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AN EXPLICIT FORMULA FOR LEBESGUE CONSTANTS ON VILENKIN GROUPS

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ABSTRACT. We derive an explicit formula for Lebesgue constants on Vilenkin groups.

1. INTRODUCTION

The following result (see [6], [8], [11]) is classical in Fourier analysis.

Theorem 1.1. *There exists a continuous function* $f : \mathbb{T} \to \mathbb{C}$ *whose Fourier series diverges at a point* $t \in \mathbb{T}$.

A quick way to prove this theorem is via the uniform boundedness principle combined with asymptotic behaviour of the sequence $(L_n)_{n=1}^{\infty}$, where L_n is given by

$$L_n = ||D_n||_{L^1(T)},$$

and D_n is the Dirichlet kernel on $\mathbb{T}$. This highlights the significance of the sequence $(L_n)_{n=1}^{\infty}$ in harmonic analysis. The constants L_n are called Lebesgue constants. Explicit formulas for Lebesgue constants were given by Fejér (see [3]), Szegő (see [9]) and Hardy (see [4]).
We give an explicit formula for Lebesgue constants on Vilenkin groups. Our derivation is based on the decomposition of the Dirichlet kernel given in [7].

2. PRELIMINARIES

We denote by $\mathbb{N}$ the set of nonnegative integers. If n is a positive integer, then

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$$

is a topological group with respect to the discrete topology and the operation of addition modulo n. Let $(m_i)_{i=0}^{\infty}$ be a sequence of positive integers greater than 1. Let

$$G = \prod_{i=0}^{\infty} \mathbb{Z}_{m_i}.$$

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We equip G with the product topology and componentwise addition. It is known that G is a Vilenkin group (see [1], [10]). We define the sequence of positive integers $(M_i)_{i=0}^{\infty}$ by

$$M_0 = 1, \quad M_{i+1} = m_i \cdot M_i,\ i \geq 0.$$

Then (see [1], [10]), every $n \in \mathbb{N}$ can be represented in a unique way as

$$n = \sum_{i=0}^{\infty} n_i \cdot M_i, \quad n_i \in \{0,1,\dots,m_i-1\}. \tag{2.1}$$

The relation (2.1) is called the representation of n. Set

$$I_0 = G, \quad I_n = \{(x_i)_{i=0}^{\infty} \in G \,|\, x_0 = \cdots = x_{n-1} = 0\}, n \geq 1.$$

Each set I_n is both open and closed in G, and they satisfy

$$I_0 \supseteq I_1 \supseteq \dots I_n \supseteq \dots.$$

It is known (see [1], [5]) that there is a unique Haar measure μ on G which satisfies $\mu(G) = 1$. In particular, we have

$$\mu(x + I_n) = \mu(I_n) = \frac{1}{M_n}, \forall x \in G, \forall n \in \mathbb{N}. \tag{2.2}$$

The integral of a measurable function $f : G \to \mathbb{C}$ over a measurable set $A \subseteq G$ (with respect to the measure μ) is denoted by

$$\int_A f(x)d\mu(x) \text{ or } \int_A f(x)dx.$$

The L^1-norm of a measurable function $f : G \to \mathbb{C}$ is denoted by $||f||_1$. For each nonnegative integer k, define the function $r_k : G \to \mathbb{C}$ as follows

$$r_k(x) = e^{\frac{2\pi i x_k}{m_k}}, \quad x = (x_i)_{i=0}^{\infty} \in G.$$

For every nonnegative integer n, define the function ψ_n as

$$\psi_n(x) = \prod_{k=0}^{\infty} r_k^{n_k}(x),\ x \in G,$$

where $n = \sum_{i=0}^{\infty} n_i \cdot M_i$ is the representation of n. Then (see [1], [10]), $(\psi_n)_{n=0}^{\infty}$ is an orthonormal system on G. The system $(\psi_n)_{n=0}^{\infty}$ is called the Vilenkin system. The Dirichlet kernel is defined as

$$D_0(x) = 0, D_n(x) = \sum_{i=0}^{n-1} \psi_i(x), n \geq 1, x \in G.$$

It is known (see [1], [2]) that

$$D_{M_n}(x) = \begin{cases} M_n, & x \in I_n, \\ 0, & x \notin I_n, \end{cases} \tag{2.3}$$

for every nonnegative integer n. Define the sequence $(e_i)_{i=0}^{\infty}$ of elements in G as

$$e_i = (\delta_{j,i})_{j=0}^{\infty},$$

where $\delta_{j,i}$ denotes the Kronecker symbol. Let $j = \sum_{i=0}^{\infty} j_i \cdot M_i$ be the representation of $j \in \mathbb{N}$. We define y_j as

$$y_j = \sum_{i=0}^{\infty} j_i e_i.$$

We will use the following lemma (see [7], Lemma 3.1):

