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ESTIMATES OF THE ORDER OF APPROXIMATION OF FUNCTIONS OF SEVERAL VARIABLES IN THE GENERALIZED LORENTZ SPACE

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ABSTRACT. In this paper, we consider an anisotropic symmetric space $X(\bar{\varphi})$ of 2π-periodic functions of m variables, in particular, the generalized Lorentz space $L^{*}_{\bar{\psi},\bar{\tau}}(\mathbb{T}^m)$ and the Nikol'skii–Besov class $S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B$. We prove an embedding theorem for the Nikol'skii–Besov class into the generalized Lorentz space and establish an upper estimate for the best approximations of functions from the class $S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B$ by trigonometric polynomials whose harmonic indices belong to the hyperbolic cross.

1. INTRODUCTION

Let $\mathbb{R}^m$ be the m–dimensional Euclidean space of points $\bar{x} = (x_1, \ldots, x_m)$ with real coordinates, and let $I^m = \{\bar{x} \in \mathbb{R}^m;\ 0 \le x_j < 1;\ j = 1, \ldots, m\} = [0,1)^m$ be the m–dimensional unit cube.

Two non-negative Lebesgue measurable functions f and g are called equimeasurable if

$$\mu\{\bar{x} