = \operatorname{diag}(-a_1, \ldots, -a_p) \in \mathbb{R}^{p \times p}$ with $a_i > 0$ represents resetting the neuron potentials to the resting states when disconnected from the network, $B \in \mathbb{R}^{p \times p}$ represents the connection weights, and $f : \mathcal{T} \times \mathbb{R}^p \to \mathbb{R}^p$ denotes a nonlinear activation function. We therefore consider the problem

$${}^{\mathrm{c}}_{k\ell}D^{\alpha_k,\rho} x(t+1-\alpha_k) = Bf(t, x(t)) - Ax(t) \quad \text{for} \quad t \in \mathcal{T}_k, \quad 0 \le k \le m-1, \tag{3.2}$$

i.e.,

$$\begin{cases} {}^{\mathrm{c}}_{0}D^{\alpha_0,\rho} x(t+1-\alpha_0) = Bf(t, x(t)) - Ax(t) & \text{for} \quad t \in \mathcal{T}_0, \\ {}^{\mathrm{c}}_{\ell}D^{\alpha_1,\rho} x(t+1-\alpha_1) = Bf(t, x(t)) - Ax(t) & \text{for} \quad t \in \mathcal{T}_1, \\ \quad \vdots & \\ {}^{\mathrm{c}}_{(m-1)\ell}D^{\alpha_{m-1},\rho} x(t+1-\alpha_{m-1}) = Bf(t, x(t)) - Ax(t) & \text{for} \quad t \in \mathcal{T}_{m-1}. \end{cases} \tag{3.3}$$

Lemma 3.1. *For $\rho \in (0,1]$ and $\alpha_0, \alpha_1, \ldots, \alpha_{m-1} \in (0,1)$, the solution x of* (3.2) *satisfies the sum equation*

$$\begin{aligned} x(t) = \rho^t x(0) &+ \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell - 1 + \alpha_{n-1}, s+1)[Bf(s, x(s)) - Ax(s)] \\ &+ \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t - 1 + \alpha_{m-1}, s+1)[Bf(s, x(s)) - Ax(s)] \quad \textit{for} \quad t \in \mathcal{T}. \end{aligned} \tag{3.4}$$

Proof. Assume that x solves (3.2). For $0 \leq k \leq m-1$, $t \in \mathcal{T} \cap [k\ell+1,(k+1)\ell]$, putting $g = {}_{k\ell}^{\mathrm{c}}D^{\alpha_k,\rho} x$, we find

$$
\begin{aligned}
x(t) &\overset{(2.5)}{=} \rho^{t-k\ell}x(k\ell) + {}_{k\ell+1-\alpha_k}I^{\alpha_k,\rho}({}_{k\ell}^{\mathrm{c}}D^{\alpha_k,\rho}x)(t) \\
&= \rho^{t-k\ell}x(k\ell) + ({}_{k\ell+1-\alpha_k}I^{\alpha_k,\rho}g)(t) \\
&\overset{(2.1)}{=} \rho^{t-k\ell}x(k\ell) + \sum_{s=k\ell+1-\alpha_k}^{t-\alpha_k} \rho^{t-s-\alpha_k} h_{\alpha_k-1}(t,s+1)g(s) \\
&= \rho^{t-k\ell}x(k\ell) + \sum_{s=k\ell}^{t-1} \rho^{t-1-s} h_{\alpha_k-1}(t,s+2-\alpha_k)g(s+1-\alpha_k) \\
&\overset{(3.2)}{=} \rho^{t-k\ell}x(k\ell) + \sum_{s=k\ell}^{t-1} \rho^{t-1-s} h_{\alpha_k-1}(t,s+2-\alpha_k)[Bf(s,x(s)) - Ax(s)] \\
&\overset{(2.2)}{=} \rho^{t-k\ell}x(k\ell) + \sum_{s=k\ell}^{t-1} \rho^{t-1-s} h_{\alpha_k-1}(t-1+\alpha_k,s+1)[Bf(s,x(s)) - Ax(s)],
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
x(t) = \rho^{t-k\ell}x(k\ell) + \sum_{s=k\ell}^{t-1} \rho^{t-1-s} h_{\alpha_k-1}(t-1+\alpha_k,s+1)[Bf(s,x(s)) - Ax(s)] \\
\text{for} \quad t \in \mathcal{T} \cap [k\ell+1,(k+1)\ell].
\end{aligned} \tag{3.5}
$$

For $k = 0$, we get from (3.5) that

$$
x(t) = \rho^{t}x(0) + \sum_{s=0}^{t-1} \rho^{t-1-s} h_{\alpha_0-1}(t-1+\alpha_0,s+1)[Bf(s,x(s)) - Ax(s)] \tag{3.6}
$$

is true for $t \in \mathcal{T} \cap [1,\ell]$, but (3.6) is also true fot $t = 0$ trivially, so (3.6) is true for $t \in [0,\ell]$. Thus, plugging ℓ into (3.6), we get

$$
x(\ell) = \rho^{\ell}x(0) + \sum_{s=0}^{\ell-1} \rho^{\ell-1-s} h_{\alpha_0-1}(\ell-1+\alpha_0,s+1)[Bf(s,x(s)) - Ax(s)]. \tag{3.7}
$$

