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SHIFT DIFFERENTIAL-DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS AND THEIR UNIQUENESS CONCERNING WEIGHTED SHARING

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ABSTRACT. The purpose of the paper is to study the uniqueness problems of certain types of shift differential-difference polynomials of meromorphic functions sharing a small function under the notion of relaxed weighted sharing.

1. INTRODUCTION AND MAIN RESULT

Let $\mathbb{C}$ denote the complex plane and let f be a non-constant meromorphic function defined on $\mathbb{C}$. We assume that the reader is familiar with the standard definitions and notations used in the Nevanlinna value distribution theory, such as $T(r,f), m(r,f), N(r,f)$ (see [6,14,15]). By $S(r,f)$, we denote any quantity satisfying the condition $S(r,f) = \circ(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set of finite linear measure. A meromorphic function a is called a small function with respect to f if $a \equiv \infty$ or $T(r,a) = S(r,f)$. We denote by $S(f)$ the collection of all small functions with respect to f. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers.

We write $E(a,f) = \{z \in \mathbb{C} : f(z) - a = 0\}$, where each zero with multiplicity m is counted m times. If we ignore the multiplicity, the set is denoted by $\overline{E}(a,f)$. For any two non-constant meromorphic functions f and g, and $a \in S(f) \cap S(g)$ we say that f and g share a IM (CM) provided that $\overline{E}(a,f) = \overline{E}(a,g)$ $(E(a,f) = E(a,g))$.

Recently, interest has been increasing in studying difference equations in the complex plane. With this development, many authors have investigated the value distribution of certain types of difference differential expression, and a handful of elegant results were published in [4,9–11,13,16].

Focusing on the k-th derivative of a more extended differential-difference expression Yi [16] proved the following results.

Theorem 1.1. *[16] Let f be a transcendental entire function of finite order, $a(z) \not\equiv 0$ be a small function with respect to $f(z)$, c_j $(j = 1,2,\ldots,s)$ be distinct*

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finite complex numbers and n, m, s and μ_j $(j = 1, 2, ..., s)$ be non-negative integers. Suppose one of the following conditions holds:
(i) $n \geq k+2$, when $m \leq k+1$,
(ii) $n \geq 2k-m+3$, when $m > k+1$.
Then, the differential-difference polynomial

$$[f^n(z)(f(z)-1)^m \prod_{j=1}^{s}(f(z+c_j))^{\mu_j}]^{(k)} - a(z)$$

has infinitely many zeros.

Theorem 1.2. *[16] Let f be a transcendental entire function of finite order, $a(z) \not\equiv 0$ be a small function with respect to $f(z)$ and c_j $(j = 1, 2, ..., s)$ be distinct finite complex numbers. Let n, m, s and μ_j $(j = 1, 2, ..., s)$ be non-negative integers and $\sigma = \sum_{j=1}^{s} \mu_j$. If $n \geq 2k+m+\sigma+5$ and the differential-difference polynomials*

$$[f^n(z)(f^m(z)-1) \prod_{j=1}^{s}(f(z+c_j))^{\mu_j}]^{(k)}$$

and

$$[g^n(z)(g^m(z)-1) \prod_{j=1}^{s}(g(z+c_j))^{\mu_j}]^{(k)}$$

share $a(z)$ CM, then $f(z) = tg(z)$, where $t^m = t^{n+\sigma} = 1$.

Relaxing the nature of sharing Lahiri [7, 8] introduced a new notion of sharing in the following definition.

Definition 1.1. *[7, 8] Let $l \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ and $a \in S(f)$. We denote by $E_l(a, f)$ the set of all zeros of $f - a$, where a zero of multiplicity m is counted m times if $m \leq l$ and $l+1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say that f, g share the function a with weight l. Moreover, we note that f and g share the function a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively.*

In 2018, Waghamore [13] studied the uniqueness problems of certain types of differential polynomials sharing a small function with finite weight as follows.

Theorem 1.3. *[13] Let f and g be two transcendental entire functions of finite order and $a(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that c_j $(j = 1, 2, ..., s)$ are non zero complex constants, μ_j $(j = 1, 2, ..., s)$ are non-negative integers, n, $m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+\sigma+5$ when $m \leq k+1$ and $n \geq 4k-m+\sigma+9$ when $m > k+1$. If $[f^n(z)(f(z)-1)^m \prod_{j=1}^{s} f(z+c_j)^{\mu_j}]^{(k)}$ and $[g^n(z)(g(z)-1)^m \prod_{j=1}^{s} g(z+c_j)^{\mu_j}]^{(k)}$ share $(a, 2)$, then either $f = g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(f, g)$ is given by*

$$R(w_1, w_2) = w_1^n(w_1-1)^m \prod_{j=1}^{s}(w_1(z+c_j))^{\mu_j} - w_2^n(w_2-1)^m \prod_{j=1}^{s}(w_2(z+c_j))^{\mu_j}.$$

Theorem 1.4. *[13] Let f and g be two transcendental entire functions of finite order and $a(z)(\not\equiv 0)$ be a small function to both f and g. Suppose that c_j $(j = 1,2,...,s)$ are non zero complex constants, μ_j $(j = 1,2,...,s)$ are non-negative integers, $n,m \geq 1$ and k (≥ 0) are integers satisfying $n \geq 5k+4m+4\sigma+8$ when $m \leq k+1$ and $n \geq 10k-m+4\sigma+15$ when $m > k+1$. If $[f^n(z)(f(z)-1)^m \prod_{j=1}^{s}(f(z+c_j))^{\mu_j}]^{(k)}$ and $[g^n(z)(g(z)-1)^m \prod_{j=1}^{s}(g(z+c_j))^{\mu_j}]^{(k)}$ share $a(z)$ IM, then the conclusion of Theorem 1.3 holds.*

Motivated by the above results, in this paper, we prove a result for transcendental meromorphic functions of finite order which improved and extended all the Theorems (1.1–1.4) to a more general case.