Lemma 2.1. *Let $j \in \mathbb{N}$. Suppose*

$$j = \sum_{i=0}^{\infty} j_i \cdot M_i, j_i \in \{0, 1, \ldots, m_i - 1\}$$

is the representation of j. Then:

i) For each positive integer k

$$j \leq M_k - 1 \Leftrightarrow y_j = j_0 \cdot e_0 + j_1 \cdot e_1 + \ldots + j_{k-1} \cdot e_{k-1}.$$

ii) For each positive integer k, the function

$$f_k : \{0, 1, \ldots, M_k - 1\} \to \{\alpha_0 \cdot e_0 + \ldots \alpha_{k-1} \cdot e_{k-1} : \alpha_i \in \{0, 1, \ldots, m_i - 1\}, i = \overline{0, k-1}\}$$

defined by

$$f_k(j) = \sum_{i=0}^{k-1} j_i \cdot e_i, \forall j = \sum_{i=0}^{\infty} j_i M_i \in \{0, 1, \ldots, M_k - 1\} \tag{2.4}$$

is bijective.

Notice that (2.4) is equivalent to $f_k(j) = y_j$, $j = \overline{0, M_k - 1}$.

iii) For each positive integer k

$$G = \biguplus_{j=0}^{M_k - 1} (y_j + I_k). \tag{2.5}$$

iv) For each $k, j \in \mathbb{N}$

$$y_j \in I_k \Leftrightarrow j \equiv 0 \,(mod\, M_k).$$

In particular,

$$y_j \in I_k \backslash I_{k+1} \Leftrightarrow j \equiv 0 \,(mod\, M_k) \wedge j \not\equiv 0 \,(mod\, M_{k+1}).$$

v) Let $i, j \in \mathbb{N}$ be such that $0 \leq i \leq n-1$. Then for every $j \in \{1, \ldots, M_n - 1\}$ we have

$$j \equiv 0 \,(mod\, M_i), j \not\equiv 0 \,(mod\, M_{i+1}) \tag{2.6}$$

if and only if the representation of j is of the form

$$j = j_i M_i + j_{i+1} M_{i+1} + \cdots + j_{n-1} M_{n-1}, j_i \neq 0. \tag{2.7}$$

Proof. Claims *i)-iv)* are proven in [7]. Therefore, we only prove claim *v)*.
Choose any $j \in \{1, \dots, M_n - 1\}$. Then, we must have $j_k = 0, \forall k \geq n$. Therefore, $j = \sum_{k=0}^{n-1} j_k M_k$ is the representation of j. Clearly, (2.7) implies (2.6). Conversely, suppose (2.6) holds. Since $\sum_{k=0}^{i-1} j_k M_k \equiv \sum_{k=0}^{n-1} j_k M_k = j \equiv 0\,(mod\, M_i)$, and $0 \leq \sum_{k=0}^{i-1} j_k M_k \leq \sum_{k=0}^{i-1}(m_k - 1)M_k = M_i - 1$, we must have $j_0 = \cdots = j_{i-1} = 0$. The condition $j \not\equiv 0\,(mod\, M_{i+1})$ implies $j_i \neq 0$. Therefore, j has the representation (2.7). □

We will also use the following decomposition of the Dirichlet kernel (see [7], Theorem 3.1):