For $k = 1$ in (3.5), we get

$$
\begin{aligned}
x(t) = \rho^{t-\ell}x(\ell) + \sum_{s=\ell}^{t-1} \rho^{t-1-s} h_{\alpha_1-1}(t-1+\alpha_1,s+1)[Bf(s,x(s)) - Ax(s)] \\
\text{for} \quad t \in \mathcal{T} \cap [\ell+1,2\ell].
\end{aligned} \tag{3.8}
$$

Substituting (3.7) in (3.8), we get

$$x(t) \overset{(3.7)}{=} \rho^t x(0) + \sum_{s=0}^{\ell-1} \rho^{t-1-s} h_{\alpha_0-1}(\ell-1+\alpha_0, s+1)[Bf(s,x(s)) - Ax(s)] \\ + \sum_{s=\ell}^{t-1} \rho^{t-1-s} h_{\alpha_1-1}(t-1+\alpha_1, s+1)[Bf(s,x(s)) - Ax(s)] \quad \text{for} \quad t \in \mathcal{T} \cap [\ell+1, 2\ell]. \tag{3.9}$$

Note that (3.9) is also true for $t \in \mathcal{T} \cap [0,\ell]$. Repeating the above step, we obtain (3.4). This completes the proof. □

4. Existence of the solution

To establish existence of a solution for the proposed variable-order discrete fractional neural network, we define $P = S + T$, where the operators S and T are given by

$$Sx(t) = \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1}, s+1)[Bf(s,x(s)) - Ax(s)],$$

$$Tx(t) = \rho^t x(0) + \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)[Bf(s,x(s)) - Ax(s)],$$

$t \in \mathcal{T}$. Because of Lemma 3.1, the function x is a solution of (3.2) if and only if x is a fixed point of the operator P. Also, observe

$$(Sx)(0) = 0 \quad \text{and} \quad (Tx)(0) = x(0). \tag{4.1}$$

Lemma 4.1. *For any $t \in \mathcal{T}$, we have*

$$\sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} \left| h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1}, s+1) \right| \\ + \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} \left| h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1) \right| \le d, \tag{4.2}$$

where

$$d := \begin{cases} \frac{1-\rho^{m\ell}}{1-\rho} & \text{for} \quad \rho \neq 1, \\ m\ell & \text{for} \quad \rho = 1. \end{cases}$$

Proof. We have

$$\sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} \left| h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1},\, s+1) \right| \\ + \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} \left| h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1},\, s+1) \right|$$

$$
\begin{aligned}
&= \sum_{n=1}^{m-1}\sum_{s=1}^{\ell} \rho^{t-s-(n-1)\ell} \left| h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1}, s+(n-1)\ell) \right| \\
&\quad + \sum_{s=1}^{t-(m-1)\ell} \rho^{t+s-(m-1)\ell} \left| h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+(m-1)\ell) \right| \\
&= \sum_{n=1}^{m-1}\sum_{s=1}^{\ell} \rho^{t+s-1-n\ell} \left| h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1}, \ell-s+1+(n-1)\ell) \right| \\
&\quad + \sum_{s=1}^{t-(m-1)\ell} \rho^{t+s-1-m\ell} \left| h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, \ell-s+1+(m-1)\ell) \right| \\
&\overset{(2.2)}{=} \sum_{n=1}^{m-1}\sum_{s=1}^{\ell} \rho^{t+s-1-n\ell} h_{\alpha_{n-1}-1}(s, 2-\alpha_{n-1}) \\
&\quad + \sum_{s=1}^{t-(m-1)\ell} \rho^{t+s-1-m\ell} h_{\alpha_{m-1}-1}(s+t-m\ell, 2-\alpha_{m-1}) \\
&\overset{(2.3)}{\leq} \sum_{n=1}^{m-1}\sum_{s=1}^{\ell} \rho^{t+s-1-n\ell} h_{\alpha_{n-1}-1}(1, 2-\alpha_{n-1}) + \sum_{s=1}^{t-(m-1)\ell} \rho^{t+s-1-m\ell} h_{\alpha_{m-1}-1}(1, 2-\alpha_{m-1}) \\
&\overset{(2.2)}{=} \sum_{n=1}^{m-1}\sum_{s=1}^{\ell} \rho^{t+s-1-n\ell} + \sum_{s=1}^{t-(m-1)\ell} \rho^{t+s-1-m\ell} \\
&\overset{(3.1)}{\leq} \sum_{n=1}^{m-1}\sum_{s=1}^{\ell} \rho^{t+s-1-n\ell} + \sum_{s=1}^{\ell} \rho^{t+s-1-m\ell} = \sum_{n=1}^{m}\sum_{s=1}^{\ell} \rho^{t+s-1-n\ell} = \rho^{t-1} \sum_{n=1}^{m} \left(\frac{1}{\rho^{\ell}}\right)^{n} \sum_{s=1}^{\ell} \rho^{s}.
\end{aligned}
$$

For $\rho = 1$, this last expression is equal to $m\ell$, while for $0 < \rho < 1$, it is equal to

$$
\rho^{t-1} \frac{\frac{1}{\rho^{\ell}}}{1-\frac{1}{\rho^{\ell}}} \left(1 - \left(\frac{1}{\rho^{\ell}}\right)^{m}\right) \frac{\rho}{1-\rho} \left(1-\rho^{\ell}\right) = \frac{\rho^{t}\left(\rho^{-\ell m}-1\right)}{1-\rho} \overset{(3.1)}{\leq} \frac{1-\rho^{\ell m}}{1-\rho}.
$$