Theorem 1.5. *Let f and g be two transcendental meromorphic functions of finite order, $a = a(z)$ $(\not\equiv 0,\infty) \in S(f)\cap S(g)$ with finitely many zeros and poles, and c_j, $j = 1,2,...,s$ $(s \in \mathbb{N})$ be non-zero complex constants such that f and g are not periodic function of period c_j. Suppose that the poles of $f(z)$ are not zeros of $f(z+c_j)$ and the poles of $g(z)$ are not zeros of $g(z+c_j)$. Let $P(w) = a_m w^m + a_{m-1}a^{m-1} + ... + a_0$, where a_m, a_{m-1}, ..., a_0 $(\neq 0) \in \mathbb{C}$ be a polynomial and $m^*(\geq 0)$, $n(\geq 1)$, $\mu_j(\geq 0)$ $(j = 1,2,...,s)$, $k(\geq 1)$ be integers, $\sigma = \sum_{j=1}^{s}\mu_j$. If $\left[f^n(z)P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\right]^{(k)}$ and $\left[g^n(z)P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}\right]^{(k)}$ share (a,l) with one of the following conditions:*

(i) $l \geq 2$ and

$$n \geq 2k+m^*+\sigma+7 \text{ when } m \leq k+1, n \geq 4k-m^*+\sigma+11 \text{ when } m > k+1 \quad (1.1)$$

(ii) $l = 1$ and

$$n \geq 2k+\frac{3m^*}{2}+\frac{3\sigma}{2}+9 \text{ when } m \leq k+1, n \geq 4k+\frac{m^*}{2}+\frac{3\sigma}{2}+13 \text{ when } m > k+1 \quad (1.2)$$

(iii) $l = 0$ and

$$n \geq 2k+4m^*+4\sigma+14 \text{ when } m \leq k+1, n \geq 4k+2m^*+4\sigma+18 \text{ when } m > k+1, \quad (1.3)$$

then one of the following cases holds:
(A) when $P(w)$ is not a monomial, then $f = tg$ for a constant t such that $t^d = 1$ where d is the GCD of the elements of J, $J = \{k \in I : a_k \neq 0\}$ and $I = \{n+\sigma, n+\sigma+1, ..., n+\sigma+m\}$, or f and g satisfy the algebraic difference equation $R(f,g) = 0$, where $R(f,g)$ is given by

$$R(w_1,w_2) = w_1^n P(w_1)\prod_{j=1}^{s}(w_1(z+c_j))^{\mu_j} - w_2^n P(w_2)\prod_{j=1}^{s}(w_2(z+c_j))^{\mu_j}.$$

(B) When $P(w) = a_i w^i$ is a non-zero monomial, then
(BI)

$$a_i^2 f^{n+i}(z) \prod_{j=1}^{s} (f(z+c_j))^{\mu_j} . g^{n+i}(z) \prod_{j=1}^{s} (g(z+c_j))^{\mu_j} = a^2.$$

(BII) $f = tg$, where t satisfies $t^{n+\sigma+i} = 1$, or else f and g satisfy the algebraic equation $R(w_1, w_2) = 0$, where

$$R(w_1, w_2) = w_1^{n+i} \prod_{j=1}^{s} (w_1(z+c_j))^{\mu_j} - w_2^{n+i} \prod_{j=1}^{s} (w_2(z+c_j))^{\mu_j}.$$

(C) When $P(w) = a_0 (\neq 0)$, then
(CI)

$$a_0^2 f^n(z) \prod_{j=1}^{s} (f(z+c_j))^{\mu_j} . g^n(z) \prod_{j=1}^{s} (g(z+c_j))^{\mu_j} = a^2.$$

(CII) $f = tg$, where t satisfies $t^{n+\sigma} = 1$, where and in the sequel

$$m^* = \begin{cases} m & \text{if } P(w) \neq a_0; \\ 0 & \text{if } P(w) = a_0. \end{cases}$$

We now recall some well known definitions and notation from value distribution theory.

Definition 1.2. *[1] Let p be a positive integer. Let $f(z)$ be a meromorphic function and $a(z) \in S(f)$.*

(i) $\overline{N}_{p)}(r, \frac{1}{f-a})$ denotes the counting function of those a-points of f whose multiplicities are not greater than p, where each a-point is counted only once.

(ii) $\overline{N}_{(p}(r, \frac{1}{f-a})$ denotes the counting function of those a-points of f whose multiplicities are not less than p, where each a-point is counted only once.

(iii) $N_p(r, \frac{1}{f-a})$ denotes the counting function of those a-points of f, where an a-point of f with multiplicity m counted m times if $m \leq p$ and p times if $m > p$.

Definition 1.3. *[1] Let f and g be two non-constant meromorphic functions, which share 1 IM. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r, \frac{1}{f-1})$ the reduced counting function of those 1-point of f and g in $\{z : |z| < r\}$ where $p > q$, denote by $N_E^{1)}(r, \frac{1}{f-1})$ the reduced counting function of those 1-point of f and g in $\{z : |z| < r\}$ where $p = q = 1$ in $\{z : |z| < 1\}$, and denote by $\overline{N}_E^{(2}(r, \frac{1}{f-1})$ the counting function counting function of those 1-point of f and g in $\{z : |z| < r\}$ where $p = q \geq 2$ in $\{z : |z| < r\}$. Similarly we define $\overline{N}_L(r, \frac{1}{g-1})$, $N_E^{1)}(r, \frac{1}{g-1})$ and $\overline{N}_E^{(2}(r, \frac{1}{g-1})$.*

Definition 1.4. *[1] Let k be a positive integer. Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p, and a 1-point of g with multiplicity q. Denote by*

$\overline{N}_{f>k}(r,\frac{1}{f-1})$ *the reduced counting function of those* 1*-points of* f *and* g *such that* $p>q=k$. $\overline{N}_{g>k}(r,\frac{1}{g-1})$ *is defined analogously.*

2. Lemmas

This section presents some lemmas needed to prove the main theorem.
Let F and G be non-constant meromorphic functions defined in the complex plane $\mathbb{C}$. We denote by H the following function:

$$H=\left(\frac{F^{(2)}}{F^{(1)}}-2\frac{F^{(1)}}{F-1}\right)-\left(\frac{G^{(2)}}{G^{(1)}}-2\frac{G^{(1)}}{G-1}\right). \tag{2.1}$$

Lemma 2.1. *[12] Let* $f(z)$ *be a meromorphic function of finite order* ρ *and let* c $(\neq 0)$ *be a fixed complex constant. Then*

$$N\left(r,\frac{1}{f(z+c)-1}\right)\leq N\left(r,\frac{1}{f-1}\right)+S(r,f),$$

outside a possible exceptional set of finite logarithmic measures.

