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BINET-CURVES FOR UNUSUAL GENERALIZED FIBONACCI SEQUENCES

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ABSTRACT. In this article, we trace Binet-curves for some unusual Generalized Fibonacci sequences. We extend their Binet-type formula to arbitrary real variables in the complex plane and use them to analyze and calculate the area under each of these curves.

1. INTRODUCTION

In recent years, various papers have been published related to different types of generalizations of the Fibonacci sequence. (See: [1–5]).

In this article, we analyze three unusual types of Generalized Fibonacci sequences by extending its Binet-type formula to arbitrary real variables in the complex plane and trace curves for the sequences mentioned below. We also obtain the area under all these curves which falls in the first quadrant. One can refer to [6, 10] for further details.

We consider the following three unusual types of Generalized Fibonacci sequences.

(i) Half-Companion Pell sequence :
The sequence of Half-Companion Pell numbers is defined recursively by the relation

$$H_n = \begin{cases} 1, & n = 0, \\ H_{n-1} + 2P_{n-1}, & n \geq 1, \end{cases} \quad \text{and} \quad P_n = \begin{cases} 1, & n = 0, \\ H_{n-1} + P_{n-1}, & n \geq 1. \end{cases}$$

The first few terms of this sequence are 1, 3, 7, 17, 41, 99,.... The Binet-type formula for this sequence $\{H_n\}_{n\geq 1}$ is $H_n = \frac{\gamma^n+\delta^n}{2}$, where $\gamma = 1+\sqrt{2}$ and $\delta = 1-\sqrt{2}$. One can refer to [7] for further details about this sequence.

(ii) We introduce a new sequence $\{S_n\}_{n\geq 1}$ defined by the relation $S_n = F_n + L_n$, where F_n, L_n are the usual n^{th} Fibonacci and Lucas numbers respectively. One can refer to [8] for further details about this sequence.

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The first few terms of the sequence $\{S_n\}_{n\geq 1}$ are 2, 2, 4, 6, 10, 16, 26. The Binet-type formula for the sequence $\{S_n\}_{n\geq 1}$ can easily be obtained from the known Binet formulae of the Fibonacci numbers and Lucas numbers. Since $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ and $L_n = \alpha^n + \beta^n$,where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$,we get

$$S_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} + (\alpha^n + \beta^n) = \frac{1}{\sqrt{5}}\left[(\sqrt{5}+1)\alpha^n + (\sqrt{5}-1)\beta^n\right].$$

(iii) We further introduce a new sequence $\{K_n\}_{n\geq 1}$ defined by the relation $K_n = P_n + H_n$, where P_n, H_n are the n^{th} Pell number and Half-Companion Pell number respectively. One can refer to [7, 9] for further details about this sequence. The first few terms of the sequence $\{K_n\}_{n\geq 1}$ are 1, 2, 5, 12, 29, 70,... The Binet-type formula for the sequence $\{K_n\}_{n\geq 1}$ can be easily obtained from the known Binet formulae of Pell numbers and Half-Companion Pell numbers. Since $P_n = \frac{\gamma^n - \delta^n}{2\sqrt{2}}$ and $H_n = \frac{\gamma^n + \delta^n}{2}$,where $\gamma = 1+\sqrt{2}$, $\delta = 1-\sqrt{2}$, we get

$$K_n = \frac{\gamma^n - \delta^n}{2\sqrt{2}} + \frac{\gamma^n + \delta^n}{2} = \left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)\gamma^n + \left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)\delta^n.$$

Thus,

$$K_n = \frac{1}{2\sqrt{2}}\left[(\sqrt{2}+1)\gamma^n + (\sqrt{2}-1)\delta^n\right].$$

In this paper, we trace Binet-Curves for each of these sequences by extending their Binet-type formulae to arbitrary real variables in the complex plane.We further obtain the area under these curves.

2. BINET-CURVE FOR $\{H_n\}_{n\geq 1}$

In the above Binet-type formula for H_n, instead of considering the index n as an integer, if we choose n to be any arbitrary real number then it is interesting to study the nature of the transformed Half-Companion Pell number H_n. We replace the integer value n by the real variable t. Then it can be easily observed that the corresponding Half-Companion Pell number H_t will be a complex number. We thus consider the Binet-type formula for these numbers as $H_t = \frac{\gamma^t + \delta^t}{2}$, where t is any arbitrary real number.

Since $(-1)^t = e^{t\log(-1)} = e^{(i\pi(2n+1))t}$, where $n = 0, \pm 1, \pm 2, \ldots$, in the following we choose $n = 0$. Then we get

$$\delta^t = \left(\frac{-1}{\gamma}\right)^t = \frac{(-1)^t}{\gamma^t} = \frac{e^{i\pi t}}{\gamma^t}.$$

Now,

$$H_t = \frac{\gamma^t + \left(\frac{-1}{\gamma}\right)^t}{2} = \frac{e^{t\ln(\gamma)} + e^{i\pi t}e^{-t\ln(\gamma)}}{2} = \frac{e^{t\ln(\gamma)} + (\cos(\pi t) + i\sin(\pi t))e^{-t\ln(\gamma)}}{2}.$$

Thus,

$$H_t = \frac{1}{2}\left(e^{t\ln(\gamma)} + \cos(\pi t)e^{-t\ln(\gamma)}\right) + i\left(\frac{\sin(\pi t)e^{-t\ln(\gamma)}}{2}\right).$$

This function describes a curve in the complex plane parameterized by a real variable t. We thus define the Binet-Curve for $-\infty < t < \infty$ in a parametric form as

$$H_t = (x(t), y(t)),$$

where

$$\mathrm{Re}(H_t) = x(t) = \frac{1}{2}\left(e^{t\ln(\gamma)} + \cos(\pi t)e^{-t\ln(\gamma)}\right),$$

$$\mathrm{Im}(H_t) = y(t) = \frac{1}{2}\left(\sin(\pi t)e^{-t\ln(\gamma)}\right).$$

We first obtain the area under the Binet-Curve for H_t for any arbitrary real number t within the fixed interval.

