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A FRACTAL DIRAC EIGENVALUE PROBLEM: SPECTRAL PROPERTIES AND NUMERICAL EXAMPLES

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ABSTRACT. In this paper, we study a Dirac boundary value problem where the operator is considered with a derivative of order $\alpha \in (0,1]$, known as the F^{α}-derivative. We prove some spectral properties of eigenvalues and eigenfunctions, and we present numerical examples to demonstrate the practical implications of our approach.

1. INTRODUCTION

The word *fractal* derives from the Latin word *fractus* which means *cracked.* Fractals exhibit unique geometric properties, showcasing a fractal dimension that surpasses their topological dimension. These intricate structures display self-similarity at varying scales, combining repetitive and random processes. The intricate nature of fractals poses challenges for conventional calculus methods in calculating derivatives and integrals. The connection between fractal geometry and natural phenomena, as seen in clouds, mountains, and lightning, underscores a complexity that defies conventional mathematical frameworks, as highlighted in [19]. Moreover, in contrast to Euclidean geometry, determining the size of fractals involves non-trivial considerations for measurements like length, surface area, and volume as discussed in [5, 16].

Fractals, characterized by intricate patterns, showcase self-similarity and often display dimensions that are non-integer and complex [18, 20]. The middle-third set of Cantor is one of the most well-known examples of fractals [4]. Fractals are often too irregular to have any smooth differentiable structure defined on them, and this results in delivering the methods and techniques of ordinary calculus as powerless and inapplicable. The techniques and methods to create calculus on the fractal sets and curves are studied in [5, 17, 26].

In recent work, Parvate and Gangal [22, 23] introduced F^{α}-calculus, a form of Riemannian-like calculus grounded in the fractal subsets of the real line. This calculus is distinguished by its algorithmic simplicity compared to other methods. F^{α}-calculus is a generalization of the ordinary calculus that addresses the cases where standard calculus is inapplicable. In this calculus, an integral of order

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$\alpha \in (0,1]$ called the F^α-integral is defined making it possible to integrate functions with fractal support F of dimension α.

Furthermore, a derivative of order $\alpha \in (0,1]$, known as the F^α-derivative, facilitates the differentiation of functions such as the Cantor staircase function and the Weierstrass function. In contrast to the classical fractional derivative, the F^α-derivative aligns the geometrical order of the derivative with the domain of the function's support, lending it a distinct physical interpretation [8, 24, 27]. Notably, the F^α-derivative is local, which is in contrast to the non-local nature of the classical fractional derivative. This locality is crucial in physics, where all measurements are inherently local. Moreover, F^α-calculus retains much of the simplicity found in ordinary calculus [6].

Recent results on the spectral analysis of Dirac systems in classical and discrete settings can be found in [15, 21, 25, 28], where issues such as spectral singularities and eigenvalue–eigenfunction behavior are investigated. In contrast, the study of Dirac systems in the context of fractal calculus is still limited.

Differential equations over fractal domains are often referred to as fractal differential equations. Studies concerning fractal differential equations have been an important area of research in recent years. For instance, in [11] Golmankhaneh and Tunç prove the existence and uniqueness theorems for the linear and non-linear fractal differential equations and they give the fractal Lipschitz condition on the F^α-calculus. In [7] Golmankhaneh and Cattani give difference equations on fractal sets and their corresponding fractal differential equations. They define an analogue of the classical Euler method in fractal calculus and they solve fractal differential equations by using this fractal Euler method. In [12] Golmankhaneh and Tunç give the analogues of Laplace and Sumudu transforms, which have an important role in control engineering problems, and they solve linear differential equations on Cantor-like sets by utilizing the fractal Sumudu transforms. Fractal differential equations were solved by defining fractal Mellin, Laplace, and Fourier transforms in [13]. Retarded, neutral, and renewal delay differential equations with constant coefficients in the fractal domain are solved through the method of steps and employing the Laplace transform in [10].