Lemma 2.2. *[5] Let* $f(z)$ *be a meromorphic function of finite order, and let* c *be a fixed non-zero complex constant. Then*

$$T(r,f(z+c))=T(r,f)+S(r,f).$$

Lemma 2.3. *[18] Let* $f(z)$ *be a meromorphic function of finite order, and let* c *be a fixed non-zero complex constant. Then*

$$m\left(r,\frac{f(z+c)}{f(z)}\right)+m\left(r,\frac{f(z)}{f(z+c)}\right)=S(r,f).$$

Lemma 2.4. *[15] Let* f *and* g *be two non-constant meromorphic functions. If* $Af+Bg=C$*, where* A*,* B*,* C *are non-zero constants, then*

$$T(r,f)\leq\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{g})+S(r,f).$$

Lemma 2.5. *[17] Let* $f(z)$ *be a non-constant meromorphic function and* s*,* k *be two positive integers. Then*

$$N_s\left(r,\frac{1}{f^{(k)}}\right)\leq T(r,f^{(k)})-T(r,f)+N_{s+k}(r,\frac{1}{f})+S(r,f),$$

and

$$N_s\left(r,\frac{1}{f^{(k)}}\right)\leq k\overline{N}(r,f)+N_{s+k}(r,\frac{1}{f})+S(r,f).$$

Lemma 2.6. *[2] If f and g share $(1,l)$ and $H \not\equiv 0$, then*

$$N(r,H) \leq \overline{N}(r,f) + \overline{N}_{(2}(r,\frac{1}{f}) + \overline{N}_{(2}(r,\frac{1}{g}) + \overline{N}_L(r,\frac{1}{f-1})$$
$$+ \overline{N}_L(r,\frac{1}{g-1}) + N_0(r,\frac{1}{f'}) + N_0(r,\frac{1}{g'}) + S(r,f).$$

Lemma 2.7. *[1] If f and g are non-constant meromorphic functions sharing $(1,1)$, then*

$$2\overline{N}_L(r,\frac{1}{f-1}) + 2\overline{N}_L(r,\frac{1}{g-1}) + \overline{N}_E^{(2}\left(r,\frac{1}{f-1}\right) - \overline{N}_{f>2}(r,\frac{1}{g-1})$$
$$\leq N(r,\frac{1}{g-1}) - \overline{N}(r,\frac{1}{g-1}) + S(r,f) + S(r,g).$$

Lemma 2.8. *[1] If f and g are non-constant meromorphic functions sharing $(1,1)$, then*
$\overline{N}_{f>2}(r,\frac{1}{g-1}) \leq \frac{1}{2}\overline{N}(r,f) + \frac{1}{2}\overline{N}(r,\frac{1}{f}) - \frac{1}{2}N_0(r,\frac{1}{f^{(1)}}) + S(r,f).$

Lemma 2.9. *[3] If f and g are non-constant meromorphic functions and f, g share $(1,0)$, then $\overline{N}_L(r,\frac{1}{f-1}) \leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{f}) + S(r,f)$.*

Lemma 2.10. *Let f and g be two transcendental meromorphic functions of finite order, $a = a(z)$ $(\not\equiv 0,\infty) \in S(f) \cap S(g)$ with finitely many zeros and poles, and c_j $(j = 1,2,....,s)$ be finite complex numbers. Let $m^*(\geq 0)$, n (≥ 1), k (>0), $\mu_j(\geq 0)$ $(j = 1,2,....,s)$ be integers such that $n > k$ and $P(w)$ be same as in Theorem 1.5. If*

$$\left[f^n(z)P(f)(z)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right]^{(k)} . \left[g^n(z)P(g)(z)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right]^{(k)} = a^2(z),$$

then $P(w)$ reduces to a non-zero monomial, namely $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0,1,..,m\}$.

Proof. Suppose $P(f)$ is not a non-zero monomial. Without loss of generality, we assume that $P(f) = a_m f^m + a_{m-1} f^{m-1} + \cdots + a_1 f + a_0$, where $a_m,\ a_{m-1},\ ....a_1$ and a_0 $(\neq 0) \in \mathbb{C}$.
Since the number of poles and zeros of $a(z)$ is finite, it follows that f as well as g has finitely many zeros and poles. Therefore $f(z) = h(z)e^{\alpha(z)}$, where α is a non-zero polynomial function and $h(z)$ is a non-zero rational function. Let

$$h_i(z) = h^{n+i}(z)\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}$$

and

$$\alpha_i(z) = (n+i)\alpha(z) + \Sigma_j^s(\alpha(z+c_j))^{\mu_j},$$

where $i = 0,\ 1,\ .....,m$. Then,

$$f^{n+i}(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j} = h_i(z)e^{\alpha_i(z)},$$

where $i = 0,\ 1,\ .....,m$. An elementary calculation shows that

$$\left[f^{n+i}(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}\right]^{(k)} = t_i\left(\alpha_i',\ \alpha_i'',....,\ \alpha_i^{(k)},\ h_i,\ h_i',....,\ h_i^{(k)}\right)e^{\alpha_i},$$

where $t_i\left(\alpha_i',\ \alpha_i'',....,\ \alpha_i^{(k)},\ h_i,\ h_i',....,\ h_i^{(k)}\right)$ for $(i = 0,\ 1,\ .....,m)$ are differential polynomials in $\alpha_i',\ \alpha_i'',....,\ \alpha_i^{(k)},\ h_i,\ h_i',....,\ h_i^{(k)}$. Therefore we have

$$t_i\left(\alpha_i',\ \alpha_i'',....,\ \alpha_i^{(k)},\ h_i,\ h_i',....,\ h_i^{(k)}\right) \not\equiv 0, \text{for } i = 0,\ 1,\ .....,m$$

and $\left[f^n(z)P(f)(z)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right]^{(k)} \not\equiv 0$. Thus we have $T(r,t_i) = S(r,f)$ for $i = 0,\ 1,\ .....m$ and

$$\overline{N}\left(r, \frac{1}{t_m e^{m\alpha(z)} + ... + t_0}\right) = S(r,f).$$

