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SELF-ORTHOGONAL CODES FROM ROW ORBIT MATRICES OF STRONGLY REGULAR GRAPHS

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ABSTRACT. We show that under certain conditions submatrices of row orbit matrices of strongly regular graphs span self-orthogonal codes. In order to demonstrate this method of construction, we construct self-orthogonal ternary linear codes from orbit matrices of the strongly regular graphs with parameters (70,27,12,9). Also we construct non self-orthogonal binary linear codes from these orbit matrices. Further, we obtain strongly regular graphs and block designs from codewords of the constructed codes.

1. INTRODUCTION

We present a method for constructing self-orthogonal codes from submatrices of row orbit matrices of strongly regular graphs. Applying this method we construct self-orthogonal ternary linear codes from orbit matrices of strongly regular graph (SRG) with parameters (70,27,12,9) for group Z_9. We also construct non self-orthogonal binary linear codes from these matrices. We use the constructed codes to obtain strongly regular graphs and block designs. More precisely, the strongly regular graphs and block designs are constructed from codewords of a given weight of the obtained binary linear codes.

The paper is organized as follows: after a brief description of the terminology and some background results in Section 2, in Section 3 we describe the concept of orbit matrices of strongly regular graphs, based on results presented in [3, 8], and in Section 4 we present obtained orbit matrices of SRG(70,27,12,9) for group Z_9. In Section 5 we present a method for construction of self-orthogonal codes from row orbit matrices of strongly regular graphs, and in Section 6 we construct binary and ternary codes from obtained orbit matrices. In Section 7 we construct strongly regular graphs and designs from codewords of the obtained codes.

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2. Background and terminology

We assume that the reader is familiar with basic notions from theory of finite groups. For basic definitions and properties of strongly regular graphs we refer the reader to [4] or [17].

A graph is *regular* if all its vertices have the same valency; a simple regular graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is *strongly regular* with parameters (v, k, λ, μ) if it has $|\mathcal{V}| = v$ vertices, valency k, and if any two adjacent vertices are together adjacent to λ vertices, while any two nonadjacent vertices are together adjacent to μ vertices. A strongly regular graph with parameters (v, k, λ, μ) is usually denoted by $\mathrm{SRG}(v, k, \lambda, \mu)$. An automorphism of a strongly regular graph Γ is a permutation of vertices of Γ, such that every two vertices are adjacent if and only if their images are adjacent.

An incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $I \subseteq \mathcal{P} \times \mathcal{B}$, is a t-(v, b, r, k, λ) *design*, if $|\mathcal{P}| = v$, $|\mathcal{B}| = b$, every block $B \in \mathcal{B}$ is incident with precisely k points, every t distinct points are together incident with precisely λ blocks and every point is incident with exactly r blocks.

A *linear q-ary (n, k) code K* over the finite field F_q of prime-power order q is a k-dimensional subspace of the n-dimensional vector space over F_q. The *weight* of a codeword is the number of its elements that are nonzero and the distance between two codewords is the *Hamming distance* between them, that is, the number of elements in which they differ. The *minimum distance* between distinct codewords is denoted by d. The minimum distance of a linear code is the minimum weight of its nonzero codewords. If a linear code K over a field of order q is of length n, dimension k, and minimum distance $d = d(K)$, then we write $[n, k, d]_q$ to show this information. An $[n, k]$ linear code K is said to be a *best known linear $[n, k]$ code* if K has the highest minimum weight among all known $[n, k]$ linear codes. An $[n, k]$ linear code K is said to be an *optimal linear $[n, k]$* code if the minimum weight of K achieves the theoretical upper bound on the minimum weight of $[n, k]$ linear codes, and *near-optimal* if its minimum distance is at most 1 less than the largest possible value.

The *dual* code $K^{\perp}$ is the orthogonal complement under the standard inner product $(\cdot, \cdot)$, *i.e.* $K^{\perp} = \{v \in F^n | (v, c) = 0 \text{ for all } c \in K\}$. If $K \subset K^{\perp}$, then K is called *self-orthogonal*.

The *support* of a nonzero codeword $x = \{x_1, \dots, x_n\}$ is the set of indices of its nonzero coordinates, i.e. $supp(x) = \{i \mid x_i \neq 0\}$. The *support design* of a code of length n for a given nonzero weight w is the design with points the n coordinate indices and blocks the supports of all codewords of weight w.

3. Orbit matrices of strongly regular graphs

In 2011 Behbahani and Lam introduced the concept of orbit matrices of SRGs (see [3]). While Behbahani and Lam were mostly focused on orbit matrices of strongly regular graphs admitting an automorphism of prime order, a general definition of an orbit matrix of a strongly regular graph is given in [8].

