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THE *OP* SUBGROUP OF A TORSION-FREE LCA GROUP

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ABSTRACT. Let G be a locally compact abelian (LCA) group. We denote by G_{op}, the intersection of all open pure subgroups of G, which we call the *OP* subgroup of G. In this paper, we prove that the *OP* subgroup of a non-discrete, torsion-free LCA group G is a non-zero pure subgroup of G.

1. INTRODUCTION

Let £ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. For a group $G \in$ £, the identity component subgroup, and the subgroup of all compact elements of G are denoted by G_0 and bG, respectively. A subgroup H of G is called pure if $nH = H \cap nG$ for all positive integers n. Recall that G_0 and bG are two important and well-known pure subgroups of G. In this paper, we produce a new and non-zero pure subgroup of a torsion-free group G. Let G_{op} be the intersection of all open pure subgroups of $G \in$ £. Then G_{op} is a closed subgroup which we call the *OP* subgroup of G. In this paper, we focus on the OP subgroup of a torsion-free group. We show that if G is a torsion-free group, then G_{op} is a non-zero pure subgroup of G (see Theorem 2.1).

The additive topological group of real numbers is denoted by $\mathbb{R}$, $\mathbb{Q}$ is the group of rationals with the discrete topology, $\mathbb{Z}$ is the group of integers, $\mathbb{Z}(n)$ the cyclic group of order n and, $\mathbb{Z}_p$ is the $p-$adic integers group for $p \in P$ where P denotes the set of all prime numbers. The dual group of G is $\hat{G} = Hom(G, \mathbb{R}/\mathbb{Z})$. For more on locally compact abelian groups, see [3].

2. THE *OP* SUBGROUP OF A TORSION-FREE LCA GROUP

In this section, we introduce the concept and study some properties of the *OP* subgroup of a torsion-free LCA group.

Definition 2.1. *[2] A subgroup H of a group G is said to be a pure subgroup if $nH = H \bigcap nG$ for all positive integers n.*

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Let $G \in \pounds$, and G_{op} be the intersection of all open pure subgroups of G. Then G_{op} is a closed subgroup of G.

Definition 2.2. *The subgroup G_{op} of a group $G \in \pounds$ is called the OP subgroup of G.*

Remark 2.1. Let D be a discrete group. It is clear that 0 is an open pure subgroup of D. So $D_{op} = 0$. The converse is not true. See Example 2.1.

Example 2.1. *Let I be an infinite set, and $G = \prod_I \mathbb{Z}(2)$ (G is equipped with the product topology). Set $H_j = \prod_{i \in I} X_i$ where $X_j = 0$ and $X_i = \mathbb{Z}(2)$ for $i \neq j$. Then H_j is pure in G for all $j \in I$, because algebraically it is a direct summand of G (letting $H'_j = \prod_{i \in I} Y_i$ where $Y_j = \mathbb{Z}(2)$ and $Y_i = 0$ if $i \neq j$, one has $G = H_j \bigoplus H'_j$) . On the other hand, H_j is an open subgroup of G for all $j \in I$. Hence $G_{op} \subseteq \bigcap_{j \in I} H_j = 0$.*

Lemma 2.1. *Let G be a group in $\pounds$. Then $G_0 \subseteq G_{op}$.*

Proof. By [3, Theorem 7.8], G_0 is the intersection of all open subgroups of G. So $G_0 \subseteq G_{op}$. □

Corollary 2.1. *Let C be a connected group in $\pounds$. Then $C = C_{op}$.*

Proof. It is clear by Lemma 2.1. □

Definition 2.3. *[1] A group $G \in \pounds$ is said to be pure-simple if and only if G contains no nontrivial closed pure subgroups.*

Example 2.2. *By [1, Theorem 1], $\mathbb{Z}_p$ is a pure-simple group for some prime number p. So $(\mathbb{Z}_p)_{op} = \mathbb{Z}_p$.*

Example 2.2 shows that the converse of Corollary 2.1 need not to be true.

Remark 2.2. Let H be an open pure subgroup of G. It is clear that every open pure subgroup of H is open pure in G. So $G_{op} \subseteq H_{op}$.