Theorem 2.1. *The area of the segment under the Binet-Curve for H_t within the interval $[n, n+1]$ is given by*

$$A_{n,n+1} = \frac{1}{8}\left\{\frac{4\ln(\gamma)(-1)^n}{\pi} + \frac{\pi}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n+1}}\right)\right\}.$$

Proof. We use Green's theorem to calculate the area value of the segment

$$A_{n,n+1} = \frac{1}{2}\int_n^{n+1}(x\,dy - y\,dx).$$

By rewriting $dy = \left(\frac{dy}{dt}\right)dt, dx = \left(\frac{dx}{dt}\right)dt$,the formula becomes

$$A_{n,n+1} = \frac{1}{2}\int_n^{n+1}(x\dot{y} - y\dot{x})\,dt. \tag{2.1}$$

We use components $x(t)$ and $y(t)$ of the Binet-type formula for the real argument and calculate this value. Now,

$$\dot{x} = \frac{dx}{dt} = \frac{1}{2}\left\{\ln\gamma e^{t\ln(\gamma)} - \ln\gamma\cos(\pi t)e^{-t\ln(\gamma)} - \pi\sin(\pi t)e^{-t\ln(\gamma)}\right\},$$

$$\dot{y} = \frac{dy}{dt} = \frac{1}{2}\left\{-\ln\gamma\sin(\pi t)e^{-t\ln(\gamma)} + \pi\cos(\pi t)e^{-t\ln(\gamma)}\right\}.$$

Thus,

$$x\dot{y} - y\dot{x} = \frac{1}{4}\left\{-2\ln\gamma\sin(\pi t) + \pi\cos(\pi t) + \pi e^{-2t\ln(\gamma)}\right\}.$$

Substituting this value in (2.1) we get

$$\begin{aligned}
A_{n,n+1} &= \frac{1}{8}\left\{-2\ln\gamma\int_n^{n+1}\sin(\pi t)\,dt+\pi\int_n^{n+1}\cos(\pi t)\,dt+\pi\int_n^{n+1}e^{-2t\ln(\gamma)}\,dt\right\}\\
&= \frac{1}{8}\left\{-2\ln\gamma\left[\frac{-\cos(\pi t)}{\pi}\right]_n^{n+1}+\pi\left[\frac{\sin(\pi t)}{\pi}\right]_n^{n+1}+\pi\left[\frac{e^{-2t\ln(\gamma)}}{-2\ln(\gamma)}\right]_n^{n+1}\right\}\\
&= \frac{1}{8}\left\{\frac{2\ln\gamma}{\pi}\left((-1)^{n+1}-(-1)^n\right)+0-\frac{\pi}{2\ln(\gamma)}\left(\gamma^{-2(n+1)}-\gamma^{-2n}\right)\right\}\\
&= \frac{1}{8}\left\{\frac{4\ln(\gamma)(-1)^n}{\pi}+\frac{\pi}{2\ln(\gamma)}\left(\frac{\gamma^2-1}{\gamma^{2n+2}}\right)\right\}.
\end{aligned}$$

Since $\gamma^2-1=2\gamma$, we obtain

$$A_{n,n+1}=\frac{1}{8}\left\{\frac{4\ln(\gamma)(-1)^n}{\pi}+\frac{\pi}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n+1}}\right)\right\}.$$

□

Remark 2.1. If we take the limit value of the area segment at infinity, we have

$$A_\infty=\lim_{n\to\infty}|A_{n,n+1}|=\left|-\frac{1}{8}\times\frac{4\ln(\gamma)(-1)^n}{\pi}\right|=\frac{1}{2\pi}\ln(\gamma).$$

Thus, we find that the sequence of the segment's area has a finite limit.

We next consider the ratio of the area of two consecutive segments and show that it tends to 1 as $n\to\infty$.

Theorem 2.2. *The ratio of the area of two consecutive segments for $\{H_n\}_{n\geq 1}$ tends to* 1 *as n tends to ∞.*

Proof. From the above theorem, the area formula for the Binet-Curve for H_t is

$$A_{n,n+1}=\frac{1}{8}\left\{\frac{4\ln(\gamma)(-1)^n}{\pi}+\frac{\pi}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n+1}}\right)\right\}.$$

Now, we replace n by $n-1$ and further $n+1$ by n to get

$$A_{n-1,n}=\frac{1}{8}\left\{\frac{4\ln(\gamma)(-1)^{n-1}}{\pi}+\frac{\pi}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n-1}}\right)\right\}.$$

We then express λ_n as the ratio of these two areas. This gives

$$\lambda_n=\left|\frac{A_{n,n+1}}{A_{n-1,n}}\right|=\left|\frac{\frac{4\ln(\gamma)(-1)^n}{\pi}+\frac{\pi}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n+1}}\right)}{\frac{4\ln(\gamma)(-1)^{n-1}}{\pi}+\frac{\pi}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n-1}}\right)}\right|=\frac{1}{\gamma^2}\left|\frac{\frac{4(\ln\gamma)^2\gamma^{2n+1}}{\pi^2}(-1)^n+\gamma^2}{\frac{4(\ln\gamma)^2\gamma^{2n+1}}{\pi^2}(-1)^n-\gamma^2}\right|. \tag{2.2}$$

If we consider the value of $n = 2m$, an even number, then we get

$$\lambda_{2m} = \left|\frac{A_{2m,2m+1}}{A_{2m-1,2m}}\right| = \left|\frac{\frac{4(\ln\gamma)^2\gamma^{4m+1}}{\pi^2} + \gamma^2}{\frac{4(\ln\gamma)^2\gamma^{4m+1}}{\pi^2} - \gamma^2}\right| > 1.$$

Also, if we take $n = 2m+1$ then we get

$$\lambda_{2m+1} = \left|\frac{A_{2m+1,2m+2}}{A_{2m,2m+1}}\right| = \left|\frac{-\frac{4(\ln\gamma)^2\gamma^{4m+3}}{\pi^2} + \gamma^2}{-\frac{4(\ln\gamma)^2\gamma^{4m+3}}{\pi^2} - \gamma^2}\right| < 1.$$

From (2.2), we get

$$\lim_{n\to\infty} \lambda_n = \frac{1}{\gamma^2}\left|\frac{\gamma^2}{1}\right| = \frac{1}{\gamma^2}\gamma^2 = 1.$$

Thus, we can interpret that the area values of consecutive segments are equal at infinity. $\square$

In the following theorem, we find the value of curvature $\kappa(t)$ of the Binet-Curve for H_t at any arbitrary point $(x(t), y(t))$.

