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# HANKEL DETERMINANTS OF SECOND AND THIRD ORDER FOR THE INVERSE OF A UNIVALENT FUNCTION

MILUTIN OBRADOVIĆ AND NIKOLA TUNESKI

ABSTRACT. In this paper, we determine upper bounds for the modulus of the Hankel determinants of second and third order for the inverse of an univalent function. The bound for the second order determinant is sharp.

## 1. INTRODUCTION AND PRELIMINARIES

As usual let $\mathcal{S}$ denote the class functions of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots, \tag{1.1}$$

which are analytic and univalent in the unit disk $\mathbb{D}:=\{z\in\mathbb{C}:\,|z|<1\}$.

For our consideration we use the definition and some results on Grunsky coefficients. Let $f\in\mathcal{S}$ and let

$$\log\frac{f(t)-f(z)}{t-z}=\sum_{p,q=0}^{\infty}\omega_{p,q}t^pz^q,$$

where $\omega_{p,q}$ are called Grunsky's coefficients with property $\omega_{p,q}=\omega_{q,p}$. For those coefficients we have the following Grunsky's inequality ( [3,9]):

$$\sum_{q=1}^{\infty}q\left|\sum_{p=1}^{\infty}\omega_{p,q}x_p\right|^2\le\sum_{p=1}^{\infty}\frac{|x_p|^2}{p}, \tag{1.2}$$

where $x_p$ are arbitrary complex numbers such that the last series converges.

Also, it is known (see [3,16]) that the square root transformation is preserved in the class $\mathcal{S}$, i.e., for a function $f$ from $\mathcal{S}$,

$$f_2(z)=\sqrt{f(z^2)}=z+b_3z^3+b_5z^5+\cdots \tag{1.3}$$

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is also in $\mathcal{S}$. For the function $f_2$ we have the appropriate Grunsky coefficients of the form $\omega^{(2)}_{2p-1,2q-1}$. Further we omit the upper index "(2)" and write only $\omega_{2p-1,2q-1}$. In that sense, the inequality (1.2) has the form

$$\sum_{q=1}^{\infty}(2q-1)\left|\sum_{p=1}^{\infty}\omega_{2p-1,2q-1}x_{2p-1}\right|^2 \le \sum_{p=1}^{\infty}\frac{|x_{2p-1}|^2}{2p-1}. \tag{1.4}$$

Similarly, we have a similar type of inequality that follows from relation (15) on page 57 in [9]:

$$\left|\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\omega_{2p-1,2q-1}x_{2p-1}x_{2q-1}\right| \le \sum_{p=1}^{\infty}\frac{|x_{2p-1}|^2}{2p-1}. \tag{1.5}$$

Choosing $x_{2p-1}=0$ and $p=3,4,\dots$, in (1.4) and (1.5) we obtain

$$|\omega_{11}x_1+\omega_{31}x_3|^2+3|\omega_{13}x_1+\omega_{33}x_3|^2+5|\omega_{15}x_1+\omega_{35}x_3|^2 \le |x_1|^2+\frac{|x_3|^2}{3} \tag{1.6}$$

and

$$|\omega_{11}x_1^2+2\omega_{13}x_1x_3+\omega_{33}x_3^2| \le |x_1|^2+\frac{|x_3|^2}{3}, \tag{1.7}$$

respectively. From (1.6), for $x_1=1$ and $x_3=0$, we obtain

$$|\omega_{11}|^2+3|\omega_{13}|^2+5|\omega_{15}|^2 \le 1, \tag{1.8}$$

and for $x_1=0$ and $x_3=1$, we obtain

$$|\omega_{13}|^2+3|\omega_{33}|^2+5|\omega_{35}|^2 \le \frac{1}{3}. \tag{1.9}$$

As it has been shown in [9, p.57], if $f\in\mathcal{S}$ is given by (1.1), then its coefficients $a_2$, $a_3$, $a_4$ and $a_5$ can be expressed in terms of the Grunsky coefficients $\omega_{2p-1,2q-1}$ of the function $f_2$ given by (1.3) in the following way:

$$\begin{aligned}
a_2 &= 2\omega_{11},\\
a_3 &= 2\omega_{13}+3\omega_{11}^2,\\
a_4 &= 2\omega_{33}+8\omega_{11}\omega_{13}+\frac{10}{3}\omega_{11}^3,\\
a_5 &= 2\omega_{35}+8\omega_{11}\omega_{33}+5\omega_{13}^2+18\omega_{11}^2\omega_{13}+\frac{7}{3}\omega_{11}^4,\\
0 &= 3\omega_{15}-3\omega_{11}\omega_{13}+\omega_{11}^3-3\omega_{33}.
\end{aligned} \tag{1.10}$$

It is important to point out that in the original source ( [9, p.57]) the third term for the coefficient $a_5$, $5\omega_{13}^2$, is written as $5\omega_{15}^2$, which turns out to be wrong.