Therefore, using the second fundamental theorem for small functions, we get

$$\begin{aligned}
&mT(r,f) = T(r,t_m e^{m\alpha(z)} + ... + t_1 e^{\alpha}) + S(r,f)\\
&\le \overline{N}\left(r, \frac{1}{t_m e^{m\alpha(z)} + ... + t_1 e^{\alpha}}\right) + \overline{N}\left(r, \frac{1}{t_m e^{m\alpha(z)} + ... + t_1 e^{\alpha} + t_0}\right) + S(r,f)\\
&\le \overline{N}\left(r, \frac{1}{t_m e^{(m-1)\alpha(z)} + ... + t_1}\right) + S(r,f) \le (m-1)T(r,f) + S(r,f),
\end{aligned}$$

which is a contradiction. Hence $P(w)$ reduces to a non-zero monomial, namely $P(w) = a_i w^i$ for some $(i = 0,\ 1,\ .....,m)$. This completes the proof. $\square$

Lemma 2.11. *Let f and g be two transcendental meromorphic functions of finite order, $a = a(z)\ (\not\equiv 0,\infty) \in S(f)\cap S(g)$ and $c_j\ (j = 1,2,....,s)$ be finite complex numbers. Let $m^*(\ge 0)$, $n\ (\ge 1)$, $\mu_j(\ge 0)\ (j = 1,2,....,s)$ be integers such that $n > 2k+m^*+\sigma+1$ and $P(w)$ be same as in Theorem 1.5. If*

$$\left[f^n(z)P(f)(z)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}\right]^{(k)} \equiv \left[g^n(z)P(g)(z)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}\right]^{(k)}, \tag{2.2}$$

then

$$f^n(z)P(f)(z)\prod_{j=1}^{s} f(z+c_j)^{\mu_j} \equiv g^n(z)P(g)(z)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}.$$

Proof. Let

$$F_1 = f^n(z)P(f)(z)\prod_{j=1}^{s} f(z+c_j)^{\mu_j}$$

and

$$G_1 = g^n(z)P(g)(z)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}.$$

Integrating (2.2) k times we get $F_1 = G_1 + R(z)$, where $R(z)$ is a polynomial of degree at most $k-1$.
If $R(z) \not\equiv 0$ then by Lemma 2.4, we have

$$T\left(r, \frac{F_1}{R(z)}\right) \le \overline{N}\left(r, \frac{F_1}{R(z)}\right) + \overline{N}\left(r, \frac{R(z)}{F_1}\right) + \overline{N}\left(r, \frac{R(z)}{G_1}\right) + S(r,f) + S(r,g).$$

Therefore using Lemma 2.5 we get

$$\begin{aligned}
&(n+m^*+\sigma)T(r,f) \le T(r,F_1) + S(r,f) \le T(r,\frac{F_1}{R}) + (k-1)\log r + O(1)\\
&\le \overline{N}(r,\frac{F_1}{R(z)}) + \overline{N}(r,\frac{R(z)}{F_1}) + \overline{N}(r,\frac{R(z)}{G_1}) + (k-1)\log r + S(r,f) + S(r,g)\\
&\le (2+m^*+\sigma)T(r,f) + (1+m^*+\sigma)T(r,g) + 2(k-1)\log r + S(r,f) + S(r,g).
\end{aligned} \tag{2.3}$$

Similarly,

$$\begin{aligned}
(n+m^*+\sigma)T(r,g) \le{}& (2+m^*+\sigma)T(r,g) + (1+m^*+\sigma)T(r,f)\\
&+ 2(k-1)\log r + S(r,f) + S(r,g).
\end{aligned} \tag{2.4}$$

Since $T(r,f) \ge \log r + O(1)$ and $T(r,g) \ge \log r + O(1)$, combining (2.3) and (2.4), we get

$$\{(n-(2k+m^*+\sigma+1)\}\,(T(r,f)+T(r,g)) \le S(r,f)+S(r,g),$$

which is a contradiction. Hence $R(z) \equiv 0$ and so, $F_1 = G_1$. This completes the proof. □

3. Proof of the Main Theorem

Proof. Let $F_1 = f^n(z)P(f)(z)\prod_{j=1}^{s}(f(z+c_j))^{\mu_j}$ and $G_1 = g^n(z)P(g)(z)\prod_{j=1}^{s}(g(z+c_j))^{\mu_j}$. Then $F = \frac{F_1^{(k)}}{a(z)}$ and $G = \frac{G_1^{(k)}}{a(z)}$. Then, from the given condition, we have F and G share $(1,l)$ except at zeros and poles of a.

Case 1: $H \not\equiv 0$. Then from (2.1) and Lemma 2.6 we have $m(r,H) = S(r,f)$ and

$$\begin{aligned}
N_E^{1)}(r,\frac{1}{F-1}) &\leq N(r,\frac{1}{H}) + S(r,f)\\
&\leq T(r,H) + S(r,f) = N(r,H) + S(r,f)\\
&\leq \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_L(r,\frac{1}{F-1})\\
&+ \overline{N}_L(r,\frac{1}{G-1}) + N_0(r,\frac{1}{F'}) + N_0(r,\frac{1}{G'}) + S(r,f). \qquad (3.1)
\end{aligned}$$

By the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned}
T(r,F) + T(r,G) \leq \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G})\\
+ \overline{N}(r,\frac{1}{G-1}) - N_0(r,\frac{1}{F'}) - N_0(r,\frac{1}{G'}) + S(r,f), \qquad (3.2)
\end{aligned}$$

where $N_0(r,\frac{1}{F'})$ denotes the counting function of zeros of F', which are not the zeros of $F(F-1)$ and $N_0(r,\frac{1}{G'})$ is defined similarly.