Theorem 2.3. *The curvature of the Binet-Curve H_t at any arbitrary point is*

$$\kappa(t) = \frac{2\Big(\sin(\pi t)\big(2(\ln\gamma)^3 - \pi^2\ln\gamma\big) - 3\pi(\ln\gamma)^2\cos(\pi t)\Big)}{\Big((\ln\gamma)^2 e^{2t\ln\gamma} + \big(\pi^2 + (\ln\gamma)^2\big)e^{-2t\ln\gamma} - 2\pi\ln\gamma\sin(\pi t) - 2(\ln\gamma)^2\cos(\pi t)\Big)^{3/2}}$$

Proof. As discussed earlier, we consider

$$\vec{r}(t) = H_t = (x(t), y(t)) = \left(\frac{1}{2}\Big(e^{t\ln(\gamma)} + \cos(\pi t)e^{-t\ln(\gamma)}\Big), \frac{1}{2}\Big(\sin(\pi t)e^{-t\ln(\gamma)}\Big)\right).$$

We use the curvature formula

$$\kappa(t) = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{3/2}}.$$

Now, it can be shown that

$$\dot{x} = \frac{1}{2}\left\{\ln\gamma e^{t\ln(\gamma)} - \ln\gamma\cos(\pi t)e^{-t\ln(\gamma)} - \pi\sin(\pi t)e^{-t\ln(\gamma)}\right\},$$

$$\ddot{x} = \frac{1}{2}\Bigg\{(\ln\gamma)^2 e^{t\ln(\gamma)} + (\ln\gamma)^2\cos(\pi t)\,e^{-t\ln(\gamma)} + 2\pi\ln\gamma\sin(\pi t)\,e^{-t\ln(\gamma)}$$
$$-\pi^2\cos(\pi t)\,e^{-t\ln(\gamma)}\Bigg\},$$

$$\dot{y} = \frac{1}{2}\left\{-\ln\gamma\sin(\pi t)e^{-t\ln(\gamma)} + \pi\cos(\pi t)e^{-t\ln(\gamma)}\right\},$$

$$\ddot{y} = -\frac{1}{2}\left\{(\ln\gamma)^2\sin(\pi t)e^{-t\ln(\gamma)} - 2\pi\ln\gamma\cos(\pi t)e^{-t\ln(\gamma)} - \pi^2\sin(\pi t)e^{-t\ln(\gamma)}\right\}.$$

Using these values, we obtain the value of $\kappa(t)$ as follows:

$$\kappa(t) = \frac{2\Big(\sin(\pi t)\big(2(\ln\gamma)^3 - \pi^2\ln\gamma\big) - 3\pi(\ln\gamma)^2\cos(\pi t)\Big)}{\Big((\ln\gamma)^2e^{2t\ln\gamma} + \big(\pi^2 + (\ln\gamma)^2\big)e^{-2t\ln\gamma} - 2\pi\ln\gamma\sin(\pi t) - 2(\ln\gamma)^2\cos(\pi t)\Big)^{3/2}}.$$

□

Note 2.1. By taking the limit as $t \to \pm\infty$, we finally get

$$lim_{t\to\infty}\kappa(t) = 0 \text{ and } \lim_{t\to-\infty}\kappa(t) = 0.$$

Thus, the curvature is zero at any point of the curve H_t when t is very large. This means the curve behaves like a straight line at distant points of H_t.This means that the curve behaves like a straight line at distant points of S_t.

When we analyze the parametric curve $(x(t), y(t))$ of S_t, we conclude the following:

(i) The curve is not symmetric about the x-axis since x is not an even function of t and also not symmetric about the y-axis since y is not an even function of t.

(ii) The curve does not pass through the origin since there does not exist any real value of t at which $x = 0$ and $y = 0$.

(iii) The curve intersects the x-axis at the points $(2,0)$, $(4,0), \ldots$

(iv) We consider

$$\frac{dy}{dx} = \frac{\frac{1}{2}\left\{-\ln\gamma\sin(\pi t)e^{-t\ln\gamma} + \pi\cos(\pi t)e^{-t\ln\gamma}\right\}}{\frac{1}{2}\left\{\ln\gamma\gamma^t - \ln\gamma\cos(\pi t)e^{-t\ln\gamma} - \pi\sin(\pi t)e^{-t\ln\gamma}\right\}}.$$

Let $\frac{dy}{dx} = 0$, we get

$$-\ln\gamma\sin(\pi t)e^{-t\ln\gamma} + \pi\cos(\pi t)e^{-t\ln\gamma} = 0.$$

Thus,

$$t = \frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{\ln\gamma}\right) = 0.41293 + m, m \in \mathbb{Z},$$

where $\gamma = 2.4142$. Hence, we conclude that the x-axis is the tangent to the curve at

$$t = 0.45162 + m, m \in \mathbb{Z}.$$

(v) We further find the maximum and minimum value of this curve. As shown above, $\frac{dy}{dx} = 0$ when $t = 0.45162 + m\ m \in \mathbb{Z}$.
If we let $m = 0$, then for $t = 0.45162$, it can be observed that
$\frac{d^2y}{dx^2} = -12.365 < 0$. Also, for $m = 1$, we get $t = 1.45162$ and in this case we observe that$\frac{d^2y}{dx^2} = 1.4230 > 0$. Thus, one of the maximum values of the

curve is at the point $(1.8657,\ 0.4395)_{t=0.45162}$, and one of the minimum values is at the point $(2.6828,\ -0.2716)_{t=1.45162}$.

(vi) It can be observed that $\lim_{t\to\infty} x(t) \to \infty$, $\lim_{t\to\infty} y(t) = 0$. Thus, $y = 0$ is an asymptote of the curve.

On the basis of these data, the S_t curve can be traced using the software MATLAB as shown in Figure 1, when $t > 0$ as well as in Figure 2, when $t < 0$.

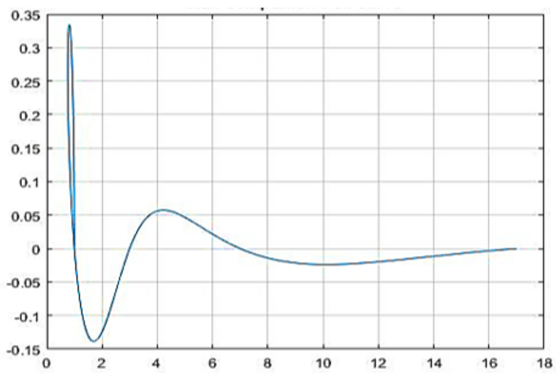

FIGURE 1. Half-Companion Pell-Binet-Curve for $t > 0$

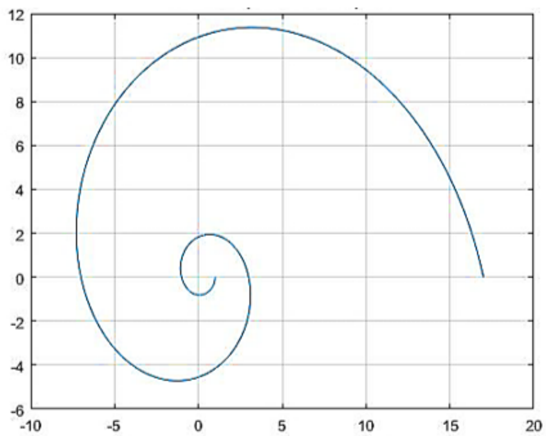

FIGURE 2. Half-Companion Pell-Binet-Curve for $t < 0$

3. Binet-Curve for $\{S_n\}_{n\geq 1}$

In the Binet-type formula for S_n, instead of considering index n as an integer, if we choose n to be any arbitrary real number then it is interesting to study the nature of this transformed sequence $\{S_n\}_{n\geq 1}$. We replace the integer value n by the real variable t. Then it can be easily observed that the corresponding number S_t will be a complex number. We thus consider the Binet-type formula for these numbers as

$$S_t = \frac{1}{\sqrt{5}}\Big\{(\sqrt{5}+1)\alpha^t + (\sqrt{5}-1)\beta^t\Big\},$$

where t is any arbitrary real number.
Now, since $(-1)^t = e^{t\log(-1)} = e^{i\pi t(2n+1)}$, where $n = 0, \pm 1, \pm 2, \ldots$, we choose only the principal value with $n = 0$. Then we get

$$\beta^t = \left(\frac{-1}{\alpha}\right)^t = \frac{(-1)^t}{\alpha^t} = \frac{e^{i\pi t}}{\alpha^t}.$$