Theorem 2.1. *Let n be a positive integer and*

$$n = n_{k_1} M_{k_1} + \ldots + n_{k_l} M_{k_l}$$

be the representation of n, where $0 \leq k_1$, $k_i < k_j$ for $i < j$, and $n_{k_i} \in \{1, \dots, m_{k_i} - 1\}, i = \overline{1, l}$. Then

$$D_n(x) = \psi_n(x) \left(\sum_{j=0}^{M_{k_l}-1} a_{n,j} \cdot D_{M_{k_l}}(x - y_j) \right), x \in G,$$

where $a_{n,j}, j = \overline{1, M_{k_l} - 1}$ are constants, and $a_{n,0}$ is a function on G. Moreover, we have:

i) If $j \not\equiv 0\,(\,mod\, M_{k_1})$, then $a_{n,j} = 0$,

ii) If $j \equiv 0\,(\,mod\, M_{k_p})$, $j \not\equiv 0\,(\,mod\, M_{k_{p+1}})$ for some $p \in \{1, \dots, l-1\}$, then

$$a_{n,j} = \frac{1}{M_{k_l}} \sum_{i=1}^{p-1} n_{k_i} M_{k_i} + \frac{M_{k_p}}{M_{k_l}} \sum_{s=m_{k_p}-n_{k_p}}^{m_{k_p}-1} e^{\frac{2\pi i s a}{m_{k_p}}},$$

where $a = \frac{j\,(\,mod\, M_{k_p+1})}{M_{k_p}}$. In particular, if $j \equiv 0\,(\,mod\, M_{k_p+1})$, then

$$a_{n,j} = \frac{1}{M_{k_l}} \sum_{i=1}^{p} n_{k_i} M_{k_i}.$$

iii)

$$a_{n,0}(x) = \frac{1}{M_{k_l}} \sum_{i=1}^{l-1} n_{k_i} M_{k_i} + \sum_{s=m_{k_l}-n_{k_l}}^{m_{k_l}-1} e^{\frac{2\pi i x_{k_l} s}{m_{k_l}}}, x \in I_{k_l}.$$

3. RESULTS

The following theorem gives us an explicit formula for Lebesgue constants on Vilenkin groups.

Theorem 3.1. *Let n be a positive integer and let*

$$n = n_{k_1}M_{k_1} + \ldots + n_{k_l}M_{k_l}$$

be the representation of n, where $0 \leq k_1$, $k_i < k_j$ *for* $i < j$, *and* $n_{k_i} \in \{1, \ldots, m_{k_i} - 1\}, i = \overline{1,l}$. *Then*

$$\begin{aligned} ||D_n||_1 = &\sum_{b=0}^{m_{k_l}-1} \left| \frac{1}{M_{k_l+1}} \sum_{i=1}^{l-1} n_{k_i}M_{k_i} + \frac{1}{m_{k_l}} \sum_{s=m_{k_l}-n_{k_l}}^{m_{k_l}-1} e^{\frac{2\pi isb}{m_{k_l}}} \right| \\ &+ \sum_{\alpha=1}^{l-1} \left(\frac{1}{M_{k_\alpha+1}} - \frac{1}{M_{k_{\alpha+1}}} \right) \cdot \sum_{i=1}^{\alpha} n_{k_i}M_{k_i} \\ &+ \sum_{\alpha=1}^{l-1} \sum_{q=1}^{m_{k_\alpha}-1} \left| \frac{1}{M_{k_\alpha+1}} \sum_{i=1}^{\alpha-1} n_{k_i}M_{k_i} + \frac{1}{m_{k_\alpha}} \sum_{s=m_{k_\alpha}-n_{k_\alpha}}^{m_{k_\alpha}-1} e^{\frac{2\pi isq}{m_{k_\alpha}}} \right|. \end{aligned}$$

Proof. By Theorem 2.1, we get

$$D_n(x) = \psi_n(x) \cdot \left(a_{n,0}(x)D_{M_{k_l}}(x) + \sum_{j=1}^{M_{k_l}-1} a_{n,j} \cdot D_{M_{k_l}}(x - y_j) \right), \forall x \in G, \tag{3.1}$$

where the constants $a_{n,j}, j \in \{1, \ldots, M_{k_l} - 1\}$ and the function $a_{n,0}(x)$ are characterized in the formulation of Theorem 2.1. Now, using (2.5) and (3.1) we obtain

$$\begin{aligned} ||D_n||_1 &= \int_G |D_n(x)|dx = \sum_{j=0}^{M_{k_l}-1} \int_{y_j+I_{k_l}} |D_n(x)|dx \\ &= \int_{I_{k_l}} |a_{n,0}(x)D_{M_{k_l}}(x)|dx + \sum_{j=1}^{M_{k_l}-1} \int_{y_j+I_{k_l}} |a_{n,j} \cdot D_{M_{k_l}}(x - y_j)|dx \\ &= M_{k_l} \cdot \int_{I_{k_l}} |a_{n,0}(x)|dx + \sum_{j=1}^{M_{k_l}-1} |a_{n,j}|, \end{aligned} \tag{3.2}$$