Let Γ be a SRG(v,k,λ,μ) and A be its adjacency matrix. Suppose an automorphism group G of Γ partitions the set of vertices V into b orbits $O_1,\dots,O_b$, with lengths $n_1,\dots,n_b$, respectively. The orbits divide A into submatrices $[A_{ij}]$, where A_{ij} is the adjacency matrix of vertices in O_i versus those in O_j. We define matrices
$C=[c_{ij}]$ and $R=[r_{ij}]$, $1\le i,j\le b$, such that
$c_{ij}=$ column sum of A_{ij},
$r_{ij}=$ row sum of A_{ij}.
The matrix R is related to C by

$$r_{ij}n_i=c_{ij}n_j. \tag{3.1}$$

Since the adjacency matrix is symmetric, it follows that

$$R=C^T. \tag{3.2}$$

The matrix R is the row orbit matrix of the graph Γ with respect to G, and the matrix C is the column orbit matrix of the graph Γ with respect to G. The matrices $C=[c_{ij}]$ and $R=[r_{ij}]$ satisfy the following conditions (see [8]):

$$\sum_{s=1}^{b}\frac{n_s}{n_j}c_{is}c_{js}=\delta_{ij}(k-\mu)+\mu n_i+(\lambda-\mu)c_{ij}$$
$$\sum_{s=1}^{b}\frac{n_s}{n_j}r_{si}r_{sj}=\delta_{ij}(k-\mu)+\mu n_i+(\lambda-\mu)r_{ji}.$$

Let us assume that a group G acts as an automorphism group of a SRG(v,k,λ,μ). Each matrix with the properties of a matrix R or C will be called a row orbit matrix or a column orbit matrix, respectively, for a strongly regular graph with parameters (v,k,λ,μ) and a group G (see [3]) although not every orbit matrix gives rise to strongly regular graphs. The following definition of orbit matrices of strongly regular graphs was introduced in [8].

Definition 3.1. *A $(b\times b)$-matrix $R=[r_{ij}]$ with entries satisfying conditions:*

$$\sum_{j=1}^{b}r_{ij}=\sum_{i=1}^{b}\frac{n_i}{n_j}r_{ij}=k$$
$$\sum_{s=1}^{b}\frac{n_s}{n_j}r_{si}r_{sj}=\delta_{ij}(k-\mu)+\mu n_i+(\lambda-\mu)r_{ji}$$

where $0\le r_{ij}\le n_j$, $0\le r_{ii}\le n_i-1$ and $\sum_{i=1}^{b}n_i=v$, is called a **row orbit matrix** *for a strongly regular graph with parameters (v,k,λ,μ) and the orbit lengths distribution $(n_1,\dots,n_b)$.*

Definition 3.2. *A $(b \times b)$-matrix $C = [c_{ij}]$ with entries satisfying conditions:*

$$\sum_{i=1}^{b} c_{ij} = \sum_{j=1}^{b} \frac{n_j}{n_i} c_{ij} = k$$

$$\sum_{s=1}^{b} \frac{n_s}{n_j} c_{is} c_{js} = \delta_{ij}(k-\mu) + \mu n_i + (\lambda - \mu) c_{ij}$$

where $0 \le c_{ij} \le n_i$, $0 \le c_{ii} \le n_i - 1$ and $\sum_{i=1}^{b} n_i = v$, is called a **column orbit matrix** *for a strongly regular graph with parameters (v,k,λ,μ) and the orbit lengths distribution $(n_1,\dots,n_b)$.*

3.1. **Orbit lengths distribution**

Suppose an automorphism group G of Γ partitions the set of vertices V into b orbits $O_1,\dots,O_b$, with sizes $n_1,\dots,n_b$. Obviously, n_i is a divisor of $|G|$, $i = 1,\dots,b$, and

$$\sum_{i=1}^{b} n_i = v.$$

When determining the orbit lengths distribution we also use the following result that can be found in [2].

Theorem 3.1. *Let $s < r < k$ be the eigenvalues of a SRG(v,k,λ,μ), then*

$$\phi \le \frac{\max(\lambda,\mu)}{k-r} v,$$

where ϕ is the number of fixed points for an nontrivial automorphism.

3.2. **Prototypes for a row of a column orbit matrix**

To construct orbit matrices with parameters (v,k,λ,μ) and the orbit lengths distribution $(n_1,\dots,n_b)$ we first need to find all prototypes.

A prototype for a row of a column orbit matrix C gives the information about the number of occurrences of each integer as an entry of a particular row of C. Behbahani and Lam [2, 3] introduced the concept of a prototype for a row of a column orbit matrix C of a strongly regular graph with a presumed automorphism group of prime order. We will generalize this concept, and describe a prototype for a row of a column orbit matrix C of a strongly regular graph under a presumed automorphism group of composite order.

Suppose an automorphism group G of a strongly regular graph Γ partitions the set of vertices V into b orbits $O_1,\dots,O_b$, of sizes $n_1,\dots,n_b$. With $l_i, i = 1,\dots,\rho$, we denote all divisors of $|G|$ in ascending order ($l_1 = 1,\dots,l_\rho = |G|$).

3.2.1. **Prototypes for a fixed row**

Consider the r-th row of a column orbit matrix C. We say that it is a fixed row of a matrix C if $n_r = 1$, i.e. if it corresponds to an orbit of length 1. The entries in this row are either 0 or 1. Let d_{l_i} denote the number of orbits whose length are l_i, $i = 1,\dots,\rho$.