Now,

$$S_t = \frac{1}{\sqrt{5}}\Big\{(\sqrt{5}+1)\alpha^t + (\sqrt{5}-1)\beta^t\Big\},$$

$$S_t = \frac{1}{\sqrt{5}}\left\{(\sqrt{5}+1)e^{t\ln(\alpha)} + (\sqrt{5}-1)\cos(\pi t)e^{-t\ln(\alpha)} + i(\sqrt{5}-1)\sin(\pi t)e^{-t\ln(\alpha)}\right\}.$$

This function describes a curve in the complex plane parameterized by the real variable t. We thus define the Binet-Curve for $-\infty < t < \infty$ in a parametric form as

$$S_t = (x(t), y(t)),$$

where

$$\operatorname{Re}(S_t) = x(t) = \frac{1}{\sqrt{5}}\left\{(\sqrt{5}+1)e^{t\ln(\alpha)} + (\sqrt{5}-1)\cos(\pi t)e^{-t\ln(\alpha)}\right\},$$

$$\operatorname{Im}(S_t) = y(t) = \frac{1}{\sqrt{5}}\left\{(\sqrt{5}-1)\sin(\pi t)e^{-t\ln(\alpha)}\right\}.$$

We obtain the area under the Binet-Curve for S_t for any arbitrary real number t within the fixed interval.

Theorem 3.1. *Area of the segment under the Binet-Curve for S_t within the interval $[n, n+1]$ is given by*

$$A_{n,n+1} = \frac{1}{10}\left\{\frac{-16\ln(\alpha)(-1)^n}{\pi} + \frac{\pi(6-2\sqrt{5})}{2\ln(\alpha)}\left(\frac{1}{\alpha^{2n+1}}\right)\right\}.$$

Proof. We use Green's theorem to calculate the area value of segments

$$A_{n,n+1} = \frac{1}{2}\int_n^{n+1}(x\dot{y} - y\dot{x})\,dt. \tag{3.1}$$

We use components $x(t)$ and $y(t)$ of the Binet-type formula for the real argument and calculate this value. Now,

$$\dot{x}=\frac{1}{\sqrt{5}}\left\{(\sqrt{5}+1)\ln\alpha e^{t\ln\alpha}-(\sqrt{5}-1)\ln\alpha\cos(\pi t)e^{-t\ln\alpha}-(\sqrt{5}-1)\pi\sin(\pi t)e^{-t\ln\alpha}\right\},$$

$$\dot{y}=\frac{1}{\sqrt{5}}\left\{-(\sqrt{5}-1)\ln\alpha\sin(\pi t)e^{-t\ln\alpha}+(\sqrt{5}-1)\pi\cos(\pi t)e^{-t\ln\alpha}\right\}.$$

Substituting these values in (3.1) we get

$$A_{n,n+1}=\frac{1}{10}\left\{\frac{-16\ln(\alpha)(-1)^n}{\pi}+\frac{\pi(6-2\sqrt{5})}{2\ln(\alpha)}\left(\frac{1}{\alpha^{2n+1}}\right)\right\}.$$

□

Remark 3.1. If we take the limit value of the area segment at infinity, then we have

$$A_\infty=\lim_{n\to\infty}|A_{n,n+1}|=\frac{8}{5\pi}\ln(\alpha).$$

Thus, we find that the sequence of the segment's areas has a finite limit.

We next consider the ratio of the area of two consecutive segments and show that it tends to 1 as $n\to\infty$.

Theorem 3.2. *The ratio of the area of two consecutive segments for $\{S_n\}_{n\geq 1}$ tends to* 1 *as n tends to* ∞*.*

Proof. From the above theorem, the area formula for the Binet-Curve for S_t is

$$A_{n,n+1}=\frac{1}{10}\left\{\frac{-16\ln(\alpha)(-1)^n}{\pi}+\frac{\pi(6-2\sqrt{5})}{2\ln(\alpha)}\left(\frac{1}{\alpha^{2n+1}}\right)\right\}.$$

Now, we replace n by $n-1$ and also $n+1$ by n to get

$$A_{n-1,n}=\frac{1}{10}\left\{\frac{-16\ln(\alpha)(-1)^{n-1}}{\pi}+\frac{\pi(6-2\sqrt{5})}{2\ln(\alpha)}\left(\frac{1}{\alpha^{2n-1}}\right)\right\}.$$

We then express λ_n as the ratio of these two areas. This gives

$$\lambda_n=\left|\frac{A_{n,n+1}}{A_{n-1,n}}\right|=\left|\frac{\frac{-16\ln(\alpha)(-1)^n}{\pi}+\frac{\pi(6-2\sqrt{5})}{2\ln(\alpha)}\left(\frac{1}{\alpha^{2n+1}}\right)}{\frac{-16\ln(\alpha)(-1)^{n-1}}{\pi}+\frac{\pi(6-2\sqrt{5})}{2\ln(\alpha)}\left(\frac{1}{\alpha^{2n-1}}\right)}\right|=\frac{1}{\alpha^2}\left|\frac{\frac{16(\ln\alpha)^2\alpha^{2n}\alpha}{\pi^2(6-2\sqrt{5})}(-1)^n+1}{\frac{16(\ln\alpha)^2\alpha^{2n}\alpha}{\pi^2(6-2\sqrt{5})}(-1)^n-\alpha^2}\right|. \tag{3.2}$$

If we consider the value of n as $n=2m$, then we have

$$\lambda_{2m}=\left|\frac{A_{2m,2m+1}}{A_{2m-1,2m}}\right|=\left|\frac{\frac{16(\ln\alpha)^2\alpha^{4m-1}}{\pi^2(6-2\sqrt{5})}+\frac{1}{\alpha^2}}{\frac{16(\ln\alpha)^2\alpha^{4m-1}}{\pi^2(6-2\sqrt{5})}-1}\right|>1.$$

Also, if we take $n = 2m+1$, we get

$$\lambda_{2m+1} = \left|\frac{A_{2m+1,2m+2}}{A_{2m,2m+1}}\right| = \left|\frac{\frac{16(\ln\alpha)^2\alpha^{4m-1}}{\pi^2(6-2\sqrt{5})} - \frac{1}{\alpha^2}}{\frac{16(\ln\alpha)^2\alpha^{4m-1}}{\pi^2(6-2\sqrt{5})}+1}\right| < 1.$$

From (3.2), we obtain

$$\lim_{n\to\infty}\lambda_n = \frac{1}{\alpha^2}\left|\frac{1}{\frac{1}{\alpha^2}}\right| = \frac{1}{\alpha^2}\alpha^2 = 1.$$

Thus, we can interpret that the area values of consecutive segments are equal at infinity. $\square$

Next, we find the value of the curvature $\kappa(t)$ of the Binet-Curve for S_t at any arbitrary point $(x(t),y(t))$.

