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# SOME RESULTS FOR ANALYTIC FUNCTIONS ASSOCIATED WITH GREGORY COEFFICIENTS

BÜLENT NAFI ÖRNEK

ABSTRACT. This paper investigates the intricate relationship between Gregory coefficients and the Schwarz Lemma, two concepts arising from different branches of mathematics—numerical analysis and complex analysis, respectively. By examining the analytic function defined by $f'(z) = \frac{\varphi(z)}{\ln(1+\varphi(z))}$ , we derive upper bounds for Taylor coefficients and analyze the modulus of the second derivative, utilizing the Schwarz Lemma. Through this exploration, we highlight the sharp inequalities and demonstrate their significance in the geometric theory of functions. The results obtained extend the classical Schwarz Lemma to boundary behaviors and offer valuable contributions to both numerical and complex analysis.

## 1. INTRODUCTION

Gregory coefficients play a significant role in numerical analysis, particularly in improving the accuracy of numerical integration. On the other hand, Schwarz Lemma is a fundamental result in complex analysis with profound implications in function theory. Mathematics encompasses a diverse array of topics, each with its specialized tools and theorems. Among these are Gregory coefficients, vital for numerical integration, and the Schwarz Lemma, a cornerstone in complex analysis. Despite belonging to different branches of mathematics, both concepts share the common goal of providing deeper insights and more accurate solutions to mathematical problems. This article explores both concepts, their origins, and their applications, and highlights their unique contributions to their respective fields of mathematics. In this article, the relationship between both concepts will be emphasized. Additionally, the application of the Schwarz Lemma will be given with the help of the analytic function expressing Gregory coefficients. Gregory coefficients [6,13], alternatively referred to as reciprocal logarithmic numbers, Bernoulli

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numbers of the second kind, or Cauchy numbers of the first kind, are rational numbers that decrease in value. That is,

$$\frac{z}{\ln(1+z)} = 1+\frac{1}{2}z-\frac{1}{12}z^2+\frac{1}{24}z^3-\frac{19}{720}z^4+\frac{3}{160}z^5-...., \ z\in D.$$

Now, we examine the function $\frac{z}{\ln(1+z)}$ defined on the open unit disc $D=\{z:|z|<1\}$. Let $\mathcal{A}$ represent the class of functions in the form $f(z)=z+b_2z^2+b_3z^3+...$ which are analytic in the unit disc $D$. Also, let $\mathcal{G}$ be the subclass of $\mathcal{A}$ satisfying the condition

$$f'(z)\prec\frac{z}{\ln(1+z)},$$

where, the symbol "$\prec$" indicates the subordinate principle [5]. Determining the upper bound of Taylor coefficients has been a crucial focus in the study of geometric properties, providing valuable insights into various subclasses of $\mathcal{G}$. Therefore, in this section, there will be an upper bound for $b_2$, one of the coefficients of the $f(z)$ function. For this we will need to use the Schwarz Lemma. Additionally, in the next section, we will evaluate the module of the second derivative of the $f(z)$ function from below. Therefore, we will need Schwarz Lemma on the boundary.

Schwarz Lemma asserts that for any analytic function $\varphi(z)$ mapping the unit disc $D$ into itself and satisfying $\varphi(0)=0$, the magnitude of $\varphi(z)$ is bounded by $|z|$ within the unit disc and $|\varphi'(0)|\leq 1$. In simpler terms, it asserts that if a function maps the unit disc into itself and has its origin mapped to the origin, then the function cannot magnify the distances between points inside the disc by more than 1 [5].

Extending the Schwarz Lemma to the boundary of the unit disc provides valuable insights into the behavior of analytic functions near this boundary, making it particularly useful for analyzing functions approaching the disc's edge. In this paper, we embark on a journey to explore the application and implications of this fascinating theorem to different classes. The Schwarz Lemma implies the boundary Schwarz Lemma $|\varphi'(1)|\geq 1$. Osserman and Unkelbach [9, 14] showed that in this case, we have in fact

$$\left|\varphi'(1)\right|\geq 1+\frac{1-|\varphi'(0)|}{1+|\varphi'(0)|}=\frac{2}{1+|\varphi'(0)|},$$

where $\varphi$ satisfies the conditions of the Schwarz Lemma and $\varphi$ extends continuously to the boundary point $1\in\partial D=\{z:|z|=1\}$, $|\varphi(1)|=1$ and $\varphi'(1)$ exist. These inequalities are sharp. In mathematical literature, these inequalities and their extensions are subjects of ongoing debate and are highly significant in the geometric theory of functions. [1–4, 7–11].