Subcase 1.1: $l \geq 1$. Then by (3.1), we get

$$\begin{aligned}
&\overline{N}\big(r,\frac{1}{F-1}\big) + \overline{N}\big(r,\frac{1}{G-1}\big) = N_E^{1)}\left(r,\frac{1}{F-1}\right) + \overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)\\
&+ \overline{N}_L\big(r,\frac{1}{F-1}\big) + \overline{N}_L\big(r,\frac{1}{G-1}\big) + \overline{N}\big(r,\frac{1}{G-1}\big) + S(r,f)\\
&\leq \overline{N}(r,F) + \overline{N}(r,G) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + 2\overline{N}_L\big(r,\frac{1}{F-1}\big) + 2\overline{N}_L\big(r,\frac{1}{G-1}\big)\\
&+ N_0(r,\frac{1}{F'}) + N_0(r,\frac{1}{G'}) + \overline{N}_E^{(2}\big(r,\frac{1}{F-1}\big) + \overline{N}\big(r,\frac{1}{G-1}\big) + S(r,f). \qquad (3.3)
\end{aligned}$$

Subcase 1.1.1: $l = 1$. Then by Lemma 2.8 we have

$$\begin{aligned}
\overline{N}_L\big(r,\frac{1}{F-1}\big) &\leq \frac{1}{2}N(r,\frac{1}{F'}|F \neq 0)\\
&\leq \frac{1}{2}\overline{N}(r,F) + \frac{1}{2}\overline{N}(r,\frac{1}{F}) + S(r,f), \qquad (3.4)
\end{aligned}$$

where $N(r,\frac{1}{F'}|F \neq 0)$ denotes the zeros of F', which are not zeros of F.

From (3.4) and Lemma 2.7 we have

$$\begin{aligned}&2\overline{N}_L(r,\frac{1}{F-1})+2\overline{N}_L(r,\frac{1}{G-1})+\overline{N}_E{}^{(2}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1})\\&\le N(r,\frac{1}{G-1})+\overline{N}_L(r,\frac{1}{F-1})+S(r,f)\\&\le N(r,\frac{1}{G-1})+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}(r,\frac{1}{F})+S(r,f).\end{aligned}\tag{3.5}$$

Using (3.5) we get from (3.3)

$$\begin{aligned}\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1})&\le\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})\\&+N(r,\frac{1}{G-1})+\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}(r,\frac{1}{F})+S(r,f)+S(r,g).\end{aligned}\tag{3.6}$$

By (3.6), we get from (3.2)

$$\begin{aligned}T(r,F)\le\frac{5}{2}\overline{N}(r,F)+2\overline{N}(r,G)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})\\+\frac{1}{2}\overline{N}(r,\frac{1}{F})+S(r,f)+S(r,g).\end{aligned}\tag{3.7}$$

Lemma 2.5 gives

$$\begin{aligned}N_2(r,\frac{1}{F})&\le N_2(r,\frac{1}{F_1^{(k)}})+S(r,f)\\&\le T(r,F_1^{(k)})-(n+m^*+\sigma)T(r,f)+N_{k+2}(r,\frac{1}{F_1})+S(r,f)\\&\le T(r,F)-(n+m^*+\sigma)T(r,f)+N_{k+2}(r,\frac{1}{F_1})+S(r,f).\end{aligned}\tag{3.8}$$

and

$$N_2(r,\frac{1}{G})\le N_2(r,\frac{1}{G_1^{(k)}})+S(r,g)\le N_{k+2}(r,\frac{1}{G_1})+S(r,g).\tag{3.9}$$

From (3.7), (3.8) and (3.9) we get

$$\begin{aligned}(n+m^*+\sigma)T(r,f)\le\frac{5}{2}\overline{N}(r,f)+2\overline{N}(r,g)+N_{k+2}(r,\frac{1}{F_1})+N_{k+2}(r,\frac{1}{G_1})\\+\frac{1}{2}\overline{N}(r,\frac{1}{F_1})+S(r,f)+S(r,g).\end{aligned}\tag{3.10}$$

If $m\le k+1$, then from (3.10) we have

$$\begin{aligned}(n+m^*+\sigma)T(r,f)\le(k+m^*+\sigma+2)\{T(r,f)+T(r,g)\}+\frac{5}{2}\overline{N}(r,f)+2\overline{N}(r,g)\\+\frac{1}{2}(m^*+\sigma+1)T(r,f)+S(r,f)+S(r,g).\end{aligned}\tag{3.11}$$

Similarly,

$$(n+m^*+\sigma)T(r,g) \le (k+m^*+\sigma+2)\{T(r,f)+T(r,g)\}+\frac{5}{2}\overline{N}(r,g)+2\overline{N}(r,f)$$
$$+\frac{1}{2}(m^*+\sigma+1)T(r,g)+S(r,f)+S(r,g). \tag{3.12}$$

Combining (3.11) and (3.12) we get

$$(n-2k-\frac{3m^*}{2}-\frac{3\sigma}{2}-9)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$$

which contradicts (1.2).

If $m > k+1$, then from (3.10) we have

$$(n+m^*+\sigma)T(r,f) \le (2k+\sigma+4)\{T(r,f)+T(r,g)\}+\frac{9}{2}T(r,f)$$
$$+\frac{1}{2}(m^*+\sigma+1)T(r,f)+S(r,f)+S(r,g). \tag{3.13}$$

Similarly,

$$(n+m^*+\sigma)T(r,g) \le (2k+\sigma+4)\{T(r,f)+T(r,g)\}+\frac{9}{2}T(r,g)$$
$$+\frac{1}{2}(m^*+\sigma+1)T(r,g)+S(r,f)+S(r,g). \tag{3.14}$$

Combining (3.13) and (3.14) we get

$$(n-4k+\frac{m^*}{2}-\frac{3\sigma}{2}-13)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

which contradicts (1.2).