Let x_e denote the number of occurrences of an element $e \in \{0,1\}$ at the positions of the r-th row which correspond to the orbits of length 1. It follows that

$$x_0 + x_1 = d_1, \tag{3.3}$$

where d_1 is the number of orbits of length 1. Since the diagonal elements of the adjacency matrix of a strongly regular graphs are equal to 0, it follows that $x_0 \geq 1$.

Let $y_e^{(l_i)}$ denote the number of occurrences of an element $e \in \{0,1\}$ at the positions of the r-th row which correspond to the orbits of length l_i $(i = 2,\dots,\rho)$. We have

$$y_0^{(l_i)} + y_1^{(l_i)} = d_{l_i}, \quad i = 2,\dots,\rho \tag{3.4}$$

Because the row sum of an adjacency matrix is equal to k, it follows that

$$x_1 + \sum_{i=2}^{\rho} l_i \cdot y_1^{(l_i)} = k. \tag{3.5}$$

The vector

$$p_1 = (x_0, x_1; y_0^{(l_2)}, y_1^{(l_2)}; \dots; y_0^{(l_\rho)}, y_1^{(l_\rho)})$$

whose components are nonnegative integer solutions of the equalities (3.3), (3.4) and (3.5) is called a prototype for a fixed row. The length of a prototype for a fixed row is 2ρ.

3.2.2. **Prototypes for a nonfixed row**

Let us consider the r-th row of a column orbit matrix C, where $n_r \neq 1$. Let d_{l_i} denote the number of orbits whose length is l_i, $i = 1,\dots,\rho$.

If a fixed vertex is adjacent to a vertex from an orbit $O_i, 1 \leq i \leq b$, then it is adjacent to all vertices from the orbit O_i. Therefore, the entries at the positions corresponding to fixed columns are either 0 or n_r. Let x_e denote the number of occurrences of an element $e \in \{0, n_r\}$ at those positions of the r-th row which correspond to the orbits of length 1. We have

$$x_0 + x_{n_r} = d_1. \tag{3.6}$$

The entries at the positions corresponding to the orbits whose lengths are greater than 1 are $0,1,\dots,n_r - 1$ or n_r. The entry at the position (r,r) is $0 \leq c_{r,r} \leq n_r - 1$, since the diagonal elements of the adjacency matrix of strongly regular graphs are 0.

Let $y_e^{(l_i)}$ denote the number of occurrences of an element $e \in \{0,\dots,n_r\}$ of r-th row at the positions which correspond to the orbits of length l_i $(i = 2,\dots,\rho)$. From (3.1) and (3.2) we conclude that

$$c_{ri} n_i = c_{ir} n_r,$$

where $c_{ir} \in \{0,\dots,n_i\}$. If $c_{ri} \cdot \frac{n_i}{n_r} \notin \{0,\dots,n_i\}$, then $y_{c_{ri}}^{(n_i)} = 0$. It follows that

$$\sum_{e=0}^{n_r} y_e^{(l_i)} = d_{l_i}, \quad i = 2,\dots,\rho. \tag{3.7}$$

Since the row sum of an adjacency matrix is equal to k, we have that

$$x_{n_r} + \sum_{i=2}^{\rho} \sum_{h=1}^{n_r} y_h^{(l_i)} \cdot h \cdot \frac{n_{l_i}}{n_r} = k, \tag{3.8}$$

If $s_{ij} = \sum_{k=1}^{b} c_{ik} c_{jk} n_k$, then $s_{rr} = \sum_{k=1}^{b} c_{rk} c_{rk} n_k$, and from the definition 3.2 we have that

$$n_r^2 x_{n_r} + \sum_{i=2}^{\rho} \sum_{h=1}^{n_r} y_h^{(l_i)} \cdot h^2 \cdot n_{l_i} = s_{rr}, \tag{3.9}$$

where $s_{rr} = (k-\mu)n_r + \mu n_r^2 + (\lambda - \mu)c_{rr}n_r$ and $c_{rr} \in \{0, \ldots, n_r - 1\}$.

The vector

$$p_{n_r} = (x_0, x_{n_r}; y_0^{l_2}, \ldots, y_{n_r}^{l_2}; \ldots; y_0^{l_\rho}, \ldots, y_{n_r}^{l_\rho}),$$

whose components are nonnegative integer solutions of equalities (3.6), (3.7), (3.8) and (3.9) is called a prototype for a row corresponding to the orbit of length n_r. The length of a prototype for a row which corresponds to the orbit of length n_r is $2+\sum_{i=2}^{\rho}(n_r + 1)$.

4. ORBIT MATRICES OF SRG(70,27,12,9)

Let Γ be a strongly regular graph with parameters (70,27,12,9). Further, let us assume that the group $G \cong \langle a \mid a^9 = 1 \rangle \cong Z_9$ acts as an automorphism group of Γ. By d_i we denote the number of G-orbits of length $i, i \in \{1,3,9\}$, and by $d = (d_1, d_3, d_9)$ we denote the corresponding orbit lengths distribution. To find all orbit matrices of SRG(70,27,12,9) for group Z_9 first we find all the orbit lengths distributions $(n_1, n_2, \ldots, n_b)$ for an action of the group Z_9 that satisfy Theorem 3.1. Using the program Mathematica we get all the prototypes for every orbit length distribution.