In light of the above-given literature, we study the Dirac eigenvalue problem generated by the fractal differential equations

$$\ell^\alpha f := \begin{cases} D_F^\alpha f_2 - p(x) f_1 = \lambda f_1 \\ -D_F^\alpha f_1 + r(x) f_2 = \lambda f_2 \end{cases}, \quad x \in [0,\pi] \tag{1.1}$$

and the boundary conditions

$$U_1(f) := f_1(0) = 0, \tag{1.2}$$

$$U_2(f) := f_1(\pi) = 0, \tag{1.3}$$

where D_F^α indicates the F^α-derivative introduced in [22], $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, λ is a spectral parameter, $p(x), r(x) \in \mathcal{L}_2^\alpha(0,\pi)$ are real valued functions where $\mathcal{L}_2^\alpha(0,\pi)$ is the space of square F^α-integrable functions on $[0,\pi]$, i.e.

$$\int_0^\pi |f(x)|^2 \,\mathrm{d}_F^\alpha x < \infty$$

holds for $f : [0,\pi] \to \mathbb{R}$ provided $\mathrm{Sch}(f)$ is an α-perfect set.

It is important to distinguish the present work from existing studies on Dirac systems that utilize classical fractional derivatives, such as the Riemann–Liouville or Caputo derivatives. Those operators are inherently non-local, meaning the state of the system at a given point depends on its history or the entire domain, often interpreted as memory effects or long-range interactions. In contrast, the F^α-derivative employed here is a local operator. It describes the scaling properties of the fractal domain itself rather than non-local interactions. Consequently, this approach generalizes the Dirac system to fractal sets while preserving the local nature of the classical differential operator, which is often physically essential for defining boundary value problems and local measurements.

Although numerous studies address various differential equations problems in the fractal calculus framework, there are relatively few that focus on eigenvalue problems in this context. For instance, in [3] Çetinkaya and Golmankhaneh explore a fractal Sturm–Liouville problem, while in [1] Allahverdiev and Tuna prove the existence and uniqueness theorem for the solutions of such problems. In a related direction, [2] studies the one-dimensional Dirac equation in the framework of fractal calculus, focusing on operator realizations, Green's functions, and eigenfunction expansions. In this work, we extend the results in [3] to the Dirac setting and provide numerical examples to demonstrate the practical implications of our approach. We believe that this work will contribute to further studies related to the eigenvalue problems generated with F^α-derivative and their applications.

The structure of the paper is as follows. In Section 2 we introduce a self-adjoint operator and we give some of the properties of eigenvalues and vector-valued eigenfunctions. In Section 3 we present numerical examples to illustrate the applicability of our conclusions. In Section 4, we close the paper with some concluding remarks.

2. Spectral Properties

We begin by recalling the necessary definitions and properties of the F^α-calculus, as developed by Parvate and Gangal [22, 23]. These tools will be used throughout this section and are essential for establishing the spectral properties of the operator ℓ^α.

Let $F \subset \mathbb{R}$ be a fractal set. The flag function (or mass function) $S_F^\alpha : \mathbb{R} \to \mathbb{R}$ is defined by

$$S_F^\alpha(x) = \sup\left\{H^\alpha\big(F \cap (-\infty,t]\big) : t \leq x\right\},$$

where, H^α denotes the α-dimensional Hausdorff measure.

For a function $f : F \to \mathbb{R}$, the F^{α}-derivative of f at $x \in F$, denoted by $D_F^{\alpha} f(x)$, is defined as

$$D_F^{\alpha} f(x) = \begin{cases} \lim\limits_{\substack{y \to x \\ y \in F}} \dfrac{f(y) - f(x)}{S_F^{\alpha}(y) - S_F^{\alpha}(x)}, & \text{if } x \in F, \\ 0, & \text{otherwise.} \end{cases}$$

provided the limit exists.

Let $f : [a,b] \to \mathbb{R}$. The F^{α}-integral of f on $[a,b]$ is defined by

$$\int_a^b f(x)\, d_F^{\alpha} x = \int_a^b f(x)\, dS_F^{\alpha}(x),$$

where the integral on the right is a Riemann-Stieltjes integral.

A crucial tool for our analysis is the integration by parts formula in F^{α}-calculus. If $f, g \in \mathcal{L}_2^{\alpha}(0,\pi)$ are functions such that $D_F^{\alpha} f$ and $D_F^{\alpha} g$ exist and are integrable, then

$$\int_0^{\pi} f(x)\, D_F^{\alpha} g(x)\, d_F^{\alpha} x = \big[f(x) g(x)\big]_0^{\pi} - \int_0^{\pi} g(x)\, D_F^{\alpha} f(x)\, d_F^{\alpha} x.$$

This property allows us to move the fractal derivative between components of the inner product, which is fundamental to proving the self-adjointness of ℓ^{α} in Theorem 2.1.

