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CESARO-LIKE OPERATORS

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ABSTRACT. In previous work it was shown that the lower triangular generalized Hausdorff matrix H_α with nonzero entries $h_{nk} = (n+\alpha+1)^{-1}$, for $\alpha \geq 0$, is subnormal on ℓ^2 if and only if $\alpha = 0,1,2,\ldots$. For $0 < h \leq 1$, the weighted Cesaro operator $C_h' : \{a_n\} \to \{b_n\}$ on ℓ^2, when $b_n = (a_0 + d_1 a_1 + \cdots + d_n a_n)/(n+1)d_n$, is subnormal when $d_j^2 = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$. In this paper we show that, when $d_j = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$, the square of the weights chosen above, then the corresponding operator C_h is bounded on ℓ^2 for $0 < h < 3/2$, that H_α is bounded on ℓ^2 for all non-integer $\alpha < 0$, and that C_h is closely related to H_{h-1}. This relationship leads to our main result that C_h is only subnormal when $h = 1$, when it corresponds to the original Cesaro operator with $\alpha = 0$ and each $d_j = 1$.

1. INTRODUCTION AND SUMMARY OF RESULTS

The Cesaro operator $C : \{a_n\} \to \{b_n\}$ on ℓ^2, where $b_n = (a_0 + a_1 + \cdots + a_n)/(n+1)$ was shown, in [6], to be subnormal, which answered a question raised in [1]. For $0 < h \leq 1$, the weighted Cesaro operator C_h' on ℓ^2, with

$$b_n = (a_0 + d_1 a_1 + \cdots + d_n a_n)/(n+1)d_n, \tag{1.1}$$

where $d_j^2 = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$, was shown to be subnormal in [5]. For $\alpha \geq 0$, another generalization of C, the lower triangular generalized Hausdorff matrix H_α, with $h_{nk} = (n+\alpha+1)^{-1}$, was shown, in [3], to be subnormal for $\alpha = 0,1,2,\ldots$. The question of the subnormality of H_α for noninteger $\alpha > 0$ was settled negatively in [7].

In this note we consider the transformation C_h in [1] with weights

$$d_j = \frac{\Gamma(j+1)\Gamma(h)}{\Gamma(j+h)}, \tag{1.2}$$

the square of the weights chosen in [5]. We show that these are also bounded operators on ℓ^2 for $0 < h < 3/2$, and that they are closely related to the operators H_α in [3], but for $-1 < \alpha \leq 0$.

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We shall show that H_α is bounded for all non-integer $\alpha < 0$, but not subnormal. The relation between C_h and H_α then yields our main result that C_h is not subnormal, except for $h = 1$, when it is the original Cesaro operator studied in [1]

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2. A WEIGHTED CESARO OPERATOR.

For each sequence $\{d_n\}, d_n > 0$, define the transformation C_d on ℓ^2 by

$$C_d\{a_j\} = \{b_j\}, \quad \text{where} \quad b_j = \frac{(a_0d_0 + a_1d_1 + \cdots + a_jd_j)}{(j+1)d_j}. \tag{1.1'}$$

If $0 < h \leq 1$ and $d_j^2 = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$, it was shown in [5] that C_d is a subnormal operator, a generalization of the result in [6] for the Cesaro operator. We now consider the case $d_j = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$, and denote the corresponding linear transformation on ℓ^2 by C_h. Our goal is to determine whether or not C_h is subnormal.

Lemma 2.1. *If* $0 < h < 3/2$ *and* $d_j = \Gamma(j+1)\Gamma(h)/\Gamma(j+h)$, $j = 0,1,2,\ldots$, *then* C_h *is bounded on* ℓ^2. *If* $h \geq 3/2$, *then* C_h *is unbounded on* ℓ^2.

Proof. If $0 < h \leq 1$, the proof in [6] also applies here. Namely, if $h = 1$, then C_h is the Cesaro operator and the result was proved in [1, p. 130]. Since $\{d_n\}$ is non-decreasing for $0 < h \leq 1$, the proof follows from the inequality

$$\left|\frac{\sum_{j=0}^n d_j a_j}{d_n(n+1)}\right| \leq \frac{\sum_{j=0}^n |a_j|)}{n+1}.$$

Since $\|C\| = 2$, [1], $\|C_h\| \leq 2$ when $0 < h \leq 1$.