Note also that for $\alpha \in (0,1)$, (2.3) implies

$$
\Delta h_{\alpha-1}(s, 2-\alpha) = h_{\alpha-2}(s, 2-\alpha) = \frac{\Gamma(s-1+\alpha)}{\Gamma(s+1)\Gamma(\alpha-1)} = \frac{(\alpha-1)\Gamma(s-1+\alpha)}{\Gamma(s+1)\Gamma(\alpha)} < 0,
$$

which means $h_{\alpha-1}(s, 2-\alpha)$ is decreasing, and this fact was used in the calculation above for the estimate that indicates that (2.3) was used. This completes the proof. □

Theorem 4.1. *Assume*

(A_1) *there exists* $L > 0$ *with*

$$
|f(t,x) - f(t,y)| \leq L|x-y| \quad \textit{and} \quad f(t,0) = 0
$$

for all $t \in \mathcal{T}$ *and all* $x, y \in \mathbb{R}^p$,

(A_2) $M \in (0,1)$, *where*

$$M = (M_A + LM_B)d \quad \text{with} \quad M_A = \|A\|_\infty, \quad M_B = \|B\|_\infty.$$

Then (3.2) *has at least one bounded solution.*

Proof. Let $r > 0$ and define

$$\Omega = \{x : \mathcal{T} \to \mathbb{R}^p : \|x\| \le r \quad \text{and} \quad |x(0)| \le r(1-M)\}.$$

Then Ω is a closed, convex, and nonempty subset of the finite-dimensional Banach space $E = \{x : \mathcal{T} \to \mathbb{R}^p\}$ equipped with the norm $\|x\| = \max_{t\in\mathcal{T}} |x(t)|$. We now prove that S and T satisfy all the required conditions of Lemma 2.3.

1. If $x, y \in \Omega$, then, for all $t \in \mathcal{T}$, we have

$$\begin{aligned}
&|(Sx)(t) - (Sy)(t)| \\
&= \Bigg| \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell - 1 + \alpha_{n-1}, s+1) \\
&\qquad \times [B(f(s,x(s)) - f(s,y(s))) - A(x(s) - y(s))] \Bigg| \\
&\le (M_A + LM_B) \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} |h_{\alpha_{n-1}-1}(n\ell - 1 + \alpha_{n-1}, s+1)| \|x - y\| \\
&\overset{(4.2)}{\le} (M_A + LM_B)d\|x-y\| = M\|x-y\|.
\end{aligned}$$

Hence $\|Sx - Sy\| \le M\|x-y\|$. By (A_2), S is a contraction mapping.

2. If $x_n \to x$ in Ω as $n \to \infty$, then, for all $t \in \mathcal{T}$, we have

$$\begin{aligned}
&|(Tx_n)(t) - (Tx)(t)| \\
&\le \rho^t |x_n(0) - x(0)| \\
&\quad + \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} |h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)| \\
&\quad \times |B(f(s,x_n(s)) - f(s,x(s))) - A(x_n(s) - x(s))| \\
&\le \rho^t \|x_n - x\| + (M_A + LM_B)d\|x_n - x\| \le (1+M)\|x_n - x\|,
\end{aligned}$$

and thus

$$0 \le \|Tx_n - Tx\| \le (1+M)\|x_n - x\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, T is continuous. If $x \in \Omega$, then, for all $t \in \mathcal{T}$, we have

$$\begin{aligned}
&|(Tx)(t)| \\
&\le \rho^t |x(0)| + \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} |h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)| |Bf(s,x(s)) - Ax(s)|
\end{aligned}$$

$$\overset{(4.2)}{\leq} r(1-M)+(M_A+LM_B)d\|x\|$$
$$\leq r(1-M)+Mr=r.$$

Hence $\|Tx\| \leq r$. So $T(\Omega)$ is bounded. As a consequence, the closure $\overline{T(\Omega)}$ is also bounded. It is also closed, and as E is of finite dimension, therefore also compact. Since $T(\Omega) \subset \overline{T(\Omega)}$, it is now established that $T(\Omega)$ resides in a compact set.

3. If $x, y \in \Omega$, then, for all $t \in \mathcal{T}$, we have

$$\begin{aligned}
&|(Sx+Ty)(t)| \leq |Sx(t)|+|Ty(t)| \\
&\leq \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} \left|h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1}, s+1)\right| |Bf(s,x(s))-Ax(s)| \\
&\quad + |y(t_0)| + \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} \left|h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)\right| |Bf(s,x(s))-Ax(s)| \\
&\overset{(4.2)}{\leq} r(1-M)+(M_A+LM_B)d\|x\| = r(1-M)+M\|x\| \leq r(1-M)+Mr=r,
\end{aligned}$$

so $\|Sx+Ty\| \leq r$. Also,

$$|(Sx)(0)+(Ty)(0)| \leq |(Sx)(0)|+|(Ty)(0)| \overset{(4.1)}{=} |y(0)| \leq r(1-M).$$

Therefore, $Sx+Ty \in \Omega$. By Lemma 2.3, $P=S+T$ has a fixed point in Ω, which is a solution of (3.2).