In the context of fractal analysis described in [6], an inner product in the Hilbert space $\mathcal{L}_2^{\alpha}(0,\pi)$ can be defined by

$$\langle f, g \rangle = \int_0^{\pi} \big(f_1(x) g_1(x) + f_2(x) g_2(x)\big)\, \mathrm{d}_F^{\alpha} x,$$

where $f = (f_1, f_2)^T \in \mathcal{L}_2^{\alpha}(0,\pi)$ and $g = (g_1, g_2)^T \in \mathcal{L}_2^{\alpha}(0,\pi)$.

We consider the Hilbert space

$$\mathcal{H} = \mathcal{L}_2^{\alpha}(0,\pi) \times \mathcal{L}_2^{\alpha}(0,\pi)$$

endowed with the inner product defined above.

We introduce the domain $L_0 \subset \mathcal{H}$ of the operator ℓ^{α} as the subspace of functions $f = (f_1, f_2)^T$ which are F^{α}-differentiable and satisfy the boundary conditions (1.2) and (1.3), i.e.,

$$L_0 = \left\{ f \in \mathcal{H} : f_1(0) = 0,\ f_1(\pi) = 0,\ \text{and } f \in AC_F^{\alpha}[0,\pi] \right\}.$$

Here $AC_F^{\alpha}[0,\pi]$ denotes the class of vector functions whose components are F^{α}-absolutely continuous on $[0,\pi]$. The operator ℓ^{α} is defined on this subspace L_0.

Theorem 2.1. *The operator ℓ^{α} is self-adjoint on the domain L_0.*

Proof. Let $f,g \in L_0$. Using the definition of the inner product, we have

$$
\begin{aligned}
&\langle \ell^\alpha f, g\rangle - \langle f, \ell^\alpha g\rangle \\
&= \int_0^\pi \big(D_F^\alpha f_2 - p(x) f_1\big) g_1 \, \mathrm{d}_F^\alpha x \\
&\quad + \int_0^\pi \big(-D_F^\alpha f_1 + r(x) f_2\big) g_2 \, \mathrm{d}_F^\alpha x \\
&\quad - \int_0^\pi f_1 \big(D_F^\alpha g_2 - p(x) g_1\big) \, \mathrm{d}_F^\alpha x \\
&\quad - \int_0^\pi f_2 \big(-D_F^\alpha g_1 + r(x) g_2\big) \, \mathrm{d}_F^\alpha x \\
&= \int_0^\pi \big(D_F^\alpha f_2 g_1 + f_2 D_F^\alpha g_1\big) \, \mathrm{d}_F^\alpha x \\
&\quad - \int_0^\pi \big(D_F^\alpha f_1 g_2 + f_1 D_F^\alpha g_2\big) \, \mathrm{d}_F^\alpha x \\
&= \int_0^\pi D_F^\alpha \big(f_2 g_1 - f_1 g_2\big) \, \mathrm{d}_F^\alpha x.
\end{aligned}
$$

Hence

$$
\langle \ell^\alpha f, g\rangle - \langle f, \ell^\alpha g\rangle = \big\{\big(f_2 g_1 - f_1 g_2\big)\chi_F(x)\big\}_{x=0}^{\pi}. \tag{2.1}
$$

Using the boundary conditions (1.2) and (1.3), we see that this term vanishes, so

$$
\langle \ell^\alpha f, g\rangle - \langle f, \ell^\alpha g\rangle = 0,
$$

completing the proof. □

Assume that the boundary value problem (1.1)–(1.3) has a nontrivial solution

$$
f(x,\lambda_0) = \begin{pmatrix} f_1(x,\lambda_0) \\ f_2(x,\lambda_0) \end{pmatrix}
$$

for a certain λ_0, called an eigenvalue, and the corresponding solution $f(x,\lambda_0)$ is called a vector-valued eigenfunction.

Lemma 2.1. *The vector-valued eigenfunctions $f(x,\lambda_1)$ and $g(x,\lambda_2)$ corresponding to different eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal.*

Proof. Since $f(x,\lambda_1)$ and $g(x,\lambda_2)$ are solutions of (1.1), we have

$$
\begin{aligned}
D_F^\alpha f_2(x,\lambda_1) - p(x) f_1(x,\lambda_1) &= \lambda_1 f_1(x,\lambda_1), \\
-D_F^\alpha f_1(x,\lambda_1) + r(x) f_2(x,\lambda_1) &= \lambda_1 f_2(x,\lambda_1), \\
D_F^\alpha g_2(x,\lambda_2) - p(x) g_1(x,\lambda_2) &= \lambda_2 g_1(x,\lambda_2), \\
-D_F^\alpha g_1(x,\lambda_2) + r(x) g_2(x,\lambda_2) &= \lambda_2 g_2(x,\lambda_2).
\end{aligned}
$$