Lemma 2.2. *Let $G \in \pounds$ be a torsion-free group and H, an open pure subgroup of G. Then $G_{op} = H_{op}$.*

Proof. By Remark 2.2, $G_{op} \subseteq H_{op}$. Let X be an arbitrary open pure subgroup of G. An easy calculation shows that $X \cap H$ is an open pure subgroup of H. So $G_{op} \subseteq H_{op}$, and the proof is complete. □

Lemma 2.3. *Let $f : G \to H$ be a morphism such that H is a torsion-free group. Then $f(G_{op}) \subseteq H_{op}$.*

Proof. Let $y \in G_{op}$ and X, be an open pure subgroup of H. It is clear that $f^{-1}(X)$ is an open subgroup of G. Now we show that $f^{-1}(X)$ is pure in G. Let n be a positive integer, and $g \in f^{-1}(X) \cap nG$. Then $f(g) \in X$, and $g = ng_1$ for some $g_1 \in G$. So $nf(g_1) \in X$. Since X is pure in H, $nf(g_1) = nx_1$ for some $x_1 \in X$. Since H is torsion-free, it follows that $f(g_1) \in X$. Hence $g \in nf^{-1}(X)$, and $f^{-1}(X)$ is pure in G. So $f(y) \in X$, and the proof is complete. □

Let $G \in \pounds$. We denote by G^*, the minimal divisible extension of G. By [3, 4.18.h], G^* is an LCA group containing G as an open subgroup.

Corollary 2.2. *Let G be a torsion-free group in $\pounds$. Then $G_{op} \subseteq G^*_{op}$.*

Proof. Since G is torsion-free, so is its divisible extension G^*. Now, it is sufficient to consider the inclusion map $i : G \hookrightarrow G^*$ and Lemma 2.3. □

Corollary 2.3. *Let G be a torsion-free group in $\pounds$ such that $G_{op} = G$. Then $G^*_{op} = G^*$.*

Proof. By Corollary 2.2, $G \subseteq G^*_{op}$. Since G^* is divisible, G^*_{op} is divisible as a pure subgroup of G^*. It follows from the minimality G^* that $G^*_{op} = G^*$. □

Lemma 2.4. *Let $G \in \pounds$ be a non discrete, totally disconnected, torsion free LCA group. Then there exists a prime q and a nonzero morphism from $\mathbb{Z}_q$ to G.*

Proof. By [3, Theorem 24.30], G contains a compact open subgroup K. By [3, Theorem 24.23 and 24.25], $\hat{K}$ is a discrete, torsion, divisible group. This implies $\hat{K} \cong \oplus_{p\in P}\mathbb{Z}(p^\infty)^{(I_p)}$ for suitable sets I_p (which for some $p \in P$ can be empty). Then $K \cong \prod_{p\in P} \mathbb{Z}_p^{I_p}$. Now choose $q \in P$ such that I_q is not empty. Thus K contains $\mathbb{Z}_q$, whose inclusion into G gives the required morphism. □

Theorem 2.1. *Let $G \in \pounds$ be a non discrete, torsion free group. Then G_{op} is a non-zero pure subgroup of G.*

Proof. Assume on the contrary, $G_{op} = 0$. Then, by Lemma 2.1, G is a totally disconnected group. By Lemma 2.4, there exists a nonzero morphism $f : \mathbb{Z}_p \to G$ for some prime number p. By Lemma 2.3, $f((\mathbb{Z}_p)_{op}) \subseteq G_{op} = 0$. On the other hand, by Example 2.2, $(\mathbb{Z}_p)_{op} = \mathbb{Z}_p$. It follows that f is a zero morphism which is a contradiction. □

References

[1] D. L. Armacost, *On pure subgroups of LCA groups*, Proc. Amer. Math. Soc., **45** (1974), pp. 414–418.

[2] L. Fuchs, *Infinite Abelian Groups*, Vol. I, Academic Press, New York, 1970.

[3] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Vol I, Second Edition, Springer-Verlag, Berlin, 1979.

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