Using our own programs written in GAP [11] we construct all orbit matrices for given orbit lengths distributions. In Table 1 we present the number of orbit matrices for Z_9 for each orbit lengths distribution.

distribution	# OM	distribution	# OM	distribution	# OM	distribution	# OM
(1, 2, 7)	10	(7, 0, 7)	1	(13, 4, 5)	0	(22, 4, 4)	0
(1, 5, 6)	8	(7, 3, 6)	1	(13, 7, 4)	0	(25, 0, 5)	0
(1, 8, 5)	0	(7, 6, 5)	1	(16, 0, 6)	0	(25, 3, 4)	0
(1, 11, 4)	7	(7, 9, 4)	4	(16, 3, 5)	0	(28, 2, 4)	0
(4, 1, 7)	2	(10, 2, 6)	0	(16, 6, 4)	0	(31, 1, 4)	0
(4, 4, 6)	4	(10, 5, 5)	0	(19, 2, 5)	0	(34, 0, 4)	0
(4, 7, 5)	3	(10, 8, 4)	0	(19, 5, 4)	0		
(4, 10, 4)	7	(13, 1, 6)	0	(22, 1, 5)	0		

TABLE 1. *Number of nonisomorphic orbit matrices of SRGs with parameters (70,27,12,9) for an automorphism group Z_9.*

5. SELF ORTHOGONAL CODES FROM ORBIT MATRICES OF STRONGLY REGULAR GRAPHS

In 2003 Harada and Tonchev introduced a method of constructing self-orthogonal codes from orbit matrices of a design (see [14]). In [8] a method for constructing self-orthogonal codes from column orbit matrices of strongly regular graphs admitting an automorphism group G which acts with all orbits of the same length is described. These codes were defined over F_q, a finite field of prime order q, such that q divides k, λ, μ. Methods of constructing self-orthogonal codes from row orbit matrices of strongly regular graphs are given in [9]. Here we give a construction of self-orthogonal codes from some submatrices of row orbit matrices of strongly regular graphs.

Theorem 5.1. *Let Γ be a $\mathrm{SRG}(v,k,\lambda,\mu)$ having an automorphism group G which acts on the set of vertices of Γ with b orbits of lengths $n_1,\dots,n_b$, respectively, such that $n_1 = n_2 = .. = n_f = 1$ and $p|n_s$ if $n_s > 1$. Further, let p be a prime dividing k, λ, μ. Let R be the row orbit matrix of the graph Γ with respect to G. If p is a prime then the code over F_p spanned by the fixed rows of R is a self-orthogonal code of length b.*

Proof. From the definition of an orbit matrix, for i^{th} and j^{th} rows of C we have

$$\sum_{s=1}^{b}\frac{n_s}{n_j}c_{js}c_{is} = \delta_{ij}(k-\mu)+\mu n_i+(\lambda-\mu)c_{ij}. \tag{5.1}$$

Let $n_i = n_j = 1$. We have

$$\sum_{s=1}^{b}\frac{n_s}{n_j}c_{js}c_{is} = \sum_{s,n_s=1}\frac{n_s}{n_j}c_{js}c_{is}+\sum_{s,n_s>1}\frac{n_s}{n_j}c_{js}c_{is}.$$

so

$$\begin{aligned}\sum_{s,n_s=1}c_{js}c_{is} &= \sum_{s=1}^{b}\frac{n_s}{n_j}c_{js}c_{is}-\sum_{s,n_s>1}\frac{n_s}{n_j}c_{js}c_{is}\\ &= \delta_{ij}(k-\mu)+\mu n_i+(\lambda-\mu)c_{ij}-\sum_{s,n_s>1}n_s c_{js}c_{is}.\end{aligned}$$

Because $p|n_s$ we conclude that $\sum_{s,n_s=1}c_{js}c_{is}$ is congruent to zero modulo p.

If $n_i = n_j = 1$ then

$$\sum_{s=1}^{b}r_{is}r_{js} = \sum_{s,n_s=1}r_{is}r_{js}+\sum_{s,n_s>1}r_{is}r_{js}$$

Since $r_{is} = c_{is}\frac{n_s}{n_i} = c_{is}n_s$ we have

$$\sum_{s=1}^{b}r_{is}r_{js} = \sum_{s,n_s=1}c_{is}c_{js}+\sum_{s,n_s>1}n_s c_{is}r_{js}. \tag{5.2}$$

From (5.1) and (5.2) and because $p|n_s$ we conclude that $\sum_{s=1}^{b}r_{is}r_{js}$ is congruent to zero modulo p. □