Further, due to the famous Koebe's 1/4 theorem, a function $f \in \mathcal{S}$ given by (1.1) has its inverse at least on the disk with radius 1/4 with an expansion

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots . \tag{1.11}$$

Using the identity $f(f^{-1}(w)) = w$, and after comparing the appropriate coefficient, we obtain

$$\begin{aligned} A_2 &= -a_2, \\ A_3 &= -a_3 + 2a_2^2, \\ A_4 &= -a_4 + 5a_2a_3 - 5a_2^3, \\ A_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{1.12}$$

In this paper we find an upper bound of the modulus of the Hankel determinant of second and third order for the inverse function of functions in the class $\mathcal{S}$. The Hankel determinant $H_q(n)(f)$ of a given function $f$, for $q \geq 1$ and $n \geq 1$, is defined by

$$H_q(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix}.$$

The general Hankel determinant is hard to deal with, so the second and the third ones,

$$H_2(2)(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$$

and

$$H_3(1)(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

respectively, are studied instead.

Even for the second and the third order determinant, little is known for the general class of univalent functions. Hayman in [4] showed that $|H_2(n)| = |a_na_{n+2} - a_{n+1}^2| \leq A\sqrt{n}$, where $A$ is an absolute constant, and that this rate of growth is the best possible. Recently, in [13] the authors obtained $|H_2(2)| \leq 1.3614356\ldots$ and $|H_3(1)| \leq 2.321434\cdots$. Current research in the area is mainly focused on the subclasses of univalent functions (starlike, convex, $\alpha$-convex, close-to-convex, spiral-like,......). More significant results can be found in [1, 2, 4–8, 10–12, 17].

Recently, the problem of finding upper bounds of the modulus of the Hankel determinant (second and third order) for the inverse function started attracting interest. In [14] the authors studied it and solved it sharply over the class $\mathcal{U}(\lambda)$, $0 < \lambda \leq 1$, of functions $f \in \mathcal{A}$ satisfying $|(z/f(z))^2 f'(z) - 1| < \lambda$, $z \in \mathbb{D}$. For the classes of starlike and convex functions, as well as for two other classes of univalent functions, the problem was studied in [15] and was solved mostly in a sharp way.

## 2. Main results

As already mentioned, in the next theorem we give the upper bound of the modulus of the second and the third Hankel determinant for the inverse functions of the functions from the class $\mathcal{S}$ (sharp for the second order determinant).

**Theorem 2.1.** *Let $f \in \mathcal{S}$ and $f^{-1}$ be its inverse function. Then*
  ($i$) $|H_2(2)(f^{-1})| \le 3$ *and the result is the best possible;*
  ($ii$) $|H_3(1)(f^{-1})| \le 2.36639\ldots$.

*Proof.* ($i$) By the definition of $H_2(2)$ and the relations (1.12) we have

$$H_2(2)(f^{-1}) = A_2A_4 - A_3^2 = a_2a_4 - a_3^2 - a_2^2(a_3 - a_2^2),$$

and from here, using (1.10) and some calculations we have

$$H_2(2)(f^{-1}) = 4\omega_{11}\left(\omega_{33} - \omega_{11}\omega_{13} + \frac{5}{12}\omega_{11}^3\right) - 4\omega_{13}^2,$$

which implies

$$|H_2(2)(f^{-1})| \le 4|\omega_{11}|\left|\omega_{33} - \omega_{11}\omega_{13} + \frac{5}{12}\omega_{11}^3\right| + 4|\omega_{13}|^2. \tag{2.1}$$

If we choose $x_1 = -\frac{1}{2}\omega_{11}$ and $x_3 = 1$ in relation (1.7), then we have

$$\left|\frac{1}{4}\omega_{11}^3 - \omega_{11}\omega_{13} + \omega_{33}\right| \le \frac{1}{4}|\omega_{11}|^2 + \frac{1}{3}, \tag{2.2}$$

i.e.,

$$\left|\left(\omega_{33} - \omega_{11}\omega_{13} + \frac{5}{12}\omega_{11}^3\right) - \frac{1}{6}\omega_{11}^3\right| \le \frac{1}{4}|\omega_{11}|^2 + \frac{1}{3},$$

and

$$\left|\omega_{33} - \omega_{11}\omega_{13} + \frac{5}{12}\omega_{11}^3\right| \le \frac{1}{6}|\omega_{11}|^3 + \frac{1}{4}|\omega_{11}|^2 + \frac{1}{3}.$$

From the last inequality and (2.1) we get

$$\left|H_2(2)(f^{-1})\right| \le 4|\omega_{11}|\left[\frac{1}{6}|\omega_{11}|^3 + \frac{1}{4}|\omega_{11}|^2 + \frac{1}{3}\right] + 4|\omega_{13}|^2. \tag{2.3}$$