Now assume that $1 < h < 3/2$. The lower triangular matrix C_h is a factorable matrix of the form $(C_h)_{nk} = a_n b_k$, where

$$a_n = \frac{1}{(n+1)d_n} = \frac{\Gamma(n+h)}{\Gamma(h)\Gamma(n+2)} = O(n^{h-2}),$$

and, by [8], page 47,

$$b_k = d_k = O(k^{1-h})$$

Thus, for sufficiently large $n > N_0$ and $k > K_0$, we may assume that C_h is a factorable matrix with entries $a'_n b'_k$, where $a'_n = n^{h-2}$ and $b_k = k^{1-h}$. By Corollary 8(iii) on page 413 of [2], with $p = q = 2$, it follows that C_h is a bounded operator on ℓ^2 if and only if $2-h > 1/2$ and $(2-h)+(h-1) \geq 1/2+1/2$.

Thus a necessary and sufficient condition for C_h to be a bounded operator on ℓ^2 is that $h < 3/2$. □

Lemma 2.2. *The point spectrum of C_h^* is the open disk*

$$\left\{\lambda : \left|\lambda - \frac{1}{3-2h}\right| < \frac{1}{3-2h}\right\},$$

for each $0 < h < 3/2$.

Proof. If $f = \{f(n)\}$ is in ℓ^2 and $C_h^* f = \lambda f$, then, as in the proof of Lemma 2.2 of [5], page 238, with $d_n = \Gamma(n+1)\Gamma(h)/\Gamma(n+h)$,

$$f(n) = \frac{\Gamma(n+1)\Gamma(h)}{\Gamma(n+h)} \prod_{j=1}^{n} \left(1 - \frac{1}{j\lambda}\right) f(0).$$

Suppose that

$$\left|\lambda - \frac{1}{3-2h}\right|^2 < \left(\frac{1}{3-2h}\right)^2,$$

or, equivalently, if $\mu = 1/\lambda$, $2Re(\mu) > 3-2h$; i.e.,
$2Re(\mu) = 3-2h+\varepsilon$ for some $\varepsilon > 0$. Then

$$|f(n)|^2 = \left|\frac{\Gamma(n+1)\Gamma(h)}{\Gamma(n+h)} \prod_{j=1}^{n} \left(1 - \frac{\mu}{j}\right) f(0)\right|^2,$$

where

$$\begin{aligned}\left|1 - \frac{\mu}{j}\right|^2 &= 1 - \frac{2Re(\mu)}{j} + \frac{|\mu|^2}{j^2} \\ &= 1 - \frac{3-2h+\varepsilon}{j} + \frac{|\mu|^2}{j^2} \\ &\le exp\left(\frac{|\mu|^2}{j^2} - \frac{3-2h+\varepsilon}{j}\right).\end{aligned}$$

It follows from the estimate $\Gamma(n+1)\Gamma(h)/\Gamma(n+h) = O(n^{1-h})$, page 57 of [8], and the argument on page 130 of [1], that f is in ℓ^2, and hence every λ satisfying $|\lambda - 1/(3-2h)| < 1/(3-2h)$ is an eigenvalue of C_h^*. That these are all of the eigenvalues follows as in [1]. □

Corollary 2.1. *Let $T_h = I - C_h$, for $0 < h < 3/2$. The point spectrum of T_h^* is*

$$\{\lambda : |\lambda - (2-2h)/(3-2h)| < 1/(3-2h)\}.$$

Proof. $C_h^* f = \lambda f$ if and only if $(I - C_h^*) f = (1-\lambda) f$. □

Following the constructions in [6], [5], and [3], for each f in ℓ^2, define F by $F(z) = \langle f, \phi_{\overline{z}} \rangle$, for all $|z - (2-2h)/(3-2h)| < 1/(3-2h)$, where $T_h^* \phi_z = z\phi_z$, and $\phi_z(0) = 1$. Let $\mathcal{H}$ denote the set of all functions F, and define $\|F\|_{\mathcal{H}} = \|f\|_{\ell_2}$.