Theorem 5.2. *Let Γ be a $\mathsf{SRG}(v,k,\lambda,\mu)$ having an automorphism group G which acts on the set of vertices of Γ with b orbits of lengths $n_1,\dots,n_b$, respectively, such that there are f fixed vertices, h orbits of length w, and $b-f-h$ orbits of lengths $n_{f+h+1},\dots,n_b$. Further, let $pw|n_s$ if $w<n_s$, and $pn_s|w$ if $n_s<w$, for $s=f+h+1,\dots,b$, where p is a prime number dividing k, λ, μ and w. Let R be the row orbit matrix of the graph Γ with respect to G. If q is a prime power such that $q=p^n$, then the code over F_q spanned by the part of the matrix R (rows and columns) determined by the orbits of length w is a self-orthogonal code of length h. If $m=min\{w,n_{f+h+1},\dots,n_b\}$, such that $p|m$ and $pm|n_s$ if $n_s\neq m$, then the code over F_q spanned by the rows of R corresponding to the orbits of length m and columns corresponding to the orbits of length greater than $m-1$ is a self-orthogonal code of length $b-f$.*

Proof. Let $n_i=n_j=w$. Then

$$\sum_{s=1}^{b}\frac{n_s}{n_j}c_{js}c_{is}=\sum_{s,n_s=1}\frac{n_s}{n_j}c_{js}c_{is}+\sum_{s,1<n_s<w}\frac{n_s}{n_j}c_{js}c_{is}+\sum_{s,n_s=w}\frac{n_s}{n_j}c_{js}c_{is}+\sum_{s,n_s>w}\frac{n_s}{n_j}c_{js}c_{is}$$

So

$$\sum_{s,n_s=w}c_{js}c_{is}=\sum_{s=1}^{b}\frac{n_s}{n_j}c_{js}c_{is}-\frac{1}{w}\sum_{s,n_s=1}c_{js}c_{is}-\sum_{s,1<n_s<w}\frac{n_s}{w}c_{js}c_{is}-\sum_{s,n_s>w}\frac{n_s}{w}c_{js}c_{is}. \tag{5.3}$$

If $s\in\{1,2,\dots,f\}$, then $c_{js},c_{is}\in\{0,w\}$, so $c_{js}c_{is}\in\{0,w^2\}$. From (5.1) and (5.3) it follows that:

$$\sum_{s,n_s=w}c_{js}c_{is}=\delta_{ij}(k-\mu)+\mu n_i+(\lambda-\mu)c_{ij}-w\cdot(f-x)-\sum_{s,n_s>w}\frac{n_s}{w}c_{js}c_{is}-\sum_{s,1<n_s<w}\frac{n_s}{w}c_{js}c_{is},$$

where $x=|\{s\in\{1,\dots,f\}:c_{js}c_{is}=0\}|$. If $n_s>w$ then $p|\frac{n_s}{w}$. If $1<n_s<w$ then $p|\frac{n_s}{w}c_{js}c_{is}$, because $c_{is}=r_{is}\frac{n_i}{n_s}=r_{is}\frac{w}{n_s}$, $c_{js}=r_{js}\frac{n_j}{n_s}=r_{js}\frac{w}{n_s}$, and $p|\frac{w}{n_s}$.
Hence $\sum_{s,n_s=w}c_{js}c_{is}$ is congruent to zero modulo p.

Since $r_{is}=c_{is}\frac{n_s}{w}$ and $r_{js}=c_{js}\frac{n_s}{w}$, we have that

$$\sum_{s,n_s=w}r_{js}r_{is}=\sum_{s,n_s=w}c_{js}\frac{n_s}{n_j}c_{is}\frac{n_s}{n_i}=\sum_{s,n_s=w}c_{js}c_{is}$$

so $\sum_{s,n_s=w}r_{js}r_{is}$ is congruent to zero modulo p. Let $n_i=n_j=m$. We have

$$\sum_{s=f+1}^{b}r_{js}r_{is}=\sum_{s,n_s=m}r_{js}r_{is}+\sum_{s,n_s>m}r_{js}r_{is}.$$

Since $r_{is}=c_{is}\frac{n_s}{n_i}$ it follows that $r_{js}r_{is}$ is divisible by p if $n_s>m$.

Also

$$\sum_{s,n_s=m}r_{js}r_{is}=\sum_{s,n_s=m}c_{js}c_{is}=\delta_{ij}(k-\mu)+\mu n_i+(\lambda-\mu)c_{ij}-m\cdot(f-y)-\sum_{s,n_s>m}\frac{n_s}{m}c_{js}c_{is},$$

where $y=|\{s\in\{1,\dots,f\}:c_{js}c_{is}=0\}|$.

Hence, we conclude that $\sum_{s,n_s=m}r_{js}r_{is}$ is divisible by p and $\sum_{s=f+1}^{b}r_{js}r_{is}$ is congruent to zero modulo p. □

6. LINEAR CODES FROM ORBIT MATRICES OF SRG(70,27,12,9)

In this section we construct self-orthogonal codes from orbit matrices of SRG(70,27,12,9) for group Z_9 presented in the Section 4 by applying theorems presented in Section 5. First we construct ternary self-orthogonal codes from the orbit matrices. Also we construct non self-orthogonal binary codes from the orbit matrices. In Tables 2, 3, 4, 5, 6, 7 and 8 we present information for the obtained codes, omitting the trivial codes. The codes were analyzed using Magma [5]. Codes marked with $*$ are optimal linear codes.