Multiplying these equations by $g_1(x,\lambda_2)$, $g_2(x,\lambda_2)$, $-f_1(x,\lambda_1)$, and $-f_2(x,\lambda_1)$, respectively, and summing, we get

$$\begin{aligned} &D_F^\alpha f_2(x,\lambda_1)\,g_1(x,\lambda_2)+f_2(x,\lambda_1)\,D_F^\alpha g_1(x,\lambda_2)\\ &\quad -D_F^\alpha f_1(x,\lambda_1)\,g_2(x,\lambda_2)-f_1(x,\lambda_1)\,D_F^\alpha g_2(x,\lambda_2)\\ &\quad =(\lambda_1-\lambda_2)\big(f_1(x,\lambda_1)g_1(x,\lambda_2)+f_2(x,\lambda_1)g_2(x,\lambda_2)\big).\end{aligned}$$

After integrating this identity over $[0,\pi]$, the integral of the left-hand side is

$$\int_0^\pi D_F^\alpha\big(f_2(x,\lambda_1)g_1(x,\lambda_2)-f_1(x,\lambda_1)g_2(x,\lambda_2)\big)\,\mathrm{d}_F^\alpha x,$$

which becomes a boundary term that vanishes by conditions (1.2), (1.3). Hence

$$(\lambda_1-\lambda_2)\int_0^\pi\big[f_1(x,\lambda_1)g_1(x,\lambda_2)+f_2(x,\lambda_1)g_2(x,\lambda_2)\big]\,\mathrm{d}_F^\alpha x=0.$$

Since $\lambda_1\neq\lambda_2$, the inner product must be zero, so the eigenfunctions are orthogonal. □

Corollary 2.1. *The eigenvalues of the boundary value problem* (1.1)–(1.3) *are real.*

Let $\varphi(\cdot,\lambda)=\begin{pmatrix}\varphi_1(\cdot,\lambda)\\ \varphi_2(\cdot,\lambda)\end{pmatrix}$ and $\psi(\cdot,\lambda)=\begin{pmatrix}\psi_1(\cdot,\lambda)\\ \psi_2(\cdot,\lambda)\end{pmatrix}$ be the solutions of (1.1) under the initial conditions

$$\varphi_1(0,\lambda)=0,\ \varphi_2(0,\lambda)=1,\ \psi_1(\pi,\lambda)=0,\ \psi_2(\pi,\lambda)=1. \tag{2.2}$$

Then,

$$U_1(\varphi)=U_2(\psi)=0. \tag{2.3}$$

Denote

$$\Delta(\lambda)=\varphi_2(\cdot,\lambda)\,\psi_1(\cdot,\lambda)-\varphi_1(\cdot,\lambda)\,\psi_2(\cdot,\lambda). \tag{2.4}$$

This function $\Delta(\lambda)$ is called the characteristic function of (1.1)–(1.3).

Owing to the linear dependence of system (1.1) on the spectral parameter λ, the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are entire functions of λ for each fixed $x\in[0,\pi]$. It follows that the function $\Delta(\lambda)$ constructed from these solutions is entire. It has an at most countable set of zeros $\{\lambda_n\}$. It can be easily seen that the characteristic function does not depend on x. Indeed, with all functions evaluated at (x,λ),

$$\begin{aligned} D_F^\alpha\Delta(\lambda)&=(D_F^\alpha\varphi_2)\psi_1+\varphi_2(D_F^\alpha\psi_1)-(D_F^\alpha\varphi_1)\psi_2-\varphi_1(D_F^\alpha\psi_2)\\ &\overset{(1.1)}{=}(\lambda+p)\varphi_1\psi_1+(r-\lambda)\varphi_2\psi_2-(r-\lambda)\varphi_2\psi_2-(\lambda+p)\varphi_1\psi_1\\ &=0.\end{aligned}$$