where, in the last step, we used the relations (2.2) and (2.3). From the obvious relation $I_{k_l} = \biguplus_{b=0}^{m_{k_l}-1} (b \cdot e_{k_l} + I_{k_l+1})$ and Theorem 2.1 (claim iii)), we get

$$\begin{aligned}\int_{I_{k_l}} |a_{n,0}(x)|dx &= \sum_{b=0}^{m_{k_l}-1} \int_{b\cdot e_{k_l}+I_{k_l+1}} |a_{n,0}(x)|dx \\ &= \sum_{b=0}^{m_{k_l}-1} \int_{b\cdot e_{k_l}+I_{k_l+1}} \left| \frac{1}{M_{k_l}} \sum_{i=1}^{l-1} n_{k_i} M_{k_i} + \sum_{s=m_{k_l}-n_{k_l}}^{m_{k_l}-1} e^{\frac{2\pi isb}{m_{k_l}}} \right| dx \\ &= \sum_{b=0}^{m_{k_l}-1} \frac{1}{M_{k_l+1}} \cdot \left| \frac{1}{M_{k_l}} \sum_{i=1}^{l-1} n_{k_i} M_{k_i} + \sum_{s=m_{k_l}-n_{k_l}}^{m_{k_l}-1} e^{\frac{2\pi isb}{m_{k_l}}} \right|,\end{aligned} \tag{3.3}$$

where, in the last step, we used (2.2). On the other hand, using Theorem 2.1 (claim i)), we get

$$\begin{aligned}\sum_{j=1}^{M_{k_l}-1} |a_{n,j}| &= \sum_{\substack{j\in\{1,\ldots,M_{k_l}-1\} \\ j\equiv 0\,(mod\,M_{k_1})}} |a_{n,j}| \\ &= \sum_{i=k_1}^{k_l-1} \sum_{\substack{j\in\{1,\ldots,M_{k_l}-1\} \\ j\equiv 0\,(mod\,M_i),\, j\not\equiv 0\,(mod\,M_{i+1})}} |a_{n,j}|.\end{aligned} \tag{3.4}$$

Now, let's calculate the inner sum appearing in (3.4). Let $\alpha \in \{1,\ldots,l-1\}$ and $i \in \{k_\alpha,\ldots,k_{\alpha+1}-1\}$ be arbitrary. We consider two cases:

i) $i = k_\alpha$.
By Lemma 2.1 (claim v)), for any $j \in \{1,\ldots,M_{k_l}-1\}$ we have $j \equiv 0\,(mod\,M_{k_\alpha})$ and $j \not\equiv 0\,(mod\,M_{k_\alpha+1})$ if and only if the representation of j is of the form $j = q\cdot M_{k_\alpha} + j_{k_\alpha+1}M_{k_\alpha+1} + \cdots + j_{k_l-1}M_{k_l-1}$, where $q \in \{1,\ldots,m_{k_\alpha}-1\}$, $j_k \in \{0,1,\ldots,m_k-1\}$ for $k \in \{k_\alpha+1,\ldots,k_l-1\}$. Now, using Theorem 2.1 (claim ii)), we get

$$\sum_{\substack{j\in\{1,\ldots,M_{k_l}-1\} \\ j\equiv 0\,(mod\,M_{k_\alpha}), \\ j\not\equiv 0\,(mod\,M_{k_\alpha+1})}} |a_{n,j}| = m_{k_\alpha+1}\cdots m_{k_l-1} \sum_{q=1}^{m_{k_\alpha}-1} \left| \frac{1}{M_{k_l}} \sum_{i=1}^{\alpha-1} n_{k_i}M_{k_i} + \frac{M_{k\alpha}}{M_{k_l}} \sum_{s=m_{k\alpha}-n_{k\alpha}}^{m_{k\alpha}-1} e^{\frac{2\pi isq}{m_{k\alpha}}} \right|. \tag{3.5}$$

ii) $k_\alpha < i < k_{\alpha+1}$.
Notice that $j\equiv 0\,(mod\,M_i),\, j\not\equiv 0\,(mod\,M_{i+1})$ implies $j\equiv 0\,(mod\,M_{k_\alpha+1}),\, j\not\equiv 0\,(mod\,M_{k_{\alpha+1}})$. By Theorem 2.1 (claim ii)), for any $j \in \{1,\ldots,M_{k_l}-1\}$