Theorem 2.1. *For all $1/2 \le h \le 3/2$, $\mathcal{H}$ is the space of functions $F(z)$ analytic for $|z - (2-2h)/(3-2h)| < 1/(3-2h)$. The operator T_h in ℓ_2 is unitarily equivalent to the operator in $\mathcal{H}$ which maps $F(z)$ into $zF(z)$ in $\mathcal{H}$. The functions $\psi_0(z) =$*

$1, \psi_n(z) = d_n(z-1)^{-n}z(z-1/2)\cdots(z-(n-1)/n), n \geq 1,$ *form an orthonormal basis for $\mathcal{H}$. The reproducing kernel function for $\mathcal{H}$ is*

$$K(w,z) = \Sigma\psi_n(z)\overline{\psi}_n(w) = {}_3F_2\Big(-\frac{\overline{w}}{1-\overline{w}} - h + 1, \frac{-z}{1-z} - h + 1; h, h, 1\Big).$$

Proof. The correspondence $f \leftrightarrow F$ is 1 - 1 since the functions ϕ_λ span a dense subset of ℓ^2 (or H^2) when $1/2 \leq h < 3/2$. The numbers $\lambda_n = 1/n, n = 1, 2, \ldots,$ are in the point spectrum of T_h^*, and ϕ_λ is a polynomial of degree $n-1$, by the proof of Lemma **??**. So, the span of all of the ϕ_λ includes all of the polynomials, which are dense in ℓ^2.

The proofs of the other statements in Theorem 1 are the same as in those on page 216 of [6] and page 238 of [5]. An orthonormal basis for $\mathcal{H}$ can also be derived by a direct computation. □

Note that, for $0 < h \leq 1/2$, 0 is not in the spectrum of T_h, so that the above proof that $f \leftrightarrow F$ is a 1-1 correspondence does not hold. This observation is due to Stefan Maurer, as is the following lemma.

Lemma 2.3. *For $0 < h < 1/2$, the correspondence $f \leftrightarrow F$ between ℓ^2 and $\mathcal{H}$ is not* $1-1$.

Proof. Since $\Gamma(n+h)/\Gamma(n+1) = O(n^{h-1})$ from page 47 of [8], $f = \{f(n+h) / \Gamma(h)\Gamma(n+1)\}$ is in ℓ^2 when $0 < h < 1/2$. We shall show that the corresponding analytic function $F(z)$ in $\mathcal{H}$ vanishes for all real z, and hence vanishes identically.

From the proof of Lemma 2.2, $F(z) = \langle f, \phi_{\overline{z}}\rangle$, where $\phi_{\overline{z}}(0) = 1$ and

$$\phi_{\overline{z}}(n) = \frac{\Gamma(n+1)\Gamma(h)}{\Gamma(n+h)}\prod_{j=1}^{n}\Big(1 - \frac{1}{j\overline{z}}\Big), \quad n = 1, 2, \ldots.$$

So, for real $c = 1/z, c = 3/2 - h + \varepsilon/2 > 1 + \varepsilon/2$, and

$$F(z) = 1 + \sum_{n=1}^{\infty}\prod_{j=1}^{n}\Big(1 - \frac{1}{jz}\Big) = 1 + \sum_{n=1}^{\infty}\prod_{j=1}^{n}\Big(1 - \frac{c}{j}\Big) = 0,$$

using the binomial expansion for $(1+x)^{c-1}$ with $x = -1$. □

A consequence of Lemma 2.3 is that T_h in ℓ^2 is not unitarily equivalent to $F(z) \to zF(z)$ in $\mathcal{H}$ for $0 < h < 1/2$. For our purposes it will be sufficient to continue with an analysis of T_h for $1/2 \leq h < 3/2$.

The mapping $z \to z/(1-z)$ takes the disk $|z - (2-2h)/(3-2h)| < 1/(3-2h)$ onto the half plane $Re(w) > -h + 1/2$. The inverse map is $w \to w/(1+w)$.