Substituting $x=0$ and $x=\pi$ in (2.4) and taking (2.2) into consideration, we have

$$\Delta(\lambda)=U_1(\psi)=-U_2(\varphi). \tag{2.5}$$

Theorem 2.2. *The zeros $\{\lambda_n\}$ of the characteristic function coincide with the eigenvalues of the boundary value problem* (1.1)–(1.3)*. The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions, and there exists a sequence $\{\beta_n\}$ such that*

$$\psi(x,\lambda_n)=\beta_n\,\varphi(x,\lambda_n),\ \beta_n\neq 0. \tag{2.6}$$

Proof. If λ_0 is a zero of $\Delta(\lambda)$, i.e. $\Delta(\lambda_0)=0$, then by construction $\psi(\cdot,\lambda_0)$ is a multiple of $\varphi(\cdot,\lambda_0)$. Both satisfy the boundary conditions, so λ_0 is indeed an eigenvalue, and $\psi(\cdot,\lambda_0)$, $\varphi(\cdot,\lambda_0)$ are eigenfunctions.

Conversely, if λ_0 is an eigenvalue and f_0 is a corresponding (nonzero) eigenfunction satisfying (1.2)–(1.3), we can match f_0 with one of $\varphi(\cdot,\lambda_0)$ or $\psi(\cdot,\lambda_0)$ (depending on initial conditions), showing $\Delta(\lambda_0)=0$. One also sees that each eigenvalue is simple from the geometric point of view. □

We define the weight numbers $\{\alpha_n\}$ of (1.1)–(1.3) by

$$\alpha_n := \int_0^{\pi} \left[\varphi_1^2(x,\lambda_n)+\varphi_2^2(x,\lambda_n)\right] \mathrm{d}_F^{\alpha}x. \tag{2.7}$$

Lemma 2.2. *The following relation holds:*

$$\beta_n\,\alpha_n = (D_{F,\lambda}^{\alpha}\Delta)(\lambda_n), \tag{2.8}$$

where β_n *are defined by* (2.6) *and* $D_{F,\lambda}^{\alpha}$ *is the* F^{α}*-derivative with respect to the spectral parameter.*

Proof. Since $\varphi(x,\lambda_n)$ and $\psi(x,\lambda)$ solve (1.1), multiply them in a standard way and integrate over $[0,\pi]$ to get

$$\begin{aligned}\left(S_F^{\alpha}(\lambda_n)-S_F^{\alpha}(\lambda)\right)\int_0^{\pi}\big[\varphi_1(x,\lambda_n)\psi_1(x,\lambda)\\ +\varphi_2(x,\lambda_n)\psi_2(x,\lambda)\big]\,\mathrm{d}_F^{\alpha}x=\Delta(\lambda_n)-\Delta(\lambda).\end{aligned}$$

As $\lambda\to\lambda_n$, the difference quotient leads to

$$(D_{F,\lambda}^{\alpha}\Delta)(\lambda_n)=\int_0^{\pi}\left[\varphi_1(x,\lambda_n)\,\psi_1(x,\lambda_n)+\varphi_2(x,\lambda_n)\,\psi_2(x,\lambda_n)\right]\mathrm{d}_F^{\alpha}x.$$

Substituting (2.6) into the expression above, we obtain

$$(D_{F,\lambda}^{\alpha}\Delta)(\lambda_n)=\beta_n\int_0^{\pi}\left[\varphi_1^2(x,\lambda_n)+\varphi_2^2(x,\lambda_n)\right]\mathrm{d}_F^{\alpha}x.$$

Now, using (2.7), we simplify this and obtain

$$(D_{F,\lambda}^{\alpha}\Delta)(\lambda_n)=\beta_n\alpha_n.$$

Thus, we arrive at (2.8). □

Corollary 2.2. *The eigenvalues of* (1.1)–(1.3) *are simple from the algebraic point of view, i.e.* $(D_{F,\lambda}^{\alpha}\Delta)(\lambda_n)\neq 0$.

Proof. Since $\alpha_n\neq 0$, $\beta_n\neq 0$, we get by virtue of (2.8) that $(D_{F,\lambda}^{\alpha}\Delta)(\lambda_n)\neq 0$. □

3. Numerical Examples

In this section, we present numerical examples to illustrate the applicability of our conclusions. We used the fourth-order classical Runge-Kutta method and the fourth-order fractal Runge-Kutta method as described in [14], with the equations given in fractal form below.