If we use the subordinate principle for the class we defined above, there exists Schwarz function $\varphi(z)$ such that

$$f'(z)=\frac{\varphi(z)}{\ln(1+\varphi(z))},$$

where we has selected the principal branches of logarithmic functions.

Here, the function $\varphi(z)$ meets the criteria of the Schwarz Lemma [5]. Therefore, applying the Schwarz Lemma, we derive

$$1+2b_2z+3b_3z^2+\ldots=\frac{\varphi(z)}{\ln(1+\varphi(z))}=1+\frac{1}{2}\varphi(z)-\frac{1}{12}\varphi(z)^2+\ldots,$$

and

$$2b_2+3b_3z+\ldots=\frac{1}{2}\frac{\varphi(z)}{z}-\frac{1}{12}\frac{\varphi(z)}{z}^2+\ldots.$$

If we take a limit as $z$ approaches zero in the last equation, we take

$$b_2=\frac{1}{4}\varphi'(0)-\frac{1}{12}\varphi'(0)\varphi(0)+\ldots.$$

and

$$|b_2|\leq\frac{1}{4}.$$

We will now demonstrate that the final inequality is sharp. Let

$$f(z)=\int_0^z\frac{t}{\ln(1+t)}dt.$$

Then

$$f'(z)=\frac{z}{\ln(1+z)}$$

$$1+2b_2z+3b_3z^2+\ldots=\frac{z}{\ln(1+z)}=1+\frac{1}{2}z-\frac{1}{12}z^2+\frac{1}{24}z^3-\ldots,$$

$$2b_2+3b_3z+\ldots=\frac{1}{2}-\frac{1}{12}z+\frac{1}{24}z^2-\ldots,$$

and

$$|b_2|=\frac{1}{4}.$$

The Schwarz lemma belonging to this class is expressed as follows.

**Lemma 1.1.** *If $f\in\mathcal{G}$, then*

$$|b_2|\leq\frac{1}{4}.$$

*This result is sharp, as demonstrated by the extremal function*

$$f(z)=\int_0^z\frac{t}{\ln(1+t)}dt=z+\frac{z^2}{4}-\frac{z^3}{36}+\frac{z^4}{96}-\ldots.$$

The subsequent lemma, referred to as the Julia-Wolff lemma, is required for the following discussion(see, [12]). The following lemma shows that simply setting the angular limit is sufficient for our hypothesis.

**Lemma 1.2** (Julia-Wolff lemma)**.** *If $\varphi$ is an analytic function in the unit disc $D$ with $\varphi(0)=0$ and $\varphi(D)\subset D$, and additionally, $\varphi$ has an angular limit $\varphi(1)$ at $1\in\partial D$ where $|\varphi(1)|=1$, then the angular derivative $\varphi'(1)$ exists and $1\le|\varphi'(1)|\le\infty$.*

The following conclusion can be directly drawn from the expression of the Julia-Wolff lemma.

**Corollary 1.1.** *The analytic function $\varphi$ has a finite angular derivative $\varphi'(1)$ if and only if $\varphi'$ has the finite angular limit $\varphi'(1)$ at $1\in\partial D$.*

## 2. Main Results

In this section, the second derivative of the analytic function $f(z)$ will be examined. In this analysis, stronger inequalities will be obtained by taking into account the coefficients in the Taylor expansion of the $f(z)$ function. Moreover, the inequality showing that there is a relationship between the coefficients will be given.

**Theorem 2.1.** *Let $f(z)\in\mathcal{G}$. Suppose that, for $1\in\partial D$, $f$ has an angular limit $f(1)$ at 1, $f'(1)=\frac{1}{\ln 2}$. Then we have the inequality*

$$\left|f''(1)\right|\ge\frac{2\ln 2-1}{2(\ln 2)^2}. \tag{2.1}$$

*This result is sharp, with equality for the function*

$$f(z)=\int_0^z\frac{t}{\ln(1+t)}dt.$$

*Proof.* Let

$$f'(z)=\frac{\varphi(z)}{\ln(1+\varphi(z))}.$$

Then

$$f''(z)=\frac{\varphi'(z)\ln(1+\varphi(z))-\frac{\varphi'(z)}{1+\varphi(z)}\varphi(z)}{\ln^2(1+\varphi(z))},$$