Theorem 3.3. *The curvature of the Binet-Curve S_t at any arbitrary point is*

$$\kappa(t) = \frac{\sqrt{5}\Big(4(\ln\alpha)^3\alpha - \pi^2\ln\alpha\sin(\pi t) - 12\pi(\ln\alpha)^2\cos(\pi t)\Big)}{\Big((6+2\sqrt{5})(\ln\alpha)^2 e^{2t\ln(\alpha)} + \big(\pi^2+\ln^2\alpha\big)(6-2\sqrt{5})e^{-2t\ln(\alpha)} - 8\pi\ln(\alpha)\sin(\pi t) - 8\ln^2\alpha\cos(\pi t)\Big)^{3/2}}.$$

Proof. As discussed earlier, we consider

$$\vec{r}(t) = S_t = (x(t),y(t))$$
$$= \left(\frac{1}{\sqrt{5}}\Big\{(\sqrt{5}+1)e^{t\ln\alpha} + (\sqrt{5}-1)\cos(\pi t)\,e^{-t\ln\alpha}\Big\}, \frac{1}{\sqrt{5}}\Big\{(\sqrt{5}-1)\sin(\pi t)\,e^{-t\ln\alpha}\Big\}\right).$$

We use the curvature formula

$$\kappa(t) = \frac{|x'y'' - x''y'|}{|r'|^3} = \frac{|x'y'' - x''y'|}{(x'^2+y'^2)^{3/2}}.$$

It is not difficult to show that

$$\dot{x} = \frac{1}{\sqrt{5}}\left\{(\sqrt{5}+1)\ln\alpha\, e^{t\ln\alpha} - (\sqrt{5}-1)\ln\alpha\cos\pi t\, e^{-t\ln\alpha} - (\sqrt{5}-1)\pi\sin\pi t\, e^{-t\ln\alpha}\right\},$$

$$\ddot{x} = \frac{1}{\sqrt{5}}\Bigg\{(\sqrt{5}+1)(\ln\alpha)^2 e^{t\ln(\alpha)} + \Big((\ln\alpha)^2+\pi^2\Big)(\sqrt{5}-1)\cos(\pi t)e^{-t\ln(\alpha)}$$
$$+ 2\pi(\sqrt{5}-1)\ln\alpha\sin(\pi t)e^{-t\ln(\alpha)}\Bigg\},$$

$$\dot{y} = \frac{1}{\sqrt{5}}\left\{-(\sqrt{5}-1)\ln\alpha\sin(\pi t)e^{-t\ln(\alpha)} + (\sqrt{5}-1)\pi\cos(\pi t)e^{-t\ln(\alpha)}\right\},$$

$$\ddot{y} = \frac{1}{\sqrt{5}}\Big((\ln\alpha)^2-\pi^2\Big)(\sqrt{5}-1)\sin(\pi t)e^{-t\ln(\alpha)} - 2\pi(\sqrt{5}-1)\ln\alpha\cos(\pi t)e^{-t\ln(\alpha)}.$$

Using these values, we obtain the value of $\kappa(t)$ as follows:

$$\kappa(t)=\frac{\sqrt{5}\Big(4(\ln\alpha)^3\alpha-\pi^2\ln\alpha\sin(\pi t)-12\pi(\ln\alpha)^2\cos(\pi t)\Big)}{\Big((6+2\sqrt{5})(\ln\alpha)^2e^{2t\ln(\alpha)}+\Big(\pi^2+\ln^2\alpha\Big)(6-2\sqrt{5})e^{-2t\ln(\alpha)}-8\pi\ln(\alpha)\sin(\pi t)-8\ln^2\alpha\cos(\pi t)\Big)^{3/2}}. \qquad \square$$

Note 3.1. Taking the limit as $t\to\pm\infty$, we finally get

$$lim_{t\to\infty}\kappa(t)=0\text{and}\lim_{t\to\infty}\kappa(t)=0.$$

Thus, the curvature is zero at any point of the curve S_t, when t is very large. This means that the curve behaves like a straight line at distant points of S_t.

When we analyze the parametric curve $(x(t),y(t))$ of S_t, we conclude the following:

(i) The curve is not symmetric about the x-axis since x is not an even function of t and also not symmetric about the y-axis since y is not an even function of t.

(ii) The curve does not pass through the origin since there does not exist any real value of t at which $x=0$ and $y=0$.

(iii) The curve intersects the x-axis at the points $(2,0),\ (4,0),\dots$

(iv) Now, $\frac{dy}{dx}$ has the form

$$=\frac{\frac{1}{\sqrt{5}}\Big\{-(\sqrt{5}-1)\ln\alpha\sin(\pi t)e^{-t\ln\alpha}+(\sqrt{5}-1)\pi\cos(\pi t)e^{-t\ln\alpha}\Big\}}{\frac{1}{\sqrt{5}}\Big\{(\sqrt{5}+1)\ln\alpha\,\alpha^t-(\sqrt{5}-1)\ln\alpha\cos(\pi t)e^{-t\ln\alpha}-(\sqrt{5}-1)\pi\sin(\pi t)e^{-t\ln\alpha}\Big\}}.$$

If $\frac{dy}{dx}=0$, we get

$$-(\sqrt{5}-1)\ln\alpha\sin(\pi t)e^{-t\ln\alpha}+(\sqrt{5}-1)\pi\cos(\pi t)e^{-t\ln\alpha}=0.$$

Thus,

$$t=\frac{1}{\pi}\tan^{-1}\Big(\frac{\pi}{\ln\alpha}\Big)=0.45162+m,m\in\mathbb{Z},$$

where $\alpha=1.618$. Hence, we conclude that the x-axis is the tangent to the curve at

$$t=0.45162+m,m\in\mathbb{Z}.$$

(v) We further find the maximum and minimum value of this curve. As shown above, $\frac{dy}{dx}=0$ when $t=0.45162+m\ m\in\mathbb{Z}$.
If we let $m=0$, then for $t=0.45162$, it can be observed that $\frac{d^2y}{dx^2}=-12.365<0$. Also, for $m=1$, we get $t=1.45162$ and in this case we observe that $\frac{d^2y}{dx^2}=1.4230>0$. Thus, one of the maximum values of the curve is at the point $(1.8657,\ 0.4395)_{t=0.45162}$, and one of the minimum values is at the point $(2.6828,\ -0.2716)_{t=1.45162}$.