$$y_{n+1} = y_n + h_F^\alpha \left(\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6} \right),$$

where

$$\begin{aligned}
k_{n1} &= f\left(S_F^\alpha(x), y_n\right), \\
k_{n2} &= f\left(S_F^\alpha(x) + \frac{1}{2}h_F^\alpha, y_n + \frac{1}{2}h_F^\alpha k_{n1}\right), \\
k_{n3} &= f\left(S_F^\alpha(x) + \frac{1}{2}h_F^\alpha, y_n + \frac{1}{2}h_F^\alpha k_{n2}\right), \\
k_{n4} &= f\left(S_F^\alpha(x) + h_F^\alpha, y_n + h_F^\alpha k_{n3}\right), \\
h_F^\alpha &= S_F^\alpha(x_{n+1}) - S_F^\alpha(x_n).
\end{aligned}$$

We approximated the integral staircase function $S_F^\alpha(x)$ by a power law x^α (an approximation valid for certain sets) described in [22]. In a problem where the exact structure of the fractal set F is more crucial than the scaling behavior of α, one can refer to the implementation in [14] of the coarse-grained mass function, $\gamma_\delta^\alpha(F,a,b)$. For stiff problems, other numerical techniques may be more suitable. We have plotted solutions to these equations using the Matplotlib Python package.

Below are three examples for (1.1)–(1.3) on $[0,\pi]$, using the approximation $S_F^\alpha(x) \approx x^\alpha$.

Example 3.1. *We define* $p(x) = \frac{1}{1+S_F^\alpha(x)}$, $r(x) = \frac{1}{1+(S_F^\alpha(x))^2}$, *and scaling indices* $\alpha = [0.8, 0.9, 1.0]$. *The numerical eigenvalues (denoted* $\tilde{\lambda}_n$*) appear in Table 1. Errors for the case* $\alpha = 1$ *are shown in Table 2.*

TABLE 1. Numerically computed eigenvalues $\tilde{\lambda}_n$ for various methods and α.

Method	α	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$
Classical	*N/A*	*0.347524*	*1.176747*	*2.055970*	*3.020643*
Fractal	*0.8*	*0.413400*	*1.438434*	*2.566015*	-
Fractal	*0.9*	*0.378385*	*1.301643*	*2.296227*	-
Fractal	*1.0*	*0.347685*	*1.176925*	*2.056040*	*3.020692*

The plots of the first eigenfunctions $f_1 = y_1(x)$ *and* $f_2 = y_2(x)$ *for the classical calculus and fractal calculus methods across the three scaling indices are shown in Figure 1.*

For easier visual verification that the classical and fractal methods agree when $\alpha = 1$, *zoomed-in plots are also provided in Figure 2.*

Example 3.2. *Let* $p(x) = S_F^\alpha(x) + 1$, $r(x) = (S_F^\alpha(x))^2 + 1$, *and* $\alpha = [0.8, 0.9, 1.0]$. *The eigenvalues and errors are shown in Tables 3 and 4.*

TABLE 2. Magnitude of error between classical and fractal methods for $\alpha = 1$.

$\tilde{\lambda}_n$	*Classical*	$\lvert\Delta\tilde{\lambda}_n\rvert$
$\tilde{\lambda}_1$	*0.347524*	1.61×10^{-4}
$\tilde{\lambda}_2$	*1.176747*	1.78×10^{-4}
$\tilde{\lambda}_3$	*2.055970*	7.00×10^{-5}
$\tilde{\lambda}_4$	*3.020643*	4.90×10^{-5}

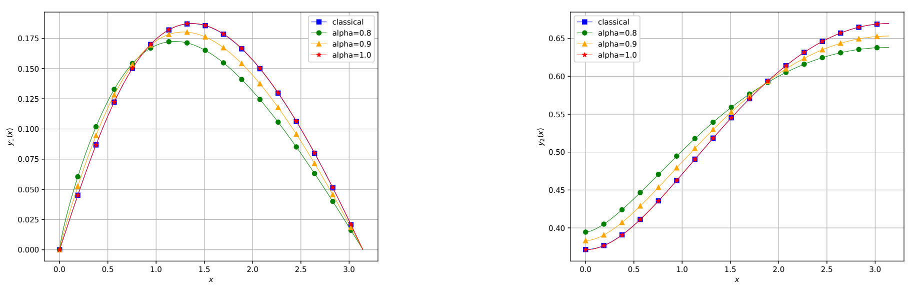

FIGURE 1. First eigenfunctions $f_1 = y_1(x)$ and $f_2 = y_2(x)$ for $p(x) = 1/(1+S_F^\alpha(x))$ and $r(x) = 1/(1+(S_F^\alpha(x))^2)$.