This completes the proof. □

5. ULAM–HYERS STABILITY

Definition 5.1 (Ulam–Hyers Stability [18])**.** *We say* (3.2) *is Ulam–Hyers stable if there exists $C>0$ such that for arbitrary $\varepsilon>0$, if y satisfies*

$$\left|{}^{\mathrm{c}}_{k\ell}D^{\alpha_k,\rho} y(t+1-\alpha_k)+Ay(t)-Bf(t,y(t))\right| \leq \varepsilon \quad \textit{for} \quad t \in \mathcal{T}_k, \quad 0 \leq k \leq m-1, \tag{5.1}$$

then there exists a solution x of (3.2) *satisfying*

$$\|x-y\| \leq C\varepsilon.$$

First, we will prove following lemma.

Lemma 5.1. *If y satisfies* (5.1)*, then*

$$\begin{aligned}
&\Bigg| y(t)-\rho^t y(0)-\sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1}, s+1)[Bf(s,y(s))-Ay(s)] \\
&- \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)[Bf(s,y(s))-Ay(s)]\Bigg| \leq d\varepsilon \quad \textit{for} \quad t \in \mathcal{T},
\end{aligned} \tag{5.2}$$

where d is given in (A_2).

Proof. If y satisfies (5.1), then there exists a function g satisfying $\|g\| \le \varepsilon$ such that

$$ {}^{\mathrm{c}}_{k\ell}D^{\alpha_k,\rho}\, y(t+1-\alpha_k) = g(t) - Ay(t) + Bf(t,y(t)) \quad \text{for} \quad t \in \mathcal{T}_k, \quad 0 \le k \le m-1. $$

Replacing x by y and $B(\cdot,x(\cdot)) - Ax$ in (3.4) by $B(\cdot,y(\cdot)) - Ay + g$, we arrive at

$$ \begin{aligned} y(t) =& \rho^t y(0) + \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1}, s+1)[Bf(s,y(s)) - Ay(s) + g(s)] \\ &+ \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)[Bf(s,y(s)) - Ay(s) + g(s)] \end{aligned} $$

for $t \in \mathcal{T}$. Rearranging and taking the norm, we have

$$ \begin{aligned} &\Bigg| y(t) - \rho^t y(0) - \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1}, s+1)[Bf(s,y(s)) - Ay(s)] \\ &\quad - \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)[Bf(s,y(s)) - Ay(s)] \Bigg| \\ &= \Bigg| \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1}, s+1) g(s) \\ &\quad + \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1) g(s) \Bigg| \\ &\le \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} \left| h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1}, s+1) \right| \|g\| \\ &\quad + \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} \left| h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1) \right| \|g\| \\ &\overset{(4.2)}{\le} d\|g\| \le d\varepsilon \quad \text{for} \quad t \in \mathcal{T}. \end{aligned} $$

This completes the proof. □

The following theorem proves the Ulam–Hyers stability of the variable-order fractional discrete neural network (3.2).

Theorem 5.1. *Under assumptions* (A_1) *and* (A_2), (3.2) *is Ulam–Hyers stable.*

Proof. Let $\varepsilon > 0$ and suppose y satisfies (5.1). Let x be the solution of (3.2) with $x(0) = y(0)$. Then

$$ |x(t) - y(t)| $$

$$
\begin{aligned}
&\overset{(3.4)}{=} \Bigg| \rho^t y(0) + \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell - 1 + \alpha_{n-1}, s+1)[Bf(s,x(s)) - Ax(s)] \\
&\quad + \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1) Bf(s,x(s)) - Ax(s)] - y(t) \Bigg| \\
&= \Bigg| y(t) - \rho^t y(0) \\
&\quad - \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell - 1 + \alpha_{n-1}, s+1)[Bf(s,x(s)) - Ax(s)] \\
&\quad - \rho^{t-1} \sum_{s=(m-1)\ell}^{t-1} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)[Bf(s,x(s)) - Ax(s)] \\
&\quad - \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell - 1 + \alpha_{n-1}, s+1)[Bf(s,y(s)) - Ay(s)] \\
&\quad - \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)[Bf(s,y(s)) - Ay(s)] \\
&\quad + \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell - 1 + \alpha_{n-1}, s+1)[Bf(s,y(s)) - Ay(s)] \\
&\quad + \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)[Bf(s,y(s)) - Ay(s)] \Bigg| \\
&\le \Bigg| y(t) - \rho^t y(0) - \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell - 1 + \alpha_{n-1}, s+1)[Bf(s,y(s)) - Ay(s)] \\
&\quad - \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)[Bf(s,y(s)) - Ay(s)] \Bigg| \\
&\quad + \Bigg| \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} \rho^{t-1-s} h_{\alpha_{n-1}-1}(n\ell - 1 + \alpha_{n-1}, s+1)[B(f(s,x(s)) - f(s,y(s))) \\
&\quad - A(x(s) - y(s))] \Bigg| \\
&\quad + \Bigg| \sum_{s=(m-1)\ell}^{t-1} \rho^{t-1-s} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1}, s+1)[B(f(s,x(s)) - f(s,y(s))) \\
&\quad - A(x(s) - y(s))] \Bigg|
\end{aligned}
$$