$$f''(1)=\frac{\varphi'(1)\ln(2)-\frac{\varphi'(1)}{2}}{\ln^2(2)},$$

and

$$\varphi'(1)=\frac{\ln^2(2)}{\ln(2)-\frac{1}{2}}f''(1).$$

Since the function $\varphi(z)$ satisfies the conditions of the Schwarz lemma at the boundary, we obtain

$$1\le\left|\varphi'(1)\right|=\frac{2\ln^2(2)}{2\ln(2)-1}\left|f''(1)\right|$$

and

$$\left|f''(1)\right| \geq \frac{2\ln(2)-1}{2\ln^2(2)}.$$

Now, we will prove that inequality (2.1) is sharp. Let

$$f(z) = \int_0^z \frac{t}{\ln(1+t)}dt.$$

Then

$$f'(z) = \frac{z}{\ln(1+z)},$$

$$f''(z) = \frac{\ln(1+z) - \frac{z}{1+z}}{\ln^2(1+z)},$$

and

$$f''(1) = \frac{2\ln 2 - 1}{2\ln^2(2)}.$$

As another example of applying this theorem, we can give the function $f(z) = z + az^2$, $a \in \mathbb{C}$. That is to say, $f'(z) = 1 + 2az$ and for sufficiently small $|a|$, the function $f'(z)$ is subordinate to $G(z)$. Hence, all assumptions of Theorem 2.1 are satisfied. Substituting this function into the inequality obtained in Theorem 2.1 yields an explicit bound on $|a|$, confirming the sharpness of the result. □

**Theorem 2.2.** *Assuming the same conditions as in Theorem 2.1, we obtain*

$$\left|f''(1)\right| \geq \left(\frac{2\ln(2)-1}{\ln^2(2)}\right)\frac{1}{1+4|b_2|}. \tag{2.2}$$

*Inequality* (2.2) *is sharp, achieving equality for the function*

$$f(z) = \int_0^z \frac{t}{\ln(1+t)}dt.$$

*Proof.* Let $\varphi(z)$ be as defined in the proof of Theorem 2.1. Thus, from the Schwarz lemma on the boundary,

$$\frac{2}{1+|\varphi'(0)|} \leq \left|\varphi'(1)\right| = \frac{2\ln^2(2)}{2\ln(2)-1}\left|f''(1)\right|.$$

Since

$$\left|\varphi'(0)\right| = 4|b_2|,$$

we take

$$\frac{2}{1+4|b_2|} \leq \frac{2\ln^2(2)}{2\ln(2)-1}\left|f''(1)\right|$$

and

$$\left|f''(1)\right| \geq \left(\frac{2\ln(2)-1}{\ln^2(2)}\right)\frac{1}{1+4|b_2|}.$$

Next, we will demonstrate that inequality (2.2) is sharp. Consider

$$f(z) = \int_0^z \frac{t}{\ln(1+t)}dt.$$

Then, we have

$$f''(1) = \frac{2\ln 2-1}{2\ln^2(2)}.$$

However, we also have

$$1+2b_2z+3b_3z^2+... = \frac{z}{\ln(1+z)} = 1+\frac{1}{2}z-\frac{1}{12}z^2+...$$

and

$$2b_2+3b_3z+... = \frac{1}{2}-\frac{1}{12}z+... .$$

Upon taking the limit as $z$ approaches 0 in the final equation, we find that $b_2 = \frac{1}{4}$. Consequently, this yields

$$\left(\frac{2\ln(2)-1}{\ln^2(2)}\right)\frac{1}{1+4|b_2|} = \left(\frac{2\ln(2)-1}{\ln^2(2)}\right)\frac{1}{2} = \frac{2\ln 2-1}{2\ln^2(2)}.$$

As a different example of applying the theorem, we can write the function $f(z)$. Let $f(z) = \frac{z}{1+\lambda z}$, $|\lambda| < 1$. Then $f'(z) = \frac{1}{(1+\lambda z)^2}$. For sufficiently small values of $|\lambda| < 1$, we obtain $f'(z) \prec \frac{z}{\ln(1+z)}$. Therefore, Theorem 2.2 is satisfied. □

**Theorem 2.3.** *Under the conditions of Theorem 2.1, we obtain*

$$\left|f''(1)\right| \geq \frac{2\ln 2-1}{2\ln^2(2)}\left(1+\frac{6(1-2|b_2|)^2}{3\left(1-16|b_2|^2\right)+2\left|9b_3+4b_2^2\right|}\right). \tag{2.3}$$