Let $\mathcal{K}$ denote the set of functions F of the form $F(z) = G(z/(1+z))$ for some G in $\mathcal{H}$, where $Re(z) > -h + 1/2$. Using the conformal mapping, the orthonormal basis $\{(-1)^n\Psi_n(z)\}$ for $\mathcal{H}$ is mapped into the orthonormal basis $\psi_0(z) = 1, \psi_n(z) = (d_n/n!)z(z-1)\cdots(z-n+1), n > 1$, for $\mathcal{K}$, and the reproducing kernel function for $\mathcal{K}$ is

$$K(w,z) = \sum\psi_n(z)\overline{\psi}_n(w) = {}_3F_2(w + h - 1, z + h - 1; 1, h, h, 1).$$

It follows, as on page 218 of [6] and page 239 of [5] that, for $0 < a, b < 1, a^z$ and b^z are in $\mathcal{K}$ and

$$\langle a^z, b^z \rangle_{\mathcal{K}} = \sum_{n=0}^{\infty} \frac{(1-a)^n (1-b)^n}{d_n^2} = \sum_{n=0}^{\infty} (1-a)^n (1-b)^n \frac{\Gamma^2(n+1)\Gamma^2(h)}{\Gamma^2(n+h)}$$
$$= F(h, h; 1; (1-a)(1-b)).$$

Let $\mathcal{K}'$ denote all functions $g(z)$ of the form $g(z) = f(z+h-1)$ for some f in $\mathcal{K}$, and define

$$\|g(z)\|_{\mathcal{K}'} = \|f(z)\|_{\mathcal{K}}.$$

Then, for $0 < a, b < 1$, a^z and b^z belong to $\mathcal{K}'$, and

$$\langle a^z, b^z \rangle_{\mathcal{K}'} = (ab)^{h-1} \langle a^z, b^z \rangle_{\mathcal{K}} = (ab)^{h-1} F(h, h, ; 1; (1-a)(1-b)).$$

This formula also appears on page 262 of [3],where it was shown to hold for $h \geq 1$. In our case, $0 < h \leq 1$, by Lemma 2.1. However, a careful review of the results on page 262 of [3] shows that they hold not only for $h \geq 1$, but also for $0 < h < 1$; i.e., $-1 < \alpha < 0$. The result that the operator H_α in [3] is bounded for $-1 < \alpha < 0$ will be proved in the next section. It follows, from a comparison of the Hilbert space $\mathcal{K}'$ above with the Hilbert space $\mathcal{K}_\alpha$ in [3], with $\alpha = h-1$ and $0 < h < 1$, that they have the same orthonormal basis. Hence they are identical.

By Theorem 1, there is a unitary operator U_1, from ℓ^2 onto $\mathcal{K}$ such that C_h is unitarily equivalent to $F(z) \to F(z)/(1+z)$ in $\mathcal{K}$, and a unitary operator U_2, from ℓ^2 onto $\mathcal{K}'$, such that C_h is unitarily equivalent to $F(z) \to F(z)/(z+2-h)$ in $\mathcal{K}'$. Since $\mathcal{K}_{h-1}$ in [3] is the same space as $\mathcal{K}'$, the operator H_{h-1} in [3] is unitarily equivalent to $F(z) \to F(z)/(1+z)$ in $\mathcal{K}'$. Therefore we have the following identities relating C_h and H_{h-1}:

$$H_{h-1}[I - (1-h)C_h] = C_h, \tag{2.1}$$

and

$$C_h[I - (h-1)H_{h-1}] = H_{h-1}. \tag{2.2}$$

For certain values of h these identities can be simplified. If $1/2 < h \leq 1$, then, by the proof of Lemma 2.1, $\|C_h\| \leq 2$ and $\|(1-h)C_h\| \leq 2(1-h) < 1$. Thus $I - (1-h)C_h$ is invertible and

$$H_{h-1} = C_h[I - (1-h)C_h]^{-1}. \tag{2.1$'$}$$

It now follows, from [4] that, for $1/2 < h \leq 1$,

$$C_h = H_{h-1}[I - (h-1)H_{h-1}]^{-1}, \tag{2.2$'$}$$

that $I - (1-h)C_h$ and $I - (h-1)H_{h-1}$ are bounded inverses of each other, and that $C_h H_{h-1} = H_{h-1} C_n$.

Since C_h is bounded for $0 < h < 3/2$ and, as we shall show in Lemma 3.1, H_α is bounded for $\alpha > -1$; i.e., $h > 0$, the operators in (2.1$'$) and (2.2$'$) are bounded for $0 < h < 3/2$. Since all of the matrix entries are polynomials in h (or rational functions of $\mathcal{H}$ with poles at the negative integers), and (2.1$'$) and (2.2$'$) hold for

$1/2 < h \leq 1$, it follows that they are also true for $0 < h < 3/2$. (Thus, as pointed out by Larry Zalcman, we need not appeal to the Identity Theorem for analytic functions.)