$$\overset{(5.2)}{\leq} \Bigg| \sum_{n=1}^{m-1} \sum_{s=(n-1)\ell}^{n\ell-1} h_{\alpha_{n-1}-1}(n\ell-1+\alpha_{n-1},s+1)[B(f(s,x(s))-f(s,y(s)))-A(x(s)-y(s))]$$
$$+ \sum_{s=(m-1)\ell}^{t-1} h_{\alpha_{m-1}-1}(t-1+\alpha_{m-1},s+1)[B(f(s,x(s)-f(s,y(s)))-A(x(s)-y(s))]\Bigg| + d\varepsilon$$
$$\overset{(A_1)}{\leq} d\varepsilon + d(M_A L\|x-y\| + M_B\|x-y\|) \overset{(A_2)}{=} d\varepsilon + M\|x-y\|$$

for $t \in \mathcal{T}$. Taking the supremum on both sides of this last inequality results in

$$\|x-y\| \leq d\varepsilon + M\|x-y\|,$$

and rearranging yields

$$\|x-y\| \leq \frac{d}{1-M}\varepsilon.$$

Hence, (3.2) is Ulam–Hyers stable. □

6. Numerical analysis

Example 6.1. *Consider*

$$\begin{aligned} {}^{\mathrm{c}}_{0}D^{\alpha_0,\rho} x(t+1-\alpha_{k\ell}) &= B\tanh x(t) - Ax(t) \quad \textit{for} \quad t \in \mathcal{T}_0, \\ {}^{\mathrm{c}}_{\ell}D^{\alpha_1,\rho} x(t+1-\alpha_1) &= B\tanh x(t) - Ax(t) \quad \textit{for} \quad t \in \mathcal{T}_1, \\ {}^{\mathrm{c}}_{2\ell}D^{\alpha_2,\rho} x(t+1-\alpha_2) &= B\tanh x(t) - Ax(t) \quad \textit{for} \quad t \in \mathcal{T}_2 \end{aligned} \tag{6.1}$$

with

$$A = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.02 \end{bmatrix}, \quad B = \begin{bmatrix} 0.002 & -0.004 & 0.0015 \\ -0.002 & 0.001 & -0.002 \\ -0.0025 & 0.0015 & -0.003 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0.9 \\ 0.6 \\ 0.3 \end{bmatrix}.$$

We see from Table 1 that the parameters satisfy assumptions (A_1) *and* (A_2) *of Theorem 4.1, which assures the existence of the solution. Also, we see that Theorem 5.1 holds, which guarantees Ulam–Hyers stability of* (6.1). *This is shown in Figure 2.*

TABLE 1. Parameters and their values.

Parameter	Value	Parameter	Value	Parameter	Value
k	$0,1,2$	m	3	ℓ	15
t	$\{0,1,\ldots,45\}$	ρ	0.5	α_k	$\{0.05,0.1,0.15\}$
L	1	$\|x(0)\|$	0.9	M_A	0.020
M_B	0.0075	d	2	M	0.0550
		$\frac{d}{1-M}$	2.1164	$r > \frac{\|x(0)\|}{1-M}$	0.95238

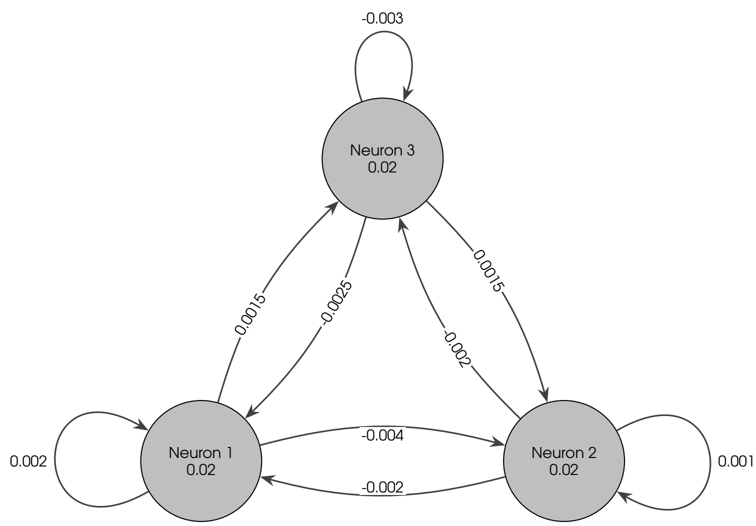

FIGURE 1. Three-neuron network for system (6.1).

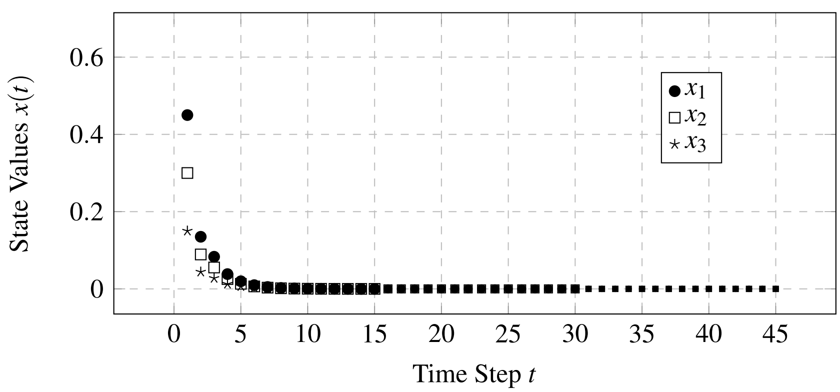

FIGURE 2. Discrete state evolution of (6.1) over time.