*The bound is sharp with the extremal function given by*

$$f(z) = \int_0^z \frac{t^2}{\ln(1+t^2)}dt.$$

*Proof.* Let $\varphi(z)$ denote the same function as in the proof of Theorem 2.1, and let $\vartheta(z) = z$. According to the maximum principle, for every $z \in D$, it follows that $|\varphi(z)| \leq |\vartheta(z)|$. Therefore, $\xi(z) = \frac{\varphi(z)}{\vartheta(z)}$ is an analytic function and $|\xi(z)| < 1$ for

$|z|<1$. By Taylor expansion of the function $\varphi(z)$, we have $\varphi(z)=a_1z+a_2z^2+a_3z^3+\dots$ . Thus, we take

$$\xi(z)=\frac{\varphi(z)}{\vartheta(z)}=\frac{a_1z+a_2z^2+a_3z^3+\dots}{z}=a_1+a_2z+a_3z^2+\dots.,$$

$$|\xi(0)|=|a_1|,$$

and

$$\left|\xi'(0)\right|=|a_2|.$$

Through straightforward computations, we obtain

$$1+2b_2z+3b_3z^2+\dots.=\frac{\varphi(z)}{\ln(1+\varphi(z))}=1+\frac{1}{2}\varphi(z)-\frac{1}{12}\varphi(z)^2+\dots,$$

$$\begin{aligned}2b_2z+3b_3z^2+\dots.&=\frac{1}{2}\left(a_1z+a_2z^2+a_3z^3+\dots\right)-\frac{1}{12}\left(a_1z+a_2z^2+a_3z^3+\dots\right)^2+\dots\\&=\frac{1}{2}z\left(a_1+a_2z+\dots\right)-\frac{1}{12}z^2\left(a_1+a_2z+\dots\right)^2+\dots,\\2b_2+3b_3z+\dots.&=\frac{1}{2}\left(a_1+a_2z+\dots\right)-\frac{1}{12}z\left(a_1+a_2z+\dots\right)^2+\dots,\\b_2&=\frac{1}{4}a_1,\end{aligned}$$

and

$$a_2=\frac{2}{3}\left(9b_3+4b_2^2\right).$$

Thus, based on the expression for $\xi(z)$, we have

$$|\xi(0)|=4|b_2| \tag{2.4}$$

and

$$\left|\xi'(0)\right|=\frac{2}{3}\left|9b_3+4b_2^2\right|.$$

The combined function

$$u(z)=\frac{\xi(z)-\xi(0)}{1-\overline{\xi(0)}\xi(z)}$$

is analytic in the unit disc $D$, $u(0)=0$, $|u(z)|<1$ for $z\in D$ and $|u(1)|=1$ for $1\in\partial D$. By the Schwarz lemma on the boundary, we obtain

$$\frac{2}{1+|u'(0)|}\leq\left|u'(1)\right|=\frac{1-|\xi(0)|^2}{\left|1-\overline{\xi(0)}\xi(1)\right|^2}\left|\xi'(1)\right|\leq\frac{1+|\xi(0)|}{1-|\xi(0)|}\left(\left|\varphi'(1)\right|-\left|\vartheta'(1)\right|\right).$$

Since

$$\left|u'(0)\right|=\frac{|\xi'(0)|}{1-|\xi(0)|^2}=\frac{\frac{2}{3}\left|9b_3+4b_2^2\right|}{1-16|b_2|^2},$$

we take

$$\frac{2}{1+\frac{2\left|9b_3+4b_2^2\right|}{3\left(1-16|b_2|^2\right)}} \leq \frac{1+4|b_2|}{1-4|b_2|}\left(\frac{2\ln^2(2)}{2\ln(2)-1}\left|f''(1)\right|-1\right)$$

and

$$\left|f'(1)\right| \geq \frac{2\ln 2-1}{2\ln^2(2)}\left(1+\frac{6\left(1-2|b_2|\right)^2}{3\left(1-16|b_2|^2\right)+2\left|9b_3+4b_2^2\right|}\right).$$