(vi) It can be observed that $\lim_{t\to\infty} x(t) \to \infty$, $\lim_{t\to\infty} y(t) = 0$. Thus, $y = 0$ is an asymptote of the curve.

On the basis of these data, the S_t curve can be traced using the software MATLAB as shown in Figure 3, when $t > 0$ as well as in Figure 4, when $t < 0$.

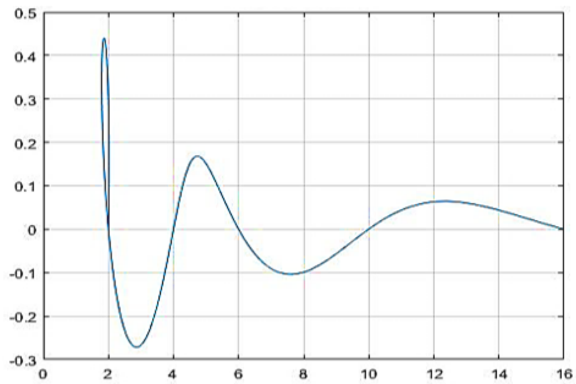

FIGURE 3. S_t-Binet-Curve for $t > 0$

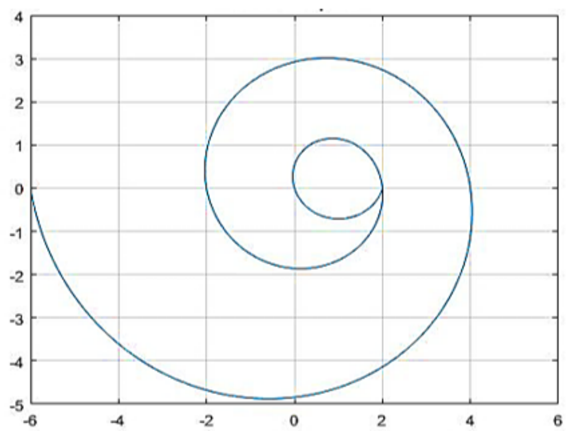

FIGURE 4. S_t-Binet spiral for $t < 0$

4. BINET-CURVE FOR $\{K_n\}_{n\geq 1}$

In the above Binet-type formula for K_n, instead of considering index n as an integer, if we choose n to be any arbitrary real number then it is interesting to study the nature of this transformed sequence $\{K_n\}_{n\geq 1}$. We replace integer value n by real variable t. Then it can be easily observed that the corresponding number K_n will be a complex number. We thus consider the Binet-type formula for these numbers as

$$K_t = \frac{1}{2\sqrt{2}}\left\{\left(\sqrt{2}+1\right)^t + \left(\sqrt{2}-1\right)\delta^t\right\},$$

where t is any arbitrary real number.

Now, since$(-1)^t = e^{\log(-1)} = e^{i\pi(2n+1)}$, where $n = 0, \pm1, \pm2, \ldots$,in the following calculations we choose only principal value with $n = 0$. Then we get

$$\delta^t = \left(-\frac{1}{\gamma}\right)^t = \frac{(-1)^t}{\gamma^t} = \frac{e^{i\pi t}}{\gamma^t}.$$

Now,

$$\begin{aligned} K_t &= \frac{1}{2\sqrt{2}}\left\{(\sqrt{2}+1)^t\gamma^t + (\sqrt{2}-1)\delta^t\right\} \\ &= \frac{1}{2\sqrt{2}}\left\{(\sqrt{2}+1)^t e^{t\ln(\gamma)} + (\sqrt{2}-1)\cos(\pi t)e^{-t\ln(\gamma)} + i(\sqrt{2}-1)\sin(\pi t)e^{-t\ln(\gamma)}\right\}. \end{aligned}$$

This function describes a curve in the complex plane parameterized by the real variable t. We define the "Binet-Curve" for $-\infty < t < \infty$ in a parametric form as

$$K_t = (x(t), y(t)),$$

where

$$\mathrm{Re}(K_t) = x(t) = \frac{1}{2\sqrt{2}}\left\{(\sqrt{2}+1)^t e^{t\ln(\gamma)} + (\sqrt{2}-1)\cos(\pi t)e^{-t\ln(\gamma)}\right\},$$

$$\mathrm{Im}(K_t) = y(t) = \frac{1}{2\sqrt{2}}\left\{(\sqrt{2}-1)\sin(\pi t)e^{-t\ln(\gamma)}\right\}.$$

We obtain the area under the Binet-Curve for K_t for any arbitrary real number t within the fixed interval.

Theorem 4.1. *The area of the segment under the Binet-Curve for K_t within the interval $[n, n+1]$ is given by*

$$A_{n,n+1} = \frac{1}{16}\left\{\frac{-4\ln(\gamma)(-1)^n}{\pi} + \frac{\pi(3-2\sqrt{2})}{\ln(\gamma)} \cdot \frac{1}{\gamma^{2n+1}}\right\}.$$

Proof. We use Green's theorem to calculate the area value of segments

$$A_{n,n+1} = \frac{1}{2}\int_n^{n+1} (x\,dy - y\,dx).$$

By rewriting $dy = \left(\frac{dy}{dt}\right)dt$ and $dx = \left(\frac{dx}{dt}\right)dt$, the formula becomes

$$A_{n,n+1} = \frac{1}{2}\int_n^{n+1} (x\dot{y} - y\dot{x})\,dt. \tag{4.1}$$

We use components $x(t)$ and $y(t)$ of the Binet-type formula for the real argument and calculate this value. Now,

$$\dot{x} = \frac{1}{2\sqrt{2}}\Bigg\{(\sqrt{2}+1)\ln\gamma e^{t\ln(\gamma)} - (\sqrt{2}-1)\ln\gamma\cos(\pi t)\,e^{-t\ln(\gamma)}$$

$$-(\sqrt{2}-1)\pi\sin(\pi t)\,e^{-t\ln(\gamma)}\Bigg\},$$

$$\dot{y} = \frac{1}{2\sqrt{2}}\left\{-(\sqrt{2}-1)\ln\gamma\sin(\pi t)e^{-t\ln(\gamma)} + \pi(\sqrt{2}-1)\cos(\pi t)e^{-t\ln(\gamma)}\right\}.$$

Substituting these values in (4.1) we get

$$A_{n,n+1} = \frac{1}{16}\left\{\frac{-4\ln(\gamma)(-1)^n}{\pi} + \frac{\pi(3-2\sqrt{2})}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n+1}}\right)\right\}.$$

□

Remark 4.1. If we take the limit value of the area segment at infinity, we have

$$A_\infty = \lim_{n\to\infty}|A_{n,n+1}| = \frac{1}{4\pi}\ln(\gamma).$$

Thus, we find that the sequence of the segment's areas has a finite limit.