Subcase 1.1.2: $l \ge 2$. In this case, we have

$$2\overline{N}_L(r,\frac{1}{F-1})+2\overline{N}_L(r,\frac{1}{G-1})+\overline{N_E}^{(2}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1})$$
$$\le N(r,\frac{1}{G-1})+S(r,f)+S(r,g). \tag{3.15}$$

From (3.3) and using (3.15), we obtain

$$\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1}) \le \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+N(r,\frac{1}{G-1})$$
$$+N_0(r,\frac{1}{F'})+N_0(r,\frac{1}{G'})+S(r,f) \le \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})$$
$$+T(r,G)+N_0(r,\frac{1}{F'})+N_0(r,\frac{1}{G'})+S(r,f). \tag{3.16}$$

Now, using (3.16) in (3.2), we get

$$\begin{aligned} T(r,F) &\le 2\overline{N}(r,F)+\overline{N}(r,G)+\overline{N}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+\overline{N}_{(2}(r,\frac{1}{G})+S(r,f) \\ &\le 2\overline{N}(r,F)+\overline{N}(r,G)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+S(r,f). \end{aligned} \tag{3.17}$$

If $m \le k+1$, then from (3.17) we have

$$\begin{aligned} (n+m^*+\sigma)T(r,f) \le (k+m^*+\sigma+2)\{T(r,f)+T(r,g)\}+2\overline{N}(r,f) \\ +\overline{N}(r,g)+S(r,f)+S(r,g). \end{aligned} \tag{3.18}$$

Similarly,

$$\begin{aligned} (n+m^*+\sigma)T(r,g) \le (k+m^*+\sigma+2)\{T(r,f)+T(r,g)\}+2\overline{N}(r,g) \\ +\overline{N}(r,f)+S(r,f)+S(r,g). \end{aligned} \tag{3.19}$$

Therefore combining (3.18) and (3.19) we get

$$(n-2k-m^*-\sigma-7)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g) \tag{3.20}$$

which contradicts $n \ge 2k+m^*+\sigma+7$.

If $m > k+1$, then from (3.10) we have

$$\begin{aligned} (n+m^*+\sigma)T(r,f) \le (2k+\sigma+4)\{T(r,f)+T(r,g)\}+2\overline{N}(r,f) \\ +\overline{N}(r,g)+S(r,f)+S(r,g). \end{aligned} \tag{3.21}$$

Similarly,

$$\begin{aligned} (n+m^*+\sigma)T(r,g) \le (2k+\sigma+4)\{T(r,f)+T(r,g)\}+2\overline{N}(r,g) \\ +\overline{N}(r,f)+S(r,f)+S(r,g). \end{aligned} \tag{3.22}$$

Again combining (3.21) and (3.22) we get

$$(n-4k+m^*-\sigma-11)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$$

which contradicts $n \ge 4k-m^*+\sigma+11$.

Subcase 1.2: $l=0$. Then we have

$$\begin{aligned} N_E^{1)}\big(r,\frac{1}{F-1}\big) &= N_E^{1)}\big(r,\frac{1}{G-1}\big)+S(r,f), \\ \overline{N}_E^{(2}\big(r,\frac{1}{F-1}\big) &= \overline{N}_E^{(2}\big(r,\frac{1}{G-1}\big)+S(r,f). \end{aligned}$$

Thus (3.1) gives

$$\begin{aligned}
&\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1})=N_E^{1)}(r,\frac{1}{F-1})+\overline{N}_E^{(2}(r,\frac{1}{F-1})\\
&+\overline{N}_L(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})+S(r,f)\\
&\leq N_E^{1)}(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{F-1})+N(r,\frac{1}{G-1})+S(r,f)\\
&\leq \overline{N}(r,F)+\overline{N}(r,G)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+2\overline{N}_L(r,\frac{1}{F-1})+\overline{N}_L(r,\frac{1}{G-1})\\
&+N(r,\frac{1}{G-1})+N_0(r,\frac{1}{F'})+N_0(r,\frac{1}{G'})+S(r,f). \qquad (3.23)
\end{aligned}$$

Therefore using (3.2), (3.23) and Lemma 2.7 we get

$$\begin{aligned}
T(r,F)\leq\ & 2\overline{N}(r,F)+2\overline{N}(r,G)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+2\overline{N}_L(r,\frac{1}{F-1})\\
&+\overline{N}_L(r,\frac{1}{G-1})+S(r,f)+S(r,g). \qquad (3.24)
\end{aligned}$$

If $m\leq k+1$, using (3.8) and (3.9) in (3.24) and Lemma 2.8 we obtain

$$\begin{aligned}
(n+m^*+\sigma)T(r,f)\leq\ & (k+m^*+\sigma+2)\{T(r,f)+T(r,g)\}+4\overline{N}(r,f)+3\overline{N}(r,g)\\
&+2(m^*+\sigma+1)T(r,f)+(m^*+\sigma+1)T(r,g)+S(r,f)+S(r,g). \qquad (3.25)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(n+m^*+\sigma)T(r,g)\leq(k+m^*+\sigma+2)\{T(r,f)+T(r,g)\}+4\overline{N}(r,g)+3\overline{N}(r,f)\\
&+2(m^*+\sigma+1)T(r,g)+(m^*+\sigma+1)T(r,f)+S(r,f)+S(r,g). \qquad (3.26)
\end{aligned}$$

Therefore (3.25) and (3.26) gives

$$(n-2k-4m^*-4\sigma-14)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

which contradicts $n\geq 2k+4m^*+4\sigma+14$.