We will now demonstrate that the inequality (2.3) achieves equality. Consider

$$f'(z)=\frac{z^2}{\ln(1+z^2)}.$$

Then

$$f''(z)=\frac{2\ln 2-1}{\ln^2(2)}.$$

On the other hand, we have

$$1+2b_2z+3b_3z^2+\ldots=\frac{z^2}{\ln(1+z^2)}=1+\frac{1}{2}z^2-\frac{1}{12}z^4+\frac{1}{24}z^6-\ldots..,$$

$$2b_2z+3b_3z^2+\ldots=\frac{1}{2}z^2-\frac{1}{12}z^4+\frac{1}{24}z^6-\ldots.,$$

and

$$2b_2+3b_3z+\ldots=\frac{1}{2}z-\frac{1}{12}z^3+\frac{1}{24}z^5-\ldots\,.$$

Passing to the limit $(z\to 0)$ in the last equality yields $b_2=0$. Similarly, using straightforward calculations, we take $b_3=\frac{1}{6}$. Therefore, we obtain

$$\frac{2\ln 2-1}{2\ln^2(2)}\left(1+\frac{6\left(1-2|b_2|\right)^2}{3\left(1-16|b_2|^2\right)+2\left|9b_3+4b_2^2\right|}\right)=\frac{2\ln 2-1}{\ln^2(2)}.$$

The example we gave in Theorem 2.2 is also valid in Theorem 2.3. That is, it satisfies the conditions of the theorem. □

**Theorem 2.4.** *Let $f\in\mathcal{G}$ and suppose $f(z)-z$ has no critical point in $D$ except $z=0$ and $b_2>0$. Then*

$$\left|9b_3+4b_2^2\right| \leq 12\left|b_2\ln(4b_2)\right|. \tag{2.5}$$

*This result is sharp.*

*Proof.* Given that $b_2>0$ in the expression of the function $f(z)$, and considering inequality (2.4) where $f(z)-z$ has no critical point in $D$ except $z=0$, we denote by $\ln\xi(z)$ the analytic branch of the logarithm, normalized under the condition

$$\ln\xi(0)=\ln(4b_2)<0.$$

The fractional function

$$\psi(z)=\frac{\ln\xi(z)-\ln\xi(0)}{\ln\xi(z)+\ln\xi(0)}$$

is analytic in the unit disc $D$, $|\psi(z)|<1$ for $z\in D$ and $\psi(0)=0$. By the Schwarz lemma, we obtain

$$1\geq\left|\psi'(0)\right|=\frac{|2\ln\xi(0)|}{|\ln\xi(0)+\ln\xi(0)|^2}\left|\frac{\xi'(0)}{\xi(0)}\right|=\frac{-1}{2\ln\xi(0)}\left|\frac{\xi'(0)}{\xi(0)}\right|=-\frac{\frac{2}{3}\left|9b_3+4b_2^2\right|}{8b_2\ln(4b_2)}$$

and

$$\left|9b_3+4b_2^2\right|\leq 12\left|b_2\ln(4b_2)\right|.$$

Now, we shall show that inequality (2.5) is sharp. Let

$$\varphi(z)=ze^{\frac{1+z}{1-z}\ln 4b_2}=zd(z),$$

where $d(z)=e^{\frac{1+z}{1-z}\ln 2b_1}$. Thus, we have

$$\begin{aligned}d(z)=\frac{\varphi(z)}{z}&=\frac{a_1z+a_2z^2+a_3z^3+\dots}{z}\\&=a_1+a_2z+a_3z^2+\dots.\end{aligned}$$

Then,

$$d(0)=a_1=\varphi'(0)=4b_2,\ d'(0)=a_2=\frac{2}{3}\left|9b_3+4b_2^2\right|.$$

Following straightforward computations, we obtain

$$d'(z)=\frac{2}{(1-z)^2}\ln(4b_2)\,e^{\frac{1+z}{1-z}\ln(4b_2)}$$

and

$$d'(0)=8b_2\ln(4b_2).$$

Thus we obtain

$$\frac{2}{3}\left|9b_3+4b_2^2\right|=8\left|b_2\ln(4b_2)\right|$$

and

$$\left|9b_3+4b_2^2\right|=12\left|b_2\ln(4b_2)\right|.$$

□

Based on the findings from [2], this theorem derives the modulus of the function's derivative at the point 1 by considering its Taylor expansions around two points.