Example 6.2. *Consider*

$$ {}_{k\ell}^{\ c}D^{\alpha_k,\rho}x(t+1-\alpha_k) = B\sin x(t) - Ax(t) \quad for \quad t \in \mathcal{T}_k \tag{6.2}$$

with

$$A = \begin{bmatrix} 0.046 & 0 \\ 0 & 0.046 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0003 & 0 \\ -0.0004 & 0.0002 \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0.9 \\ 0.6 \end{bmatrix}.$$

We see from Table 2 that the parameters satisfy assumptions (A_1) and (A_2) of Theorem 4.1, which assures the existence of the solution. Also, we see that Theorem 5.1 holds, which *guarantees Ulam–Hyers stability of* (6.2). *This is shown in Figure 4.*

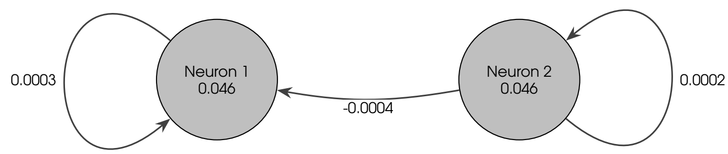

FIGURE 3. Two-neuron network for system (6.2).

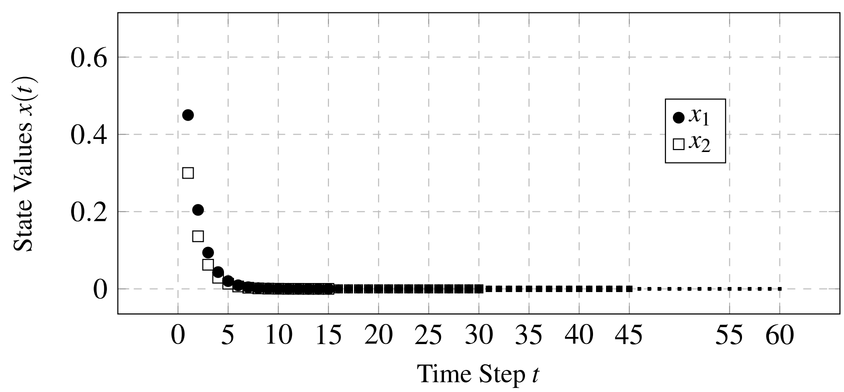

FIGURE 4. Discrete state evolution of (6.2) over time.

Example 6.3. *Consider*

$$ {}_{k\ell}^{\mathrm{c}}D^{\alpha_k,\rho} x(t+1-\alpha_k) = B\sin x(t) - Ax(t) \quad \textit{for} \quad t \in \mathcal{T}_k \tag{6.3}$$

with

$$A = \begin{bmatrix} 0.015 & 0 \\ 0 & 0.015 \end{bmatrix}, \quad B = \begin{bmatrix} -0.0003 & 0 \\ -0.0004 & 0.0002 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -0.001 \\ 0.0009 \end{bmatrix}.$$

We see from Table 3 that the parameters satisfy assumptions (A_1) *and* (A_2) *of Theorem 4.1, which assures the existence of the solution. Also, we see that Theorem 5.1 holds, which guarantees Ulam–Hyers stability of* (6.3). *This is shown in Figure 6.*

TABLE 2. Parameters and their values.

Parameter	Value	Parameter	Value	Parameter	Value
k	$0,1,2,3$	m	4	ℓ	15
t	$\{0,1,\ldots,60\}$	ρ	*0.5*	α_k	$\{0.5,0.4,0.3,0.2\}$
L	*1*	$\lvert x(0)\rvert$	*0.9*	M_A	0.046
M_B	0.0006	d	2	M	0.0932
		$\dfrac{d}{1-M}$	2.205558	$r > \dfrac{\lvert x(0)\rvert}{1-M}$	0.9925

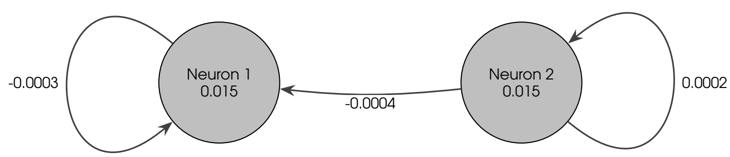

FIGURE 5. Two-neuron network for system (6.3).

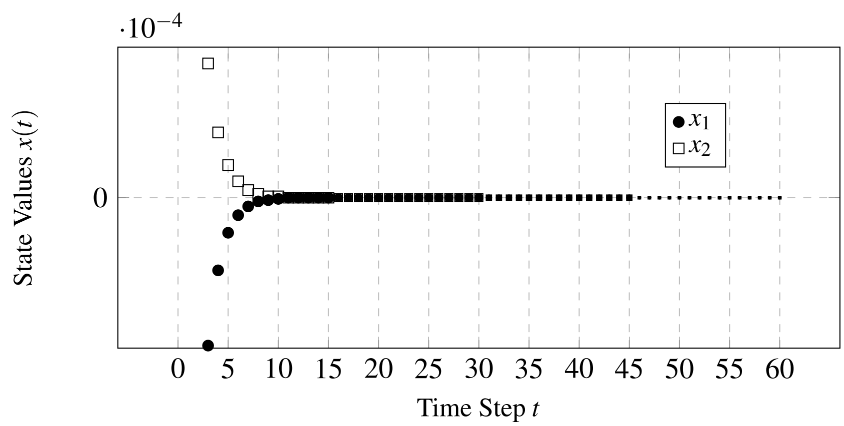

FIGURE 6. Discrete state evolution of (6.3) over time.