distribution	$[n,k,d]$	$\|\text{Aut}(K)\|$	distribution	$[n,k,d]$	$\|\text{Aut}(K)\|$
(1,11,4)	[11,5,3]	12	(1,11,4)	[11,4,6]*	24
(1,11,4)	[11,4,6]*	72	(4,4,6)	[4,1,3]	6
(4,7,5)	[7,3,3]	18	(4,10,4)	[10,3,3]	72
(4,10,4)	[10,4,3]	216	(7,6,5)	[6,2,3]	72
(7,9,4)	[9,3,3]	1296			

TABLE 2. *Ternary codes from orbit matrices of SRG(70,24,12,9) for Z_9 obtained from part corresponding to the orbits of length 3*

distribution	$[n,k,d]$	$\|\text{Aut}(K)\|$	distribution	$[n,k,d]$	$\|\text{Aut}(K)\|$
(1,11,4)	[15,5,3]	288	(1,11,4)	[15,4,6]	576
(1,11,4)	[15,4,6]	1728	(4,4,6)	[10,1,3]	30240
(4,7,5)	[12,3,3]	2160	(7,6,5)	[11,2,3]	8640
(7,9,4)	[14,3,3]	1728			

TABLE 3. *Ternary codes from orbit matrices of SRG(70,24,12,9) for Z_9 obtained from rows corresponding to the orbits of length 3 and columns corresponding to the orbits of length 3 and 9*

distribution	$[n,k,d]$	$\|\text{Aut}(K)\|$	distribution	$[n,k,d]$	$\|\text{Aut}(K)\|$
(1,2,7)	[7,3,3]	6	(1,2,7)	[7,2,3]	12
(1,2,7)	[7,2,3]	18	(1,2,7)	[7,2,3]	24
(1,2,7)	[7,2,3]	72	(4,1,7)	[7,1,6]	72
(4,1,7)	[7,1,6]	720	(4,4,6)	[6,1,6]*	72

TABLE 4. *Ternary codes from orbit matrices of SRG(70,24,12,9) for Z_9 obtained from part corresponding to the orbits of length 9*

distribution	$[n,k,d]$	$\|\text{Aut}(K)\|$	distribution	$[n,k,d]$	$\|\text{Aut}(K)\|$
(1,11,4)	[11,8,1]	288	(1,11,4)	[11,8,1]	192
(1,11,4)	[11,8,1]	288	(1,11,4)	[11,8,1]	288
(1,11,4)	[11,8,1]	576	(1,11,4)	[11,8,1]	1152
(1,11,4)	[11,8,2]*	5760	(4,4,6)	[4,2,1]	6
(4,7,5)	[7,4,1]	24	(4,7,5)	[7,4,1]	144
(4,7,5)	[7,6,1]	240	(4,10,4)	[10,8,1]	4320
(4,10,4)	[10,8,1]	10080	(4,10,4)	[10,8,1]	10080
(4,10,4)	[10,8,1]	80640	(7,9,4)	[9,6,2]*	1296
(7,9,4)	[9,8,2]*	362880			

TABLE 5. *Binary codes from orbit matrices of SRG(70,24,12,9) for Z_9 obtained from part corresponding to the orbits of length 3*

distribution	$[n,k,d]$	$\|\mathrm{Aut}(K)\|$	distribution	$[n,k,d]$	$\|\mathrm{Aut}(K)\|$
(1,2,7)	[10,1,3]	30240	(1,11,4)	[16,1,7]	1828915200
(1,11,4)	[16,1,9]	1828915200	(1,11,4)	[16,1,3]	37362124800
(4,1,7)	[12,3,3]	17280	(4,1,7)	[12,4,3]	20160
(4,4,6)	[14,4,3]	17280	(4,4,6)	[14,3,3]	967680
(7,0,7)	[14,4,3]	846720	(4,7,5)	[16,4,3]	34560
(4,7,5)	[16,3,3]	967680	(4,10,4)	[18,4,3]	155520
(4,10,4)	[18,3,3]	2903040	(4,10,4)	[18,3,4]	11612160
(4,10,4)	[18,4,4]	622080	(7,3,6)	[16,4,3]	725760
(7,6,5)	[18,4,3]	1451520	(7,9,4)	[20,4,3]	6531840
(7,9,4)	[20,4,4]	26127360			

TABLE 6. *Binary codes from orbit matrices of SRG(70,24,12,9) obtained from fixed rows of row orbit matrices*

distribution	$[n,k,d]$	$\|\mathrm{Aut}(K)\|$	distribution	$[n,k,d]$	$\|\mathrm{Aut}(K)\|$
(1,2,7)	[9,6,2]*	96	(1,2,7)	[9,6,1]	216
(1,2,7)	[9,4,2]	216	(1,2,7)	[9,6,2]*	4320
(1,2,7)	[9,8,1]	4320	(1,2,7)	[9,8,1]	4320
(1,2,7)	[9,6,2]*	4320	(1,2,7)	[9,8,1]	4320
(1,11,4)	[15,10,2]	8	(1,11,4)	[15,10,2]	16
(1,11,4)	[15,10,2]	72	(1,11,4)	[15,10,2]	72
(4,4,6)	[10,8,1]	1296	(4,7,5)	[12,10,1]	8640
(4,10,4)	[14,12,1]	172800	(4,10,4)	[14,12,1]	362880
(7,0,7)	[7,6,2]	5040	(7,3,6)	[9,8,2]*	362880
(7,6,5)	[11,10,2]*	39916800	(7,9,4)	[13,10,2]	31104
(7,9,4)	[13,12,2]*	6227020800			