We next consider the ratio of the area of two consecutive segments and show that it tends to 1 as $n\to\infty$.

Theorem 4.2. *The ratio of the area of two consecutive segments for $\{K_n\}_{n\geq 1}$ tends to* 1 *as n tends to* ∞.

Proof. From the above theorem, the area formula for the Binet-Curve for S_t is

$$A_{n,n+1} = \frac{1}{16}\left\{\frac{-4\ln(\gamma)(-1)^n}{\pi} + \frac{\pi(3-2\sqrt{2})}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n+1}}\right)\right\}.$$

Now we replace n by $n-1$ and $n+1$ by n to get

$$A_{n-1,n} = \frac{1}{16}\left\{\frac{-4\ln(\gamma)(-1)^{n-1}}{\pi} + \frac{\pi(3-2\sqrt{2})}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n-1}}\right)\right\}.$$

We then express λ_n as the ratio of these two areas. This gives

$$\lambda_n = \left|\frac{A_{n,n+1}}{A_{n-1,n}}\right| = \left|\frac{\frac{-4\ln(\gamma)(-1)^n}{\pi} + \frac{\pi(3-2\sqrt{2})}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n+1}}\right)}{\frac{-4\ln(\gamma)(-1)^{n-1}}{\pi} + \frac{\pi(3-2\sqrt{2})}{\ln(\gamma)}\left(\frac{1}{\gamma^{2n-1}}\right)}\right| = \frac{1}{\gamma^2}\left|\frac{\frac{4(\ln\gamma)^2\gamma^{2n}\gamma}{\pi^2(3-2\sqrt{2})}(-1)^n+1}{\frac{4(\ln\gamma)^2\gamma^{2n}}{\pi^2(3-2\sqrt{2})}(-1)^n-1}\right|. \tag{4.2}$$

If we consider the value of n as $n = 2m$ and $n = 2m+1$, then we get

$$\lambda_{2m} = \left|\frac{A_{2m,2m+1}}{A_{2m-1,2m}}\right| = \left|\frac{\dfrac{4(\ln\gamma)^2\gamma^{4m+1}}{\pi^2(3-2\sqrt{2})}+1}{\dfrac{4(\ln\gamma)^2\gamma^{4m-1}}{\pi^2(3-2\sqrt{2})}-1}\right| > 1.$$

$$\lambda_{2m+1} = \left|\frac{A_{2m+1,2m+2}}{A_{2m,2m+1}}\right| = \left|\frac{-\dfrac{4(\ln\gamma)^2\gamma^{4m+1}}{\pi^2(3-2\sqrt{2})}+\gamma^2}{\dfrac{4(\ln\gamma)^2\gamma^{4m-1}}{\pi^2(3-2\sqrt{2})}-1}\right| < 1.$$

From (4.2), we get

$$\lim_{n\to\infty}\lambda_n = \frac{1}{\gamma^2}\left|\frac{\gamma}{1}\right| = \frac{1}{\gamma^2}\gamma^2 = 1.$$

Thus, we can interpret that the area values of consecutive segments are equal at infinity. $\square$

We find the value of the curvature $\kappa(t)$ of the Binet-Curve for K_t at any arbitrary point $(x(t), y(t))$.

Theorem 4.3. *The curvature of the Binet-Curve K_t at any given point is*

$$\kappa(t) = \frac{2\sqrt{2}\left\{\left[2(\ln\gamma)^3-\pi^2\ln(\gamma)\right]\sin(\pi t)-3\pi(\ln\gamma)^2\cos(\pi t)\right\}+\left\{\pi(\ln\gamma)^2+\pi^3\right\}(3-2\sqrt{2})e^{-2t\ln(\gamma)}}{\left((3+2\sqrt{2})(\ln\gamma)^2e^{2t\ln(\gamma)}+(3-2\sqrt{2})(\pi^2+(\ln\gamma)^2)e^{-2t\ln(\gamma)}-2\pi\ln(\gamma)\sin(\pi t)-2(\ln\gamma)^2\cos(\pi t)\right)^{3/2}}.$$

Proof. As discussed earlier, we consider

$$\begin{aligned}\vec{r}(t) &= K_t = (x(t), y(t))\\ &= \left(\frac{1}{2\sqrt{2}}\left\{(\sqrt{2}+1)e^{t\ln\gamma}+(\sqrt{2}-1)\cos\pi t\, e^{-t\ln\gamma}\right\}, \frac{1}{2\sqrt{2}}\left\{(\sqrt{2}-1)\sin\pi t\, e^{-t\ln\gamma}\right\}\right).\end{aligned}$$

We use the curvature formula

$$\kappa(t) = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{|\dot{r}|^3} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(x^2+y^2)^{3/2}}.$$

Now, it can be shown that

$$\dot{x} = \frac{1}{2\sqrt{2}}\left\{(\sqrt{2}+1)\ln\gamma e^{t\ln\gamma}-(\sqrt{2}-1)\ln\gamma\cos\pi t\, e^{-t\ln\gamma}-\pi(\sqrt{2}-1)\sin\pi t\, e^{-t\ln\gamma}\right\},$$

$$\ddot{x} = \frac{1}{2\sqrt{2}}\Bigg\{(\sqrt{2}+1)(\ln\gamma)^2e^{t\ln(\gamma)}+(\sqrt{2}-1)\left((\ln\gamma)^2+\pi^2\right)\cos(\pi t)e^{-t\ln(\gamma)}$$

$$+2\pi(\sqrt{2}-1)(\ln\gamma)\sin(\pi t)e^{-t\ln(\gamma)}\Bigg\},$$

$$\dot{y}=\frac{1}{2\sqrt{2}}\left\{-(\sqrt{2}-1)\ln\gamma\sin(\pi t)e^{-t\ln(\gamma)}+\pi(\sqrt{2}-1)\cos(\pi t)e^{-t\ln(\gamma)}\right\},$$

$$\ddot{y}=\frac{1}{2\sqrt{2}}\left\{\left((\ln\gamma)^2-\pi^2\right)(\sqrt{2}-1)\sin(\pi t)e^{-t\ln(\gamma)}-2\pi(\sqrt{2}-1)\ln\gamma\cos(\pi t)e^{-t\ln(\gamma)}\right\}.$$