In the next section we shall use identities (2.1′) and (2.2′) to show that C_h is not subnormal, except for $h = 1$.

3. Generalized Hausdorff Operators H_α for $\alpha < 0$.

As in [3], let H_α denote the lower triangular matrix with entries $h_{nk} = 1/(n+\alpha+1)$. From [4], H_α is bounded on ℓ^2 for $\alpha \geq 0$.

Lemma 3.1. *For $-1 < \alpha < 0, H_\alpha$ is bounded on ℓ^2*

Proof. Note that the nonzero terms of H_α are

$$h_{nk} = \frac{d_n}{n+1}, \quad \text{where} \quad d_n = \frac{(n+1)}{(n+\alpha+1)}.$$

Therefore $H_\alpha = DC$, where D is the diagonal matrix with diagonal entries (d_n) and C is the Cesaro matrix of order 1.

Since C is known to be a bounded operator on ℓ^2 ([1]), to prove the lemma it will be sufficient to show that the sequence $\{d_n\}$ is bounded.

By inspection, for each $n \geq 0$,

$$|d_n| \leq \max\left\{\frac{1}{\varepsilon}, 2\right\},$$

where $\varepsilon = \alpha + 1$ denotes the distance from α to -1. Therefore D is bounded by $1/\varepsilon$ and H_α is bounded on ℓ^2 for $-1 < \alpha < 0$. □

For completeness we present a more general result. The symbol $\mathbb{N}$ denotes the set of positive integers.

Lemma 3.2. *For any $k \in \mathbb{N}$, if $-k < \alpha < -k+1$, then H_α is bounded on ℓ^2.*

Proof. As in the proof of Lemma 3.1, define $\varepsilon_1 = \alpha + k$, the distance from α to $-k$, and $\varepsilon_2 = \alpha + k - 1$, the distance from α to $-k+1$. Then D is bounded by $\max\{k/\varepsilon_1, (k+1)/\varepsilon_2\}$, and H_α is bounded on ℓ^2 for $-k < \alpha < -k+1$. □

Remark. It is clear from the above arguments that H_α is also bounded on ℓ^p for $p > 1$, and for all non-integer $\alpha < 0$.

The next lemma is needed to show that H_α is not subnormal on ℓ^2 for non-integer $\alpha < 0$.

Lemma 3.3. *If H_α is subnormal on ℓ^2 for any $\alpha > -1$, and $n \in \mathbb{N}$, then $H_{\alpha+n}$ is also subnormal on ℓ^2.*

Proof. Since the proof is by induction, it is sufficient to provide a proof for $n = 1$. If the first row and column of H_α are deleted, the resulting matrix is $H_{\alpha+1}$. Thus, $H_{\alpha+1}$ may be regarded as the restriction of H_α to the closed invariant subspace of ℓ^2 consisting of all sequences $\{a_n\} \in \ell^2$ of the form $\{0, a_1, a_2, \ldots\}$. Since the restriction of a subnormal operator to a closed invariant subspace is clearly also subnormal, $H_{\alpha+1}$ is subnormal. □

Corollary 3.1. *H_α is not subnormal for $-1 < \alpha < 0$.*

Proof. The proof is by contradiction. Let $-1 < \alpha < 0$ and assume that H_α is subnormal. Then $0 < \alpha + 1 < 1$, and, by Lemma 3.3, $H_{\alpha+1}$ is subnormal. But this contradicts the result in [7] that H_α is not subnormal for any non-integer $\alpha > 0$. □

Corollary 3.2. *H_α is not subnormal for any non-integer $\alpha < 0$.*

Proof. The result clearly follows from Corollary 2, Lemma 3.3, and an induction argument. □

Theorem 3.1. *C_h is not subnormal for any $0 < h < 3/2$, except for $h = 1$, the Cesaro operator.*

Proof. If $h \neq 1$, and C_h is subnormal, then so is $[I - (1-h)C_h]^{-1}$. Thus, by (2.1′), so is H_{h-1}. If $0 < h < 1$, we have a contradiction to Corollary 2. If $1 < h < 3/2$, we have a contradiction to the result in [7] that H_α is not subnormal for non-integer $\alpha > 0$. □

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