TABLE 7. *Binary codes from orbit matrices of SRG(70,24,12,9) for Z_9 obtained from rows corresponding to the orbits of length 3 and columns corresponding to the orbits of length 3 and 9*

distribution	$[n,k,d]$	$\|\mathrm{Aut}(K)\|$	distribution	$[n,k,d]$	$\|\mathrm{Aut}(K)\|$
(1,2,7)	[7,2,4]*	72	(1,2,7)	[7,4,1]	144
(1,2,7)	[7,4,2]	144	(1,2,7)	[7,4,2]	144
(1,2,7)	[7,6,1]	144	(1,2,7),(7,0,7),(4,1,7)	[7,6,2]*	5040
(4,7,5),(7,6,5)	[5,4,2]*	120			

TABLE 8. *Binary codes from orbit matrices of SRG(70,24,12,9) for Z_9 obtained from rows corresponding to the orbits of length 9*

7. SRGs AND DESIGNS CONSTRUCTED FROM CODES

In this section we use the codes constructed in Section 6 to obtain strongly regular graphs and block designs. In order to construct strongly regular graphs we consider a set of codewords of certain weight w and look at the pairwise distances of the codewords. We identify the vertices of the graph by the codewords of weight w and define adjacency with respect to the Hamming distance of the codewords. In some cases the constructed graphs are strongly regular. We observe the cases when the distances between two codewords take two, three or four values.

If there are two possible values for a distance between two codewords, denoted by d_1 and d_2, then we define two vertices x and y to be adjacent if and only if

$d(x,y) = d_1$ (for the complementary graph we define that x and y are adjacent if and only if $d(x,y) = d_2$).

If there are three possible values for a distance between two codewords, namely d_1 and d_2 and d_3, we have more possibilities to define adjacency. Firstly, we define two vertices x and y to be adjacent if and only if $d(x,y) = d_1$, Secondly, two vertices x and y are adjacent if and only if $d(x,y) = d_2$, and thirdly, two vertices x and y are adjacent if and only if $d(x,y) = d_3$.

Let there be four values for a distance between two codewords, d_1 and d_2, d_3 and d_4. Firstly, we define two vertices x and y to be adjacent if and only if $d(x,y) = d_1$, or $d(x,y) = d_2$, secondly, two vertices x and y are adjacent if and only if $d(x,y) = d_1$ or $d(x,y) = d_3$ and thirdly, two vertices x and y are adjacent if and only if $d(x,y) = d_1$ or $d(x,y) = d_4$. Further, we define adjacency taking into cosideration only one intersection (d_1, d_2, d_3 or d_4). The construction is conducted using the GAP package Grape [16]. The obtained strongly regular graphs are presented in Tables 9, 10, 11.

(v,k,λ,μ)	$\lvert\mathrm{Aut}(G)\rvert$	Distribution
(15,6,1,3)	720	(7,6,5)
(21,10,3,6)	5040	(4,10,4)
(27,10,1,5)	51840	(7,9,4)
(28,12,6,4)	40320	(4,10,4)
(36,14,7,4)	362880	(7,9,4)
(45,16,8,4)	3628800	(4,10,4)
(120,56,28,24)	3628800	(4,10,4)
(126,100,78,84)	3628800	(7,9,4)

TABLE 9. *SRGs from binary codes obtained from part corresponding to the orbits of length 3*

(v,k,λ,μ)	$\lvert\mathrm{Aut}(G)\rvert$	Distribution
(10,3,0,1)	120	(1,5,6)
(15,6,1,3)	720	(1,2,7),
(28,12,6,4)	40320	(1,5,6),(4,1,7)
(36,14,7,4)	362880	(7,3,6)
(45,16,8,4)	3628800	(4,4,6)
(55,18,9,4)	39916800	(7,6,5)
(78,22,11,4)	6227020800	(7,9,4)
(119,54,21,27)	394813440	(4,7,5)
(120,56,28,24)	3628800	(4,4,6)
(126,100,78,84)	3628800	(7,3,6)
(330,266,211,228)	39916800	(7,6,5)

TABLE 10. *SRGs from binary codes obtained from rows corresponding to the orbits of length 3 and columns corresponding to the orbits of length 3 and 9*

(v,k,λ,μ)	$\lvert\mathrm{Aut}(G)\rvert$	Distribution
(10,3,0,1)	120	(4,7,5),(7,6,5)
(15,6,1,3)	720	(1,5,6),(4,4,6),(7,3,6)
(21,10,3,6)	5040	(1,2,7),(4,1,7),(7,0,7)
(35,16,6,8)	40320	(1,2,7),(4,1,7),(7,0,7)

TABLE 11. *SRGs from binary codes obtained from part corresponding to the orbits of length 9*