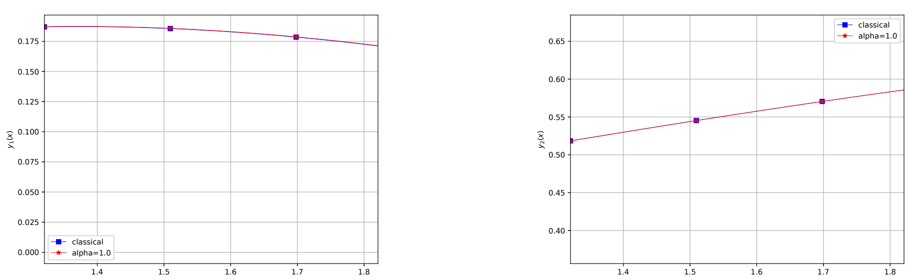

FIGURE 2. Zoomed-in comparison of the first eigenfunctions for $\alpha = 1$ in the first numerical example.

The plots of the first eigenfunctions $f_1 = y_1(x)$ *and* $f_2 = y_2(x)$ *for the classical calculus and fractal calculus methods across the three scaling indices are shown in Figure 3.*

For easier visual verification that the classical and fractal methods nearly agree when $\alpha = 1$, *zoomed-in plots are also provided in Figure 4.*

TABLE 3. Numerically computed eigenvalues $\tilde{\lambda}_n$ for various methods and α.

Method	α	$\tilde{\lambda}_1$
Classical	*N/A*	*1.544759*
Fractal	*0.8*	*1.516625*
Fractal	*0.9*	*1.530339*
Fractal	*1.0*	*1.544186*

TABLE 4. Magnitude of error between classical and fractal methods for $\alpha = 1$.

$\tilde{\lambda}_n$	*Classical*	$\lvert\Delta\tilde{\lambda}_n\rvert$
$\tilde{\lambda}_1$	*1.544759*	5.73×10^{-4}

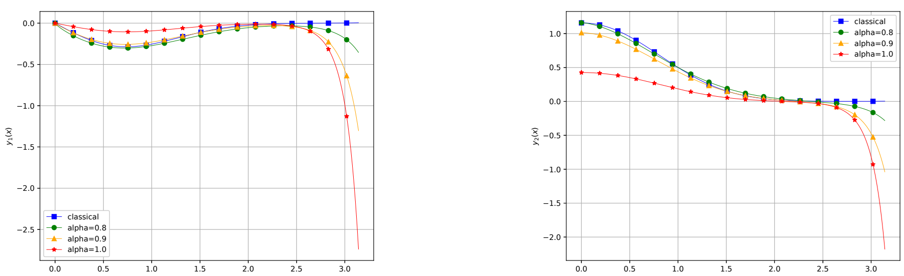

FIGURE 3. First eigenfunctions $f_1 = y_1(x)$ and $f_2 = y_2(x)$ for $p(x) = S_F^{\alpha}(x) + 1$ and $r(x) = (S_F^{\alpha}(x))^2 + 1$.

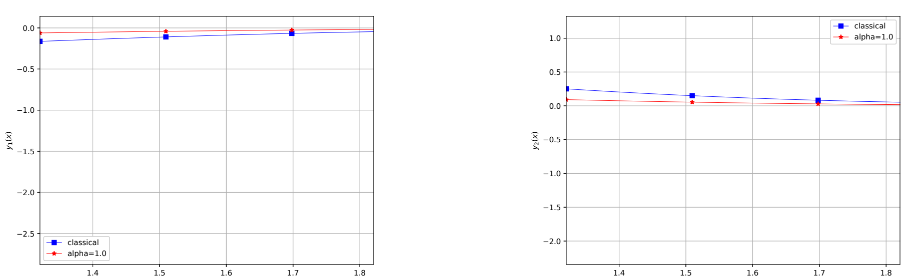

FIGURE 4. Zoomed-in comparison of the first eigenfunctions for $\alpha = 1$ in the second numerical example.

Example 3.3. *Let $p(x) = e^{S_F^{\alpha}(x)}$, $r(x) = e^{-S_F^{\alpha}(x)}$, and $\alpha = [0.8, 0.9, 1.0]$. Tables 5 and 6 show the eigenvalues and errors.*

TABLE 5. Numerically computed eigenvalues $\tilde{\lambda}_n$ for various methods and α.