**Theorem 2.5.** *Let $f\in\mathcal{G}$ and $f'(\gamma_0)=1$ for $0<|\gamma_0|<1$. Suppose that, for $1\in\partial D$, $f$ has an angular limit $f(1)$ at 1, $f'(1)=\frac{1}{\ln 2}$. Then we have the inequality*

$$
\begin{aligned}
|f''(1)| \geq & \tfrac{2\ln 2-1}{2(\ln 2)^2}\left(1+\tfrac{1-|\gamma_0|^2}{|1-\gamma_0|^2}+\tfrac{|\gamma_0|-2|f''(0)|}{|\gamma_0|+2|f''(0)|}\right. \\
& \left.\times\left[1+\tfrac{|\gamma_0|^2+4|f''(0)||f''(\gamma_0)|\left(1-|\gamma_0|^2\right)-2|f''(\gamma_0)|\left(1-|\gamma_0|^2\right)-2|f''(0)|}{|\gamma_0|^2+4|f''(0)||f''(\gamma_0)|\left(1-|\gamma_0|^2\right)+2|f''(\gamma_0)|\left(1-|\gamma_0|^2\right)+2|f''(0)|}\tfrac{1-|\alpha_0|^2}{|1-\alpha_0|^2}\right]\right).
\end{aligned} \tag{2.6}
$$

*Inequality (2.6) is sharp, with equality for each possible value of* $|f''(0)|$ *and* $|f''(\gamma_0)|$.

*Proof.* According to the Schwarz-Pick Lemma [5], we have

$$\left|\frac{g(z)-g(\gamma_0)}{1-\overline{g(\gamma_0)}g(z)}\right| \leq \left|\frac{z-\gamma_0}{1-\overline{\gamma_0}z}\right| = |\tau(z)|$$

and

$$|g(z)| \leq \frac{|g(\gamma_0)|+|\tau(z)|}{1+|g(\gamma_0)|\,|\tau(z)|}, \tag{2.7}$$

where $g : D \to D$ is an anlytic function and $\gamma_0 \in D$. If $H : D \to D$ is analytic and $0 < |\gamma_0| < 1$, letting $g(z) = \frac{H(z)-H(0)}{z\left(1-\overline{H(0)}H(z)\right)}$ in (2.7) we obtain

$$\left|\frac{H(z)-H(0)}{z\left(1-\overline{H(0)}H(z)\right)}\right| \leq \frac{\left|\frac{H(\gamma_0)-H(0)}{\gamma_0\left(1-\overline{H(0)}H(\gamma_0)\right)}\right|+|\tau(z)|}{1+\left|\frac{H(\gamma_0)-H(0)}{\gamma_0\left(1-\overline{H(0)}H(\gamma_0)\right)}\right||\tau(z)|}$$

and

$$|H(z)| \leq \frac{|H(0)|+|z|\,\frac{|\alpha|+|\tau(z)|}{1+|\alpha||\tau(z)|}}{1+|H(0)|\,|z|\,\frac{|\alpha|+|\tau(z)|}{1+|\alpha||\tau(z)|}}, \tag{2.8}$$

where

$$\alpha = \frac{H(\gamma_0)-H(0)}{\gamma_0\left(1-\overline{H(0)}H(\gamma_0)\right)}.$$

If we take

$$H(z) = \frac{\varphi(z)}{z\frac{z-\gamma_0}{1-\overline{\gamma_0}z}},$$

then we have

$$H(0) = \frac{\varphi'(0)}{-\gamma_0}, H(\gamma_0) = \frac{\varphi'(\gamma_0)\left(1-|\gamma_0|^2\right)}{\gamma_0}$$

and

$$\alpha = \frac{\frac{\varphi'(\gamma_0)\left(1-|\gamma_0|^2\right)}{\gamma_0}+\frac{\varphi'(0)}{\gamma_0}}{\alpha_0\left(1+\frac{\overline{\varphi'(0)}}{\gamma_0}\frac{\varphi'(\gamma_0)\left(1-|\gamma_0|^2\right)}{\gamma_0}\right)},$$

where $|B| \leq 1$. Let $|H(0)| = \beta$ and

$$m = \frac{\left|\frac{\varphi'(\gamma_0)\left(1-|\gamma_0|^2\right)}{\gamma_0}\right| + \left|\frac{\varphi'(0)}{\gamma_0}\right|}{|\gamma_0|\left(1+\left|\frac{\varphi'(0)}{\gamma_0}\right|\left|\frac{\varphi'(\gamma_0)\left(1-|\gamma_0|^2\right)}{\gamma_0}\right|\right)}.$$