Example 6.4. *Consider*

$$ {}_{k\ell}^{c}D^{\alpha_k,\rho}\, x(t+1-\alpha_1) = B\sin x(t) - Ax(t) \quad \textit{for} \quad t \in \mathcal{T}_k \tag{6.4} $$

TABLE 3. Parameters and their values.

Parameter	Value	Parameter	Value	Parameter	Value
k	$0,1,2,3$	m	4	ℓ	15
t	$\{0,1,\ldots,60\}$	ρ	0.5	α_k	$\{0.8,0.6,0.3,0.4\}$
L	1	$\lvert x(0)\rvert$	0.001	M_A	0.015
M_B	0.0006	d	2	M	0.0312
		$\dfrac{d}{1-M}$	2.0644	$r > \dfrac{\lvert x(0)\rvert}{1-M}$	0.00103

with

$$A = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad B = \begin{bmatrix} -0.03 & 0.02 \\ -0.01 & 0.06 \end{bmatrix}, \quad x(0) = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}.$$

We see from Table 4 that the parameters satisfy assumptions (A_1) *and* (A_2) *of Theorem 4.1, which assures the existence of the solution. Also, we see that Theorem 5.1 holds, which guarantees Ulam–Hyers stability of 6.4. This is shown in Figure 8.*

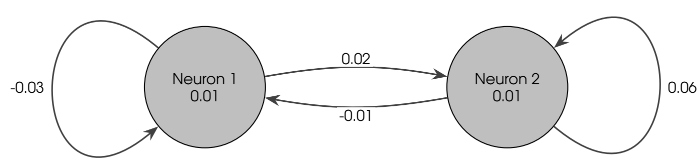

FIGURE 7. Two-neuron network for system (6.4)

7. CONCLUSION

This study introduced a discrete neural network model based on the generalized proportional Caputo fractional operator. Using Krasnosel′skiĭ's fixed-point theorem, we established the existence and Ulam–Hyers stability of the solutions. The results demonstrate that the model is both mathematically well posed and robust under perturbations, offering a solid foundation for memory-based neural computation. Future work may explore extensions with delays, impulses, and real-world applications.

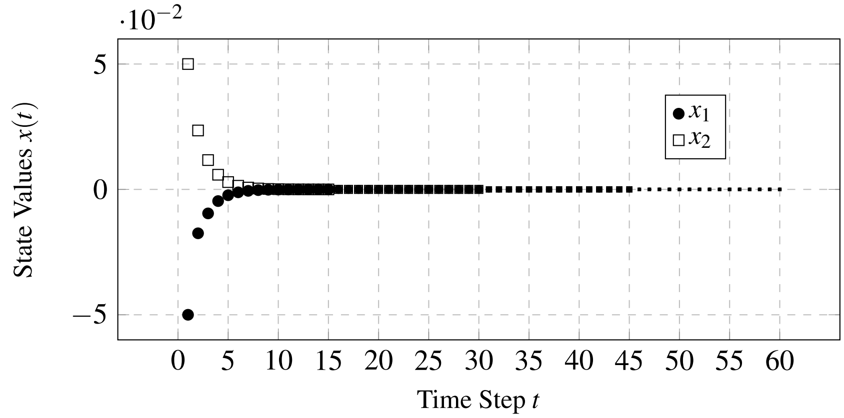

FIGURE 8. Discrete state evolution of (6.4) over time.

TABLE 4. Parameters and their values.

Parameter	Value	Parameter	Value	Parameter	Value
k	$0,1,2,3$	m	4	ℓ	15
t	$\{0,1,\ldots,60\}$	ρ	*0.5*	α_k	$\{0.05,0.05,0.05,0.05\}$
L	*1*	$\lvert x(0)\rvert$	0.1	M_A	0.01
M_B	0.07	d	2	M	0.16
		$\dfrac{d}{1-M}$	2.3809	$r > \dfrac{\lvert x(0)\rvert}{1-M}$	0.11904

REFERENCES

[1] Thabet Abdeljawad, *On Riemann and Caputo fractional differences*, Computers and Mathematics with Applications, **62** (2011), no. 3, 1602–1611.

[2] Ravi P. Agarwal, Juan J. Nieto, and Donal O'Regan, *Existence and stability results for a class of fractional difference equations*, Advances in Difference Equations, **2018** (2018), no. 1, 1–14.

[3] Bilal Ahmad, Shahbaz Mohsin, and M. Ilyas Khan, *Ulam–Hyers–Rassias stability for discrete fractional systems with memory*, Mathematics, **8** (2020), no. 11, 1991.

[4] Anwar Ali, Sabir Hussain, and Amir Khan, *Existence and stability results for impulsive Caputo–Fabrizio neural networks with time delay*, Mathematics and Computers in Simulation, **196** (2022), 328–343.

[5] Serkan Araci, Mehmet Acikgoz, and Waseem Ahmad Khan, *Generalized Ulam–Hyers–Rassias stability of nonlinear fractional discrete equations via fixed point method*, Symmetry, **14** (2022), no. 5, 1015.

[6] Abdon Atangana and Dumitru Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model*, Thermal Science, **20** (2016), no. 2, 763–769.

[7] Dumitru Baleanu and Guillermo Fernández, *Recent advances in discrete fractional calculus and applications*, Mathematics, **9** (2021), no. 2, 138.