such that $j \equiv 0\,(mod\,M_i),\, j \not\equiv 0\,(mod\,M_{i+1})$ we have

$$a_{n,j} = \frac{1}{M_{k_l}} \sum_{r=1}^{\alpha} n_{k_r} M_{k_r}. \tag{3.6}$$

On the other hand, by Lemma 2.1 (claim v)), for any $j \in \{1,\ldots,M_{k_l}-1\}$ we have $j \equiv 0\,(mod\,M_i)$ and $j \not\equiv 0\,(mod\,M_{i+1})$ if and only if the representation of j is of the form $j = q\cdot M_i + j_{i+1}M_{i+1} + \cdots + j_{k_l-1}M_{k_l-1}$, where $q \in \{1,\ldots,m_i-1\}$, $j_s \in \{0,1,\ldots,m_s-1\}$ for $s \in \{i+1,\ldots,k_l-1\}$. Therefore, we have

$$\sum_{\substack{j\in\{1,\ldots,M_{k_l}-1\}\\ j\equiv 0\,(mod\,M_i),\, j\not\equiv 0\,(mod\,M_{i+1})}} 1 = (m_i-1)\cdot m_{i+1}\cdots m_{k_l-1} = (m_i-1)\frac{M_{k_l}}{M_{i+1}}. \tag{3.7}$$

Now, using (3.6) and (3.7), we get

$$\begin{aligned}\sum_{\substack{j\in\{1,\ldots,M_{k_l}-1\},\\ j\equiv 0\,(mod\,M_i),\\ j\not\equiv 0\,(mod\,M_{i+1})}} |a_{n,j}| &= \frac{m_i-1}{M_{i+1}}\cdot\sum_{r=1}^{\alpha} n_{k_r}M_{k_r} \\ &= \left(\frac{1}{M_i}-\frac{1}{M_{i+1}}\right)\cdot\sum_{r=1}^{\alpha} n_{k_r}M_{k_r}.\end{aligned} \tag{3.8}$$

Applying (3.5) and (3.8) for every $\alpha \in \{1,\ldots,l-1\}$ we have

$$\begin{aligned}\sum_{k_\alpha\le i<k_{\alpha+1}}\ \sum_{\substack{j\in\{1,\ldots,M_{k_l}-1\},\\ j\equiv 0\,(mod\,M_i),\\ j\not\equiv 0\,(mod\,M_{i+1})}} |a_{n,j}| &= \left(\frac{1}{M_{k_\alpha+1}}-\frac{1}{M_{k_{\alpha+1}}}\right)\cdot\sum_{i=1}^{\alpha} n_{k_i}M_{k_i} \\ &+ \frac{1}{M_{k_\alpha+1}}\sum_{q=1}^{m_{k_\alpha}-1}\left|\sum_{i=1}^{\alpha-1} n_{k_i}M_{k_i} + M_{k_\alpha}\sum_{s=m_{k_\alpha}-n_{k_\alpha}}^{m_{k_\alpha}-1} e^{\frac{2\pi i s q}{m_{k_\alpha}}}\right|.\end{aligned} \tag{3.9}$$

From (3.9) and (3.4) we get

$$\begin{aligned}\sum_{j=1}^{M_{k_l}-1}|a_{n,j}| &= \sum_{\alpha=1}^{l-1}\left(\frac{1}{M_{k_\alpha+1}}-\frac{1}{M_{k_{\alpha+1}}}\right)\cdot\sum_{i=1}^{\alpha} n_{k_i}M_{k_i} \\ &+ \sum_{\alpha=1}^{l-1}\frac{1}{M_{k_\alpha+1}}\sum_{q=1}^{m_{k_\alpha}-1}\left|\sum_{i=1}^{\alpha-1} n_{k_i}M_{k_i} + M_{k_\alpha}\sum_{s=m_{k_\alpha}-n_{k_\alpha}}^{m_{k_\alpha}-1} e^{\frac{2\pi i s q}{m_{k_\alpha}}}\right|.\end{aligned} \tag{3.10}$$

Finally, from (3.2), (3.3) and (3.10) we get the claim of the theorem. □

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