If $m>k+1$, using (3.8) and (3.9) in (3.24) we have

$$\begin{aligned}
&(n+m^*+\sigma)T(r,f)\leq(2k+\sigma+4)\{T(r,f)+T(r,g)\}+4\overline{N}(r,f)+3\overline{N}(r,g)\\
&+2(m^*+\sigma+1)T(r,f)+(m^*+\sigma+1)T(r,g)+S(r,f)+S(r,g). \qquad (3.27)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&(n+m^*+\sigma)T(r,g)\leq(2k+\sigma+4)\{T(r,f)+T(r,g)\}+4\overline{N}(r,g)+3\overline{N}(r,f)\\
&+2(m^*+\sigma+1)T(r,g)+(m^*+\sigma+1)T(r,f)+S(r,f)+S(r,g). \qquad (3.28)
\end{aligned}$$

Thus combining (3.27) and (3.28) we get

$$(n-4k-2m^*-4\sigma-18)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

which contradicts $n \geq 4k + 2m^* + 4\sigma + 18$.

Case 2: $H \equiv 0$. Integrating (2.1) twice we get

$$\frac{1}{F-1} = \frac{C}{G-1} + D, \tag{3.29}$$

where $C \neq 0$ and D are constants. From 3.29 it is clear that F, G share the value 1 CM and hence they share $(1,l)$. Also $T(r,F) = T(r,G) + O(1)$. Here, the following subcases can arise.

Subcase 2.1: $D \neq 0$ and $C \neq D$. If $D = -1$, then from (3.29) we have $F = \frac{-C}{G-C-1}$. Therefore

$$\overline{N}\left(r, \frac{1}{G-(C+1)}\right) = \overline{N}(r,F).$$

Because of the Second Fundamental Theorem of Nevanlinna and Lemma 2.5, we obtain

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,G) + \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{G-(C+1)}\right) + S(r,G) \leq \overline{N}(r,G) \\ &+ T(r,G) - (n+m^*+\sigma)T(r,g) + N_{k+1}(r,0;G_1) + \overline{N}(r,G) + S(r,g). \end{aligned} \tag{3.30}$$

If $m \leq k+1$, then from (3.30) we have

$$(n+m^*+\sigma)T(r,g) \leq (k+m^*+\sigma+1)T(r,g) + 2\overline{N}(r,g) + S(r,g). \tag{3.31}$$

Similarly,

$$(n+m^*+\sigma)T(r,f) \leq (k+m^*+\sigma+1)T(r,f) + 2\overline{N}(r,f) + S(r,f). \tag{3.32}$$

Therefore using (3.31) and (3.32) we get

$$(n-k-3)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g)$$

contradicting $n \geq k+3$.

If $m > k+1$, then from (3.30) we have

$$(n+m^*+\sigma)T(r,f) \leq (2k+\sigma+4)T(r,g) + 2\overline{N}(r,g) + S(r,g). \tag{3.33}$$

Similarly,

$$(n+m^*+\sigma)T(r,g) \leq (2k+\sigma+4)T(r,f) + 2\overline{N}(r,f) + S(r,f). \tag{3.34}$$

Thus (3.33) and (3.34) gives

$$(n-2k+m^*-6)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

which contradicts $n \geq 2k - m^* + 6$.

Subcase 2.2: $D \neq 0$ and $C = D$.
Subcase 2.2.1: If $D \neq -1$, then from (3.29) we have $\frac{1}{F} = \frac{DG}{(1+D)G-1}$. Therefore

$$\overline{N}\left(r, \frac{1}{G - \frac{1}{D+1}}\right) = \overline{N}(r, \frac{1}{F}).$$

In a similar way as in Subcase 2.1, we get a contradiction.

Subcase 2.2.2 Suppose $D = -1$. Then

$$FG = 1. \tag{3.35}$$

Subcase 2.2.2.1 Suppose $P(w)$ is not a monomial.
Since $n > k$, by Lemma 2.10, $P(w)$ is reduced to a non-zero monomial, which is a contradiction. Therefore we must have $FG \not\equiv 1$.

Subcase 2.2.2.2 Suppose $P(w)$ is a non-zero monomial, namely, $P(w) = a_i w^i \not\equiv 0$ for some $i \in \{0, 1, ..., m\}$. Then it follows from (3.35) that

$$a_i^2 f^{n+i}(z) \prod_{j=1}^{s} f(z+c_j)^{\mu_j} . g^{n+i}(z) \prod_{j=1}^{s} g(z+c_j)^{\mu_j} = a^2.$$

Subcase 2.2.2.3 Suppose $P(w) = a_0$, a non-zero constant.
Then, proceeding in a similar manner as done in Subcase 2.2.2.2, we conclude that

$$a_0^2 f^n(z) \prod_{j=1}^{s} f(z+c_j)^{\mu_j} . g^n(z) \prod_{j=1}^{s} g(z+c_j)^{\mu_j} = a^2.$$

Subcase 2.3. Suppose $D = 0$.
Then

$$F = \frac{G + C - 1}{C}.$$

If $C \neq 1$, then we have

$$\overline{N}\left(r, \frac{1}{G - (1 - C)}\right) = \overline{N}(r, \frac{1}{F}).$$

Because of the Second Fundamental Theorem of Nevanlinna, we obtain

$$T(r,G) \leq \overline{N}(r,G) + \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{G-(1-C)}\right) + S(r,G) \leq \overline{N}(r,G)$$
$$+ T(r,G) - (n + m^* + \sigma)T(r,g) + N_{k+1}(r,0;G_1) + N_{k+1}(r,0,F_1) + S(r,f+)S(r,g).$$

If $m \leq k+1$, then from (3.17) we have

$$(n+m^*+\sigma)T(r,g) \leq (k+m^*+\sigma+1)\{T(r,f+T(r,g)\}+\overline{N}(r,g)+S(r,g). \tag{3.36}$$

Similarly,

$$(n+m^*+\sigma)T(r,f) \leq (k+m^*+\sigma+1)\{T(r,f)+T(r,g)\}+\overline{N}(r,f)+S(r,f). \tag{3.37}$$

Now, by 3.36 and (3.37), we get

$$(n-2k-m^*-\sigma-3)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g), \tag{3.38}$$

contradicting $n \geq 2k+m^*+\sigma+3$.