Method	α	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	$\tilde{\lambda}_3$	$\tilde{\lambda}_4$	$\tilde{\lambda}_5$	$\tilde{\lambda}_6$
Classical	*N/A*	*0.148677*	*0.458639*	*0.865004*	*1.452401*	*2.170184*	*2.965759*
Fractal	*0.8*	*0.210897*	*0.644622*	*1.301299*	*2.201887*	-	-
Fractal	*0.9*	*0.175896*	*0.542309*	*1.057334*	*1.790641*	*2.656072*	-
Fractal	*1.0*	*0.148792*	*0.458986*	*0.865601*	*1.453235*	*2.171232*	*2.966991*

TABLE 6. Magnitude of error between classical and fractal methods for $\alpha = 1$.

$\tilde{\lambda}_n$	*Classical*	$\lvert\Delta\tilde{\lambda}_n\rvert$
$\tilde{\lambda}_1$	*0.148677*	1.15×10^{-4}
$\tilde{\lambda}_2$	*0.458639*	3.47×10^{-4}
$\tilde{\lambda}_3$	*0.865004*	5.97×10^{-4}
$\tilde{\lambda}_4$	*1.452401*	8.34×10^{-4}
$\tilde{\lambda}_5$	*2.170184*	1.05×10^{-3}
$\tilde{\lambda}_6$	*2.965759*	1.23×10^{-3}

The plots of the first eigenfunctions $f_1 = y_1(x)$ and $f_2 = y_2(x)$ for the classical calculus and fractal calculus methods across the three scaling indices are shown in Figure 5.

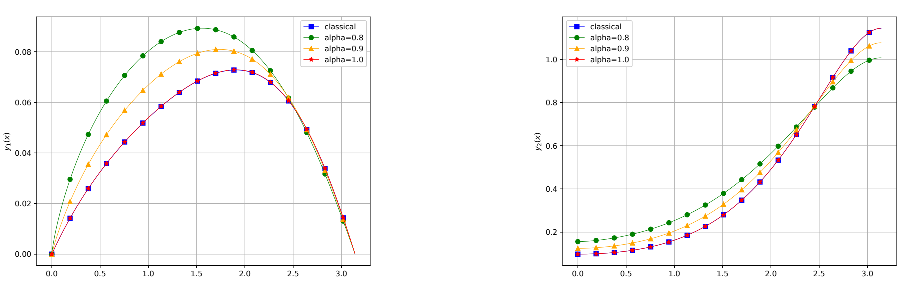

FIGURE 5. First eigenfunctions $f_1 = y_1(x)$ and $f_2 = y_2(x)$ for $p(x) = e^{S_F^{\alpha}(x)}$ and $r(x) = e^{-S_F^{\alpha}(x)}$.

For easier visual verification that the classical and fractal methods agree when $\alpha = 1$, zoomed-in plots are also provided in Figure 6.

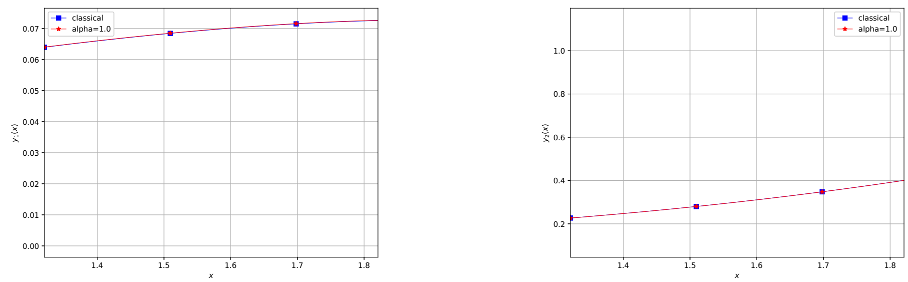

FIGURE 6. Zoomed-in comparison of the first eigenfunctions for $\alpha = 1$ in the third numerical example.

4. CONCLUSION

In this paper, we have extended the results in [3] by introducing and investigating a fractal Dirac problem on a finite interval. We have shown some properties of eigenvalues and eigenfunctions. From the numerical method presented, we see that our theoretical findings are validated. The results are presented for various fractal derivative orders; as the order tends to 1, the eigenvalues approach those of the corresponding integer-order problem. The method is simple to implement and can be adapted for other problems.

Some possible future directions include studying the spectral properties of boundary value problems with fractal delay Sturm–Liouville and Dirac equations [9,10], as well as implementing the coarse-grained mass function without approximation in the numerical method to see how the errors improve versus the increased computational expense.

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