Using these we obtain the value of $\kappa(t)$ as follows:

$$\kappa(t)=\frac{2\sqrt{2}\left\{\left[2(\ln\gamma)^3-\pi^2\ln(\gamma)\right]\sin(\pi t)-3\pi(\ln\gamma)^2\cos(\pi t)\right\}+\left\{\pi(\ln\gamma)^2+\pi^3\right\}(3-2\sqrt{2})e^{-2t\ln(\gamma)}}{\left((3+2\sqrt{2})(\ln\gamma)^2e^{2t\ln(\gamma)}+(3-2\sqrt{2})(\pi^2+(\ln\gamma)^2)e^{-2t\ln(\gamma)}-2\pi\ln(\gamma)\sin(\pi t)-2(\ln\gamma)^2\cos(\pi t)\right)^{3/2}}. \qquad \square$$

Note 4.1. By taking the limit as $t\to\pm\infty$, we finally get

$$\lim_{t\to\infty}\kappa(t)=0 \text{and} \lim_{t\to-\infty}\kappa(t)=0.$$

Thus, the curvature is zero at any point of the curve K_t, when t is very large. This means the curve behaves like a straight line at distant points of K_t.

Now, we analyze the parametric curve $(x(t),y(t))$ of K_t numbers and we conclude the following:

(i) The curve is not symmetric about the x-axis since x is not an even function of t and not symmetric about the y-axis since y is not an even function of t.

(ii) The curve does not pass through the origin since there does not exist any real value of t at which $x=0$ and $y=0$.

(iii) The curve intersects the x-axis at the points $(1,0),(2,0),(5,0),\ldots$.

(iv) When we consider

$$\frac{dy}{dx}=\frac{\frac{1}{2\sqrt{2}}\left\{-(\sqrt{2}-1)\ln y\sin\pi te^{-t\ln\gamma}+\pi(\sqrt{2}-1)\cos\pi te^{-t\ln\gamma}\right\}}{\frac{1}{2\sqrt{2}}\left\{(\sqrt{2}+1)\ln\gamma e^{t\ln\gamma}-(\sqrt{2}-1)\ln y\cos\pi te^{-t\ln\gamma}-\pi(\sqrt{2}-1)\sin\pi te^{-t\ln\gamma}\right\}},$$

then $\frac{dy}{dx}=0$ implies that

$$t=\frac{1}{\pi}\tan^{-1}\left(\frac{\pi}{\ln y}\right)=0.412936+m, m\in\mathbb{Z},$$

where $y=2.4142$. Hence, we conclude that the x-axis is the tangent to the curve at $t=0.412936+m, m\in\mathbb{Z}$.

(v) We further find the maximum and minimum value of this curve. As shown above, $\frac{dy}{dx}=0$ when $t=0.412936+m,\ m\in\mathbb{Z}$.
If we let $m=0$, then for $t=0.41293$ it can be observed that
$\frac{d^2y}{dx^2}=-11.892<0$. Also, for $m=1$, we get $t=1.412936$ and in this case we observe that $\frac{d^2y}{dx^2}=0.06276>0$. Thus, one of the maximum values of the curve is at the point $(1.25574,\ 0.09798)_{t=0.412936}$ and one of the minimum values is at the point $(2.95384,\ -0.04058)_{t=1.412936}$.

(vi) It can be observed that $\lim_{t\to\infty} x(t) \to \infty$ when $y(t) = 0$. Thus, $y = 0$ is an asymptote of the curve.

On the basis of these data, the K_t curve can be traced using the software MATLAB as shown in Figure 5, when $t > 0$ as well as in Figure 6, when $t < 0$.

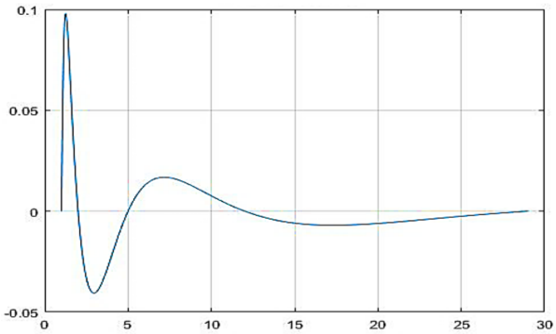

FIGURE 5. K_t-Binet-Curve for $t > 0$

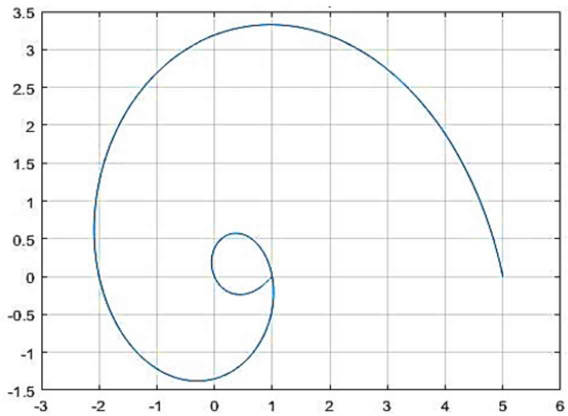

FIGURE 6. K_t-Binet spiral for $t < 0$

REFERENCES

[1] M. Agnes, N. Buenaventura, J. J. Labao, C. K. Soria, K. A. Limbaco, and L. Natividad, *Inclusion of inserted terms (Fibonacci mean) in a Fibonacci sequence*, Math Investigatory Project, Central Luzon State University, 2010.

[2] S. Arvadia, *Left k-Fibonacci sequence and related identities*, Journal Club for Applied Sciences, Vol. 2, no. 1 (2015), 20–26.
[3] S. Arvadia, *Right k-Fibonacci sequence and related identities*, International Research Journal of Mathematics, Engineering and IT, Vol. 2, no. 4 (2015), 25–39.
[4] G. Bilgici, *New generalizations of Fibonacci and Lucas Sequences*, Applied Mathematical Sciences, Vol. 8, no. 29 (2014), 1429–1437.
[5] D. Burton, *Elementary Number Theory*, 7th Edition, McGraw Hill Education (India) Private Limited, New Delhi, 2015.
[6] H. S. Dhami, *Differential Calculus*, New Age International (P) Limited Publishers, 2007.
[7] M. A. Gopalan and A. Geetha, *Observations on half-companion Pell numbers*, International Journal of Scientific and Research Publications, Vol. 4, no. 2 (2014).
[8] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Pure and Applied Mathematics, Wiley-Interscience, New York, NY, USA, 2001.
[9] T. Koshy, *Pell and Pell–Lucas Numbers with Applications*, Springer-Verlag, New York, 2014.
[10] M. Özvatan and O. K. Pashaev, *Generalized Fibonacci Sequences and Binet-Fibonacci Curves*, `https://www.researchgate.net/publication/318785379`.

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