If $m > k+1$, then from (3.10) we have

$$(n+m^*+\sigma)T(r,f) \leq (2k+\sigma+4)\{T(r,f)+T(r,g)\}+\overline{N}(r,g)+S(r,g). \tag{3.39}$$

Similarly,

$$(n+m^*+\sigma)T(r,g) \leq (2k+\sigma+4)\{T(r,f)+T(r,g)\}+\overline{N}(r,f)+S(r,f). \tag{3.40}$$

Adding (3.39) and (3.40) we get

$$(n-4k+m^*-\sigma-9)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

contradicting $n \geq 4k-m^*+\sigma+9$.
Therefore, $C=1$ and $F \equiv G$. In view of Lemma 2.11, we conclude that $F_1 \equiv G_1$, i.e.,

$$f^n(z)P(f)(z)\prod_{j=1}^{s} f(z+c_j)^{\mu_j} \equiv g^n(z)P(g)(z)\prod_{j=1}^{s} g(z+c_j)^{\mu_j}. \tag{3.41}$$

Suppose (3.41) holds. We have the following subcases.

Subcase 2.3.1. $P(w)$ is not a monomial. Suppose $h=\frac{f}{g}$ is a constant. By (3.41) putting $f=hg$ we get

$$a_m g^m(z)(h^{n+m+\sigma}-1)+a_{m-1}g^{m-1}(z)(h^{n+m+\sigma-1}-1)+\ldots\ldots+a_0(h^{n+\sigma}-1) \equiv 0,$$

which implies that $h^d=1$, where d is the GCD of the elements of J, $J=\{k \in I : a_k \neq 0\}$ and $I=\{n+\sigma, n+\sigma+1, \ldots\ldots, n+\sigma+m\}$. Thus $f=tg$ for a constant t. If h is non-constant, then f and g satisfy the algebraic difference equation $R(f,g)=0$ where $R(f,g)$ is given by

$$R(w_1,w_2)=w_1^n P(w_1)\prod_{j=1}^{s} w_1(z+c_j)^{\mu_j}-w_2^n P(w_2)\prod_{j=1}^{s} w_2(z+c_j)^{\mu_j}.$$

Subcase 2.3.2. Suppose $P(w)$ is a non-zero monomial, say $P(w)=a_i w^i$, for some $i \in \{0,1,\ldots m\}$. Then, by the same argument as in Subcase 2.3.1, we conclude that if h is constant, then $f=tg$ where t satisfies $t^{n+\sigma+i}=1$, or else f and g satisfy the algebraic equation $R(w_1,w_2)=0$, where $R(w_1,w_2)=w_1^{n+i}\prod_{j=1}^{s}(w_1(z+c_j))^{\mu_j}-w_2^{n+i}\prod_{j=1}^{s}(w_2(z+c_j))^{\mu_j}$.

Subcase 2.3.3. $P(w) = a_0(\neq 0)$, a constant. Let $h = \frac{f}{g}$. Then from (3.41) we have

$$h^n(z) \equiv \frac{1}{\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}}.$$

Therefore by Lemmas 2.1, 2.2 and 2.3 we have

$$\begin{aligned} nT(r,h) = T(r,h^n) + S(r,h) &= T\left(r, \frac{1}{\prod_{j=1}^{s}(h(z+c_j))^{\mu_j}}\right) + S(r,h) \\ &\leq \sigma T(r,h) + S(r,h), \end{aligned}$$

which is a contradiction since $n > \sigma$. Hence h must be constant, then $f = tg$ where t satisfies $t^{n+\sigma} = 1$. □

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Conflicts of interest

The authors declare no conflicts of interest.

REFERENCES

[1] A. Banerjee, *Meromorphic functions sharing one value*, Int. J. Math. Sci., 22 (2005), 3587–3598.

[2] A. Banerjee and S. Majumder, *On the uniqueness of a power of a meromorphic function sharing a small function with the power of its derivative*, Comment. Math. Univ. Carolin., 51(4) (2010), 565–576.

[3] A. Banerjee and B. Chakraborty, *Further investigations on a questions of Zhang and Lu*, Ann. Univ. Paedagog. Crac. Stud. Math., 14 (2015), 105–119.

[4] M. R. Chen and Z. X. Chen, *Properties of difference polynomials of entire functions with finite order*, Chinese Ann. Math., 33A (2012), 359–374.

[5] Y. M. Chiang and S. J. Feng, *On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J., 16 (2008), no. 1, 105–129.

[6] W. K. Hayman, *Meromorphic function, Clarendon Press*, Oxford, 1964.

[7] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl. 46 (2001), 241–253.

[8] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J., 161(2001), 193–206.

[9] I. Laine and C. C. Yang, *Value distribution of difference polynomials*, Proc. Japan Acad. Ser. A, 83 (2007), 148–151.

[10] K. Liu and L. Z. Yang, *Value distribution of the difference operator*, Arch. Math., 92 (2009), 270–278.

[11] F. H. Liu and H. X. Yi, *Uniqueness of meromorphic functions concerning the difference polynomials*, Kyungpook Math. J., 55 (2015), 411–427.

[12] X. Luo and W. C. Lin, *Value sharing results for shifts of meromorphic functions*, J. Math. Anal. Appl., 377 (2011), 441–449.

[13] H. P. Waghamore, *Generalization of value distribution and uniqueness of certain types of difference polynomials*, Tbilisi Mathematical Journal, 11(1) (2018), 1–13.
[14] L. Yang, *Value distributions theory*, Springer-Verlag, Berlin, 1993.
[15] H. X. Yi and C. C. Yang, *Uniqueness theory of meromorphic functions (in Chinese)*, Science Press, Beijing, 1995.
[16] K. Zhang and H. X. Yi, *The value distribution and uniqueness of one certain type of differential-difference polynomials*, Acta Mathematica Scientia Series Manuscript, 34 (2014), no. 3, 719–728.
[17] J. L. Zhang and L. Z. Yang, *Some results related to a conjecture of R. Bruck*, J. Inequal. Pure Appl. Math., 8 (2007), Art. 18.
[18] J. L. Zhang, *Value distribution and shared sets of differences of meromorphic functions*, J. Math. Anal. Appl. 367 (2010), 401–408.

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