From (2.8), we take

$$|\varphi(z)| \leq |z|\,|\tau(z)|\,\frac{\beta+|z|\,\frac{m+|\tau(z)|}{1+m|\tau(z)|}}{1+\beta\,|z|\,\frac{m+|\tau(z)|}{1+m|\tau(z)|}}$$

and

$$\frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1+\beta\,|z|\,\frac{m+|\tau(z)|}{1+m|\tau(z)|} - \beta\,|z|\,|\tau(z)| - |z|^2\,|\tau(z)|\,\frac{m+|\tau(z)|}{1+m|\tau(z)|}}{(1-|z|)\left(1+\beta\,|z|\,\frac{m+|\tau(z)|}{1+m|\tau(z)|}\right)} = J.$$

Let $s(z) = 1+\beta\,|z|\,\frac{m+|\tau(z)|}{1+m|\tau(z)|}$ and $r(z) = 1+m\,|\tau(z)|$. Considering the functions $s(z)$ and $r(z)$ in the earlier inequality, we obtain

$$J = \frac{1}{s(z)r(z)}\left\{\frac{1-|z|^2\,|\tau(z)|^2}{1-|z|} + m\,|\tau(z)|\,\frac{1-|z|^2}{1-|z|} + \beta m\,|z|\,\frac{1-|\tau(z)|^2}{1-|z|}\right\}. \qquad (2.9)$$

Since

$$\lim_{z\to 1} s(z) = \lim_{z\to 1}\left(1+\beta\,|z|\,\frac{m+|\tau(z)|}{1+m\,|\tau(z)|}\right) = 1+\beta,$$

$$\lim_{z\to 1} b(z) = \lim_{z\to 1}\left(1+m\,|\tau(z)|\right) = 1+m,$$

and

$$1-|\tau(z)|^2 = 1-\left|\frac{z-\gamma_0}{1-\overline{\gamma_0}z}\right|^2 = \frac{\left(1-|\gamma_0|^2\right)\left(1-|z|^2\right)}{|1-\overline{\gamma_0}z|^2},$$

passing to the angular limit in (2.9) gives

$$\left|\varphi'(1)\right| \geq 1+\frac{1-|\gamma_0|^2}{|1-\gamma_0|^2} + \frac{1-\beta}{1+\beta}\left[1+\frac{1-m}{1+m}\,\frac{1-|\gamma_0|^2}{|1-\gamma_0|^2}\right].$$

Moreover, since

$$\frac{1-\beta}{1+\beta} = \frac{1-|H(0)|}{1+|H(0)|} = \frac{1-\left|\frac{\varphi'(0)}{\gamma_0}\right|}{1+\left|\frac{\varphi'(0)}{\gamma_0}\right|} = \frac{|\gamma_0|-|\varphi'(0)|}{|\gamma_0|+|\varphi'(0)|} = \frac{|\gamma_0|-4\,|b_2|}{|\gamma_0|+4\,|b_2|},$$

$$\frac{1-m}{1+m}=\frac{1-\frac{\left|\frac{\varphi'(\gamma_0)\left(1-|\gamma_0|^2\right)}{\gamma_0}\right|+\left|\frac{\varphi'(0)}{\gamma_0}\right|}{|\gamma_0|\left(1+\left|\frac{\varphi'(0)}{\gamma_0}\right|\left|\frac{\varphi'(\gamma_0)\left(1-|\gamma_0|^2\right)}{\gamma_0}\right|\right)}}{1+\frac{\left|\frac{\varphi'(\gamma_0)\left(1-|\gamma_0|^2\right)}{\gamma_0}\right|+\left|\frac{\varphi'(0)}{\gamma_0}\right|}{|\gamma_0|\left(1+\left|\frac{\varphi'(0)}{\gamma_0}\right|\left|\frac{\varphi'(\gamma_0)\left(1-|\gamma_0|^2\right)}{\gamma_0}\right|\right)}},$$

$$\frac{1-m}{1+m}=\frac{|\gamma_0|^2+2|f''(0)|2|f''(\gamma_0)|\left(1-|\gamma_0|^2\right)-2|f''(\gamma_0)|\left(1-|\gamma_0|^2\right)-2|f''(0)|}{|\gamma_0|^2+2|f''(0)|2|f''(\gamma_0)|\left(1-|\gamma_0|^2\right)+2|f''(\gamma_0)|\left(1-|\gamma_0|^2\right)+2|f''(0)|},$$