From (1.8) we have $|\omega_{13}|^2 \le \frac{1}{3}(1 - |\omega_{11}|^2)$, and from (2.3):

$$\left|H_2(2)(f^{-1})\right| \le \frac{4}{3} + \frac{4}{3}|\omega_{11}| - \frac{4}{3}|\omega_{11}|^2 + |\omega_{11}|^3 + \frac{2}{3}|\omega_{11}|^4 = \psi(|\omega_{11}|),$$

where $\psi(t)=\frac{4}{3}+\frac{4}{3}t-\frac{4}{3}t^2+t^3+\frac{2}{3}t^4$ and $t=|\omega_{11}|\in[0,1]$ since by (1.10), for the class $\mathcal{S}$, $2|\omega_{11}|=|a_2|\leq 2$. Then,

$$\psi'(t)=\frac{4}{3}-\frac{8}{3}t+3t^2+\frac{8}{3}t^3=\frac{4}{3}(1-t)^2+\frac{5}{3}t^2+\frac{8}{3}t^3>0,$$

$t\in[0,1]$, meaning that $\psi(t)\leq\psi(1)=3$, $t\in[0,1]$. Therefore, $\left|H_2(2)(f^{-1})\right|\leq 3$.

The estimate is sharp due to the Koebe function $k(z)=\frac{z}{(1-z)^2}$ such that

$$\begin{aligned}k^{-1}(w)&=\frac{2w+1-\sqrt{1+4w}}{2w}=w+\sum_{2}^{\infty}(-1)^{n+1}\frac{(2n)!}{(n+1)!n!}w^n\\&=w-2w^2+5w^3-14w^4+\cdots,\end{aligned}$$

and $H_2(2)(k^{-1})=A_2A_4-A_3^2=3$.

$(ii)$ From the definition of the third Hankel determonant (1.11) we have

$$H_3(1)(f^{-1})=A_3(A_2A_4-A_3^2)-A_4(A_4-A_2A_3)+A_5(A_3-A_2^2).$$

Using the relations (1.12), after some calculations, we obtain

$$\begin{aligned}H_3(1)(f^{-1})&=a_3(a_2a_4-a_3^2)-a_4(a_4-a_2a_3)+a_5(a_3-a_2^2)-(a_3-a_2^2)^3\\&=H_3(1)(f)-(a_3-a_2^2)^3.\end{aligned}\tag{2.4}$$

Now, using (1.10) and (2.4), after some calculations and rearrangements, we have

$$\begin{aligned}H_3(1)(f^{-1})=\omega_{11}^2\left(2\omega_{13}-\omega_{11}^2\right)^2+\left(2\omega_{35}+3\omega_{13}^2\right)\left(2\omega_{13}-\omega_{11}^2\right)\\-\left(2\omega_{33}-\frac{2}{3}\omega_{11}^3\right)^2,\end{aligned}$$

which implies

$$\begin{aligned}\left|H_3(1)(f^{-1})\right|=|\omega_{11}|^2\left|2\omega_{13}-\omega_{11}^2\right|^2+\left(2|\omega_{35}|+3|\omega_{13}|^2\right)\left|2\omega_{13}-\omega_{11}^2\right|\\+\left|2\omega_{33}-\frac{2}{3}\omega_{11}^3\right|^2.\end{aligned}$$

Since $|2\omega_{13}-\omega_{11}^2|=|a_3-a_2^2|\leq 1$ (see [3] for $|a_3-a_2^2|\leq 1$):

$$\left|H_3(1)(f^{-1})\right|\leq\underbrace{|\omega_{11}|^2+3|\omega_{13}|^2+2|\omega_{35}|}_{=B_1}+\underbrace{\left|2\omega_{33}-\frac{2}{3}\omega_{11}^3\right|^2}_{=B_2}.\tag{2.5}$$

For the expression $B_1$, from the inequalities (1.8) and (1.9), we have

$$|\omega_{11}|^2+3|\omega_{13}|^2\leq 1$$

and

$$|\omega_{35}|\leq\frac{1}{\sqrt{15}}.$$

From all these facts we have
$$B_1 \le 1+\frac{2}{\sqrt{15}}. \tag{2.6}$$

For the expression $B_2$, we use the last relation from (1.10) and we get $\omega_{33} = \omega_{15}-\omega_{11}\omega_{13}+\frac{1}{3}\omega_{11}^3$, which, together with (2.5), further leads to $B_2 = 4|\omega_{15}-\omega_{11}\omega_{13}|^2$. Therefore

$$\begin{aligned} B_2 &\le 4\left(|\omega_{15}|+|\omega_{11}||\omega_{13}|\right)^2 \\ &= 4\left(\frac{1}{\sqrt{5}}\sqrt{1-|\omega_{11}|^2-3|\omega_{13}|^2}+|\omega_{11}||\omega_{13}|\right)^2 = 4\varphi_2^2(|\omega_{11}|,|\omega_{13}|), \end{aligned}$$

where $\varphi_2(x,y) = \frac{1}{\sqrt{5}}\sqrt{1-x^2-3y^2}+xy$, with $0\le x\le 1$ and $0\le y\le \frac{1}{\sqrt{3}}\sqrt{1-x^2}$.