The strongly regular graph with parameters $(10,3,0,1)$ is the Petersen graph, the unique SRG with these parameters. The constructed strongly regular graphs with parameters $(15,6,1,3)$ and $(21,10,3,6)$ are complement of the triangular graphs $T(6)$ and $T(7)$. Strongly regular graphs with parameters $(28,12,6,4)$, $(36,14,7,4)$, $(45,16,8,4)$, $(55,18,9,4)$ and $(78,22,11,4)$ are the triangular graphs $T(8)$, $T(9)$, $T(10)$, $T(11)$ and $T(13)$ respectively. The constructed strongly regular graph with parameters $(35,16,6,8)$ is the complement of the distance 2 graph in the Johnson graph $J(7,4)$ and $(27,10,1,5)$ is a complement of a Schläfli graph. The constructed strongly regular graph with parameters $(120,56,28,24)$ is the complement of the distance 2 graph in the Johnson graph $J(10,3)$ and strongly regular graph with parameters $(126,100,78,84)$ is the complement of the distance 1 or 4 in the Johnson graph $J(9,4)$. The constructed strongly regular graph with parameters $(330,266,211,228)$ is the complement of the distance 1 or 4 in the Johnson graph $J(11,4)$ and the strongly regular graph with parameters $(119,54,21,27)$ is $O^-(8,2)$ polar graph.

We also use the codes from Section 6 to construct designs, taking into consideration a set of codewords of certain weight. Identifying coordinate positions of the codes with the points, and codewords of a weight k with blocks, we obtain an incidence structure having all blocks of size k. In other words, we consider the support designs of the constructed codes. In some cases these support designs are designs. In Tables 12, 13 and 14 we present information on the obtained $t-(v,b,r,k,\lambda)$ designs for which $t<k<v-t$.

$t-(v,b,r,k,\lambda)$	$\lvert\mathrm{Aut}(D)\rvert$	Distribution	$t-(v,b,r,k,\lambda)$	$\lvert\mathrm{Aut}(D)\rvert$	Distribution
2-(10,120,36,3,8)	3628800	(4,10,4)	2-(10,120,84,7,56)	3628800	(4,10,4)
2-(10,210,126,6,70)	3628800	(4,10,4)	2-(10,210,84,4,28)	3628800	(4,10,4)
2-(10,252,126,5,56)	3628800	(4,10,4)	3-(10,210,126,6,35)	3628800	(4,10,4)
3-(10,210,84,4,7)	3628800	(4,10,4)	3-(10,252,126,5,21)	3628800	(4,10,4)
2-(6,20,10,3,4)	720	(7,6,5)	2-(9,126,56,4,21)	362880	(7,9,4)
2-(9,84,56,6,35)	362880	(7,9,4)	3-(9,126,56,4,6)	362880	(7,9,4)

TABLE 12. *Designs from binary codes obtained from part corresponding to the orbits of length 3*

$t-(v,b,r,k,\lambda)$	$\lvert\mathrm{Aut}(D)\rvert$	Distribution	$t-(v,b,r,k,\lambda)$	$\lvert\mathrm{Aut}(D)\rvert$	Distribution
2-(8,56,21,3,6)	40320	(4,1,7)	2-(8,70,35,4,15)	40320	(4,1,7)
3-(8,70,35,4,5)	40320	(4,1,7)	2-(10,120,36,3,8)	3628800	(4,4,6)
2-(10,210,84,4,28)	3628800	(4,4,6)	2-(10,252,126,5,56)	3628800	(4,4,6)
3-(10,210,126,6,35)	3628800	(4,4,6)	3-(10,210,84,4,7)	3628800	(4,4,6)
3-(10,252,126,5,21)	3628800	(4,4,6)	2-(9,84,56,6,35)	362880	(7,3,6)
3-(9,126,56,4,6)	362880	(7,3,6)	2-(11,165,120,8,84)	39916800	(7,6,5)
2-(11,330,120,4,36)	39916800	(7,6,5)	2-(11,462,252,6,126)	39916800	(7,6,5)
3-(11,330,120,4,8)	39916800	(7,6,5)	3-(11,462,252,6,56)	39916800	(7,6,5)
2-(13,1287,792,8,462)	6227020800	(7,9,4)	2-(13,1716,792,6,330)	6227020800	(7,9,4)
2-(13,286,220,10,165)	6227020800	(7,9,4)	2-(13,715,220,4,55)	6227020800	(7,9,4)
3-(13,1287,792,8,252)	6227020800	(7,9,4)	3-(13,1716,792,6,120)	6227020800	(7,9,4)
3-(13,715,220,4,10)	6227020800	(7,9,4)			

TABLE 13. *Designs from binary codes obtained from rows corresponding to the orbits of length 3 and columns corresponding to the orbits of length 3 and 9*

$t-(v,b,r,k,\lambda)$	$\|\mathrm{Aut}(D)\|$	Distribution
2-(7,35,20,4,10)	5040	(1,2,7),(4,1,7),(7,0,7)
2-(6,20,10,3,4)	720	(1,5,6),(4,4,6),(7,3,6)

TABLE 14. *Designs from binary codes obtained from part corresponding to the orbits of length 9*

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