and $\frac{1-m}{1+m}=\frac{|\gamma_0|^2+4|f''(0)||f''(\gamma_0)|\left(1-|\gamma_0|^2\right)-2|f''(\gamma_0)|\left(1-|\gamma_0|^2\right)-2|f''(0)|}{|\gamma_0|^2+4|f''(0)||f''(\gamma_0)|\left(1-|\gamma_0|^2\right)+2|f''(\gamma_0)|\left(1-|\gamma_0|^2\right)+2|f''(0)|}$,
we obtain

$$|\varphi'(1)|\geq 1+\frac{1-|\gamma_0|^2}{|1-\gamma_0|^2}+\frac{|\gamma_0|-4|b_2|}{|\gamma_0|+4|b_2|}$$
$$\times\left[1+\frac{|\gamma_0|^2+4|f''(0)||f''(\gamma_0)|\left(1-|\gamma_0|^2\right)-2|f''(\gamma_0)|\left(1-|\gamma_0|^2\right)-2|f''(0)|}{|\gamma_0|^2+4|f''(0)||f''(\gamma_0)|\left(1-|\gamma_0|^2\right)+2|f''(\gamma_0)|\left(1-|\gamma_0|^2\right)+2|f''(0)|}\frac{1-|\gamma_0|^2}{|1-\gamma_0|^2}\right].$$

From the definition of $\varphi(z)$, we have

$$\varphi'(1)=\frac{\ln^2(2)}{\ln(2)-\frac{1}{2}}f''(1).$$

We thus take inequality (2.6).

Let $\gamma_0$ be any real number in the interval $(-1,0)$, and let $\varkappa$ and $\rho$ be arbitrary real numbers such that $0<\varkappa=|\varphi'(0)|<|\gamma_0|$, $0<\rho=|\varphi'(\gamma_0)|<\frac{|\gamma_0|}{\left(1-|\gamma_0|^2\right)}$ to show that inequality (2.6) is sharp. Let

$$\mathbb{K}=\frac{\frac{\varkappa}{\gamma_0}+\frac{\rho\left(1-|\gamma_0|^2\right)}{\gamma_0}}{\gamma_0\left(1+\varkappa\rho\frac{1-|\gamma_0|^2}{\gamma_0^2}\right)}=\frac{1}{\gamma_0^2}\frac{\rho\left(1-|\gamma_0|^2\right)+\varkappa}{1+\varkappa\rho\frac{1-|\gamma_0|^2}{\gamma_0^2}}.$$

Consider the function

$$\varphi(z)=z\frac{z-\gamma_0}{1-\overline{\gamma_0}z}\frac{\frac{-\varkappa}{\gamma_0}+z\frac{\mathbb{K}+\tau(z)}{1+\tau(z)\mathbb{D}}}{1-\frac{\varkappa}{\gamma_0}z\frac{\mathbb{K}+\tau(z)}{1+\tau(z)\mathbb{K}}}. \tag{2.10}$$

From equation (2.10), after performing straightforward calculations, we derive $\varphi'(0)=\varkappa$ and $\varphi'(\gamma_0)=\rho$. Therefore, we obtain

$$\varphi'(1)=1+\frac{1-\gamma_0^2}{(1-\gamma_0)^2}+\frac{\gamma_0+\varkappa}{\gamma_0-\varkappa}\left(1+\frac{\gamma_0^2+\varkappa\rho\left(1-|\gamma_0|^2\right)-\rho\left(1-|\gamma_0|^2\right)-\varkappa}{\gamma_0^2+\varkappa\rho\left(1-|\gamma_0|^2\right)+\rho\left(1-|\gamma_0|^2\right)+\varkappa}\frac{1-\gamma_0^2}{(1-\gamma_0)^2}\right).$$

By selecting appropriate signs for the numbers $\gamma_0$, $\varkappa$ and $\rho$, we can infer from the final equation that inequality (2.6) is sharp. $\square$

## 3. CONCLUSION

These results enhance our understanding of the geometric properties of the subclass $\mathcal{G}$ of analytic functions. The inequalities derived provide tools for analyzing the distortion and behavior of these functions, particularly in relation to the unit disc and its boundary.

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Bülent Nafi Örnek
Amasya University
Department of Computer Engineering
Merkez-Amasya 05100, Türkiye
e-mail: *nafiornek@gmail.com*,
*nafi.ornek@amasya.edu.tr*