From the system of equations $[\varphi_2(x,y)]'_x = 0$ and $[\varphi_2(x,y)]'_y = 0$ we receive the unique interior critical point $\left(\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}\right)$ and $\varphi_2\left(\frac{1}{\sqrt{5}},\frac{1}{\sqrt{5}}\right) = \frac{4}{5\sqrt{3}} = 0.46188\ldots$. On the boundary we have

- $\varphi_2(x,0) = \frac{1}{\sqrt{5}}\sqrt{1-x^2} \le \frac{1}{\sqrt{5}} = 0.44721\ldots$;
- $\varphi_2(0,y) = \frac{1}{\sqrt{5}}\sqrt{1-3y^2} \le \frac{1}{\sqrt{5}}$;
- $\varphi_2\left(x,\frac{1}{\sqrt{3}}\sqrt{1-x^2}\right) = \frac{1}{\sqrt{3}}x\sqrt{1-x^2} \le \frac{1}{2\sqrt{3}} < \frac{1}{\sqrt{5}}$, for $0\le x\le 1$.

From all this we conclude that
$$B_2 \le 4\cdot\left(\frac{4}{5\sqrt{3}}\right)^2 = \frac{64}{75} = 0.85(3). \tag{2.7}$$

Finally, from (2.5), (2.6) and (2.7) we obtain
$$\left|H_3(1)(f^{-1}\right| \le 1+\frac{2}{5} = 2.36639\ldots.$$

□

**Corollary 2.1.** *Let $f\in\mathcal{S}$. Then the following estimates are sharp:*

(*i*) $\left|H_2(2)(f^{-1})-H_2(2)(f)\right| \le 4$;

(*ii*) $\left|H_3(1)(f^{-1})-H_3(1)(f)\right| \le 1$.

*Proof.*

(*i*) From (2.2) we obtain that for any function $f\in\mathcal{S}$,
$$\left|H_2(2)(f^{-1})-H_2(2)(f)\right| = |a_2|^2|a_3-a_2^2| \le 4,$$
since $|a_2|\le 2$ and $|a_3-a_2^2|\le 1$ (see [3]). The above estimate is sharp since for the Koebe function we have $H_2(2)(k) = 2\cdot 4-3^2 = -1$ and $H_2(2)(k^{-1}) = (-2)\cdot(-14)-5^2 = 3$, so that $H_2(2)(k^{-1})-H_2(2)(k) = 4$.

(*ii*) From (2.4), for $f \in \mathcal{S}$ we have

$$\left|H_3(1)(f^{-1}) - H_3(1)(f)\right| = |a_3 - a_2^2|^3 \leq 1,$$

for every $f \in \mathcal{S}$. The above inequality is also sharp as the function $f_1(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + \cdots$ with $a_2 = a_4 = 0$ and $a_3 = a_5 = 1$ shows. Indeed, from (1.12) we obtain $A_2 = A_4 = 0$, $A_3 = -1$, $A_5 = 2$, i.e., $H_3(1)(f_1) = 0$, $H_3(1)(f_1^{-1}) = -1$ and $|H_3(1)(f_1^{-1}) - H_3(1)(f_1)| = 1$.

□

***Remark*** 2.1. The estimates for the second Hankel determinant (Theorem 2.1(i) and Corollary 2.1(i)) are of course valid, but even more sharp for all classes of univalent functions containing the Koebe function, such as the classes of starlike functions, the class $\mathcal{U}$, et c.. This goes in line with Theorem 1(i) from [14] with $\lambda = 1$ in it.

Additionally, the estimate 3 for the second Hankel determinant for starlike functions was obtained in [15] using a different method based on coefficients of Schwartz functions.

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Milutin Obradović
Department of Mathematics
Faculty of Civil Engineering
University of Belgrade
Bulevar Kralja Aleksandra 73, 11000, Belgrade
Serbia.
e-mail: *obrad@grf.bg.ac.rs*

*and*

* Corresponding author

Nikola Tuneski*
Department of Mathematics and Informatics
Faculty of Mechanical Engineering
Ss. Cyril and Methodius University in Skopje
Karpoš II b.b., 1000 Skopje
Republic of North Macedonia
e-mail: *nikola.tuneski@mf.ukim.edu.mk*