[8] Dumitru Baleanu, Guo–Cheng Wu, Yun–Ru Bai, and Fu–Lai Chen, *Stability analysis of Caputo–like discrete fractional systems*, Communications in Nonlinear Science and Numerical Simulation, **48** (2017), 520–530.
[9] Martin Bohner and Rajrani Gupta, *The discrete generalized proportional fractional derivatives*, Journal of Nonlinear and Convex Analysis, **26** (2025), no. 8, 2427–2448.
[10] T.A Burton, *A fixed-point theorem of krasnoselskii*, Applied Mathematics Letters, **11** (1998), no. 1, 85–88.
[11] Xiaoli Cai and Lin Zhao, *New stability criteria for fractional-order neural systems with impulses and delays*, Nonlinear Analysis: Hybrid Systems, **49** (2023), 101366.
[12] Churong Chen, Martin Bohner, and Baoguo Jia, *Ulam-Hyers stability of Caputo fractional difference equations*, Math. Methods Appl. Sci., **42** (2019), no. 18, 7461–7470. MR 4037983
[13] Guillermo Fernández-Anaya and Dumitru Baleanu, *Dynamics of a discrete-time system with proportional memory: Stability and bifurcation analysis*, Mathematics, **10** (2022), no. 5, 705.
[14] Ahlem Gasri, Adel Ouannas, Amina Aicha Khennaoui, Giuseppe Grassi, Taki-Eddine Oussaeif, and Viet-Thanh Pham, *Chaotic fractional discrete neural networks based on the Caputo h-difference operator: stabilization and linear control laws for synchronization*, The European Physical Journal Special Topics, **231** (2022), no. 10, 1815–1829.
[15] Christopher Goodrich and Allan C. Peterson, *Discrete fractional calculus*, Springer, Cham, 2015. MR 3445243
[16] Amel Hioual, Adel Ouannas, Taki-Eddine Oussaeif, Giuseppe Grassi, Iqbal M. Batiha, and Shaher Momani, *On variable-order fractional discrete neural networks: Solvability and stability*, Fractal and Fractional, **6** (2022), no. 2.
[17] Amin Jajarmi, Dumitru Baleanu, and Mehdi Jafari, *A new fractional-order mathematical model for tumor-immune interactions and control strategy using Caputo–Fabrizio derivative*, Chaos, Solitons and Fractals, **122** (2019), 29–40.
[18] Soon-Mo Jung, *Ulam stability of differential equations*, Nonlinear Analysis: Theory, Methods and Applications, **71** (2009), no. 12, e363–e368.
[19] Anatoly A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Preface*, Theory and Applications of Fractional Differential Equations (Anatoly A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, eds.), North-Holland Mathematics Studies, vol. 204, North-Holland, 2006, pp. vii–x.
[20] Chen Li and Guanrong Chen, *Chaos synchronization of fractional-order neural networks with application to secure communication*, IEEE Transactions on Neural Networks and Learning Systems, **29** (2018), no. 9, 4364–4375.
[21] Wenjun Lin, Qing Wang, and Guoqiang Chen, *A novel fractional-order controller for a class of discrete-time chaotic systems*, Chaos, Solitons and Fractals, **130** (2020), 109417.
[22] J. Tenreiro Machado, Fernando B. Duarte, and Alexandra M. Galhano, *A review on fractional order models of neurons and synapses*, Nonlinear Dynamics, **63** (2011), 615–623.
[23] Bijan Mandal and Sudhakar Singh, *Generalized Ulam–Hyers stability for discrete time Caputo-type impulsive systems*, Journal of Computational and Applied Mathematics, **387** (2021), 112609.
[24] C.A. Monje, Y.Q. Chen, B.M. Vinagre, D. Xue, and V. Feliu-Batlle, *Fractional-order systems and controls: Fundamentals and applications*, Advances in Industrial Control, Springer London, 2010.
[25] Sumati Kumari Panda, A.M. Nagy, Velusamy Vijayakumar, and Bipan Hazarika, *Stability analysis for complex-valued neural networks with fractional order*, Chaos, Solitons and Fractals, **175** (2023), 114045.
[26] Igor Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Mathematics in science and engineering, Academic Press, London, 1999.
[27] Sajjad Ullah, Bilal Ahmad, and Gul Rahman, *Analysis and control of a discrete Caputo–Fabrizio neural network with delay*, Alexandria Engineering Journal, **72** (2023), 99–108.
[28] Feng Wang and Xiaolong Guo, *Stability and bifurcation analysis of discrete-time fractional neural networks with delays*, Mathematical Methods in the Applied Sciences, **45** (2022), no. 13, 7764–7780.

[29] Xing Yang, Bin Wang, and Anmin Luo, *Modeling cognitive memory with fractional-order recurrent neural networks*, Cognitive Computation, **14** (2022), no. 1, 105–117.
[30] Xiaowei Yu and Jin Wu, *On a discrete-time neural network model with memory and fractional dynamics*, Applied Mathematics and Computation, **403** (2021), 126217.
[31] Jiguang Zhang and Yaping Ma, *Solvability of a fractional neural network model with delay via fixed point approach*, Discrete Dynamics in Nature and Society, **2021** (2021), 1–10.
[32] Hao Zhou and Xian Chen, *Solvability analysis of impulsive discrete fractional boundary value problems with multiple delays*, Advances in Difference Equations, **2020** (2020), no. 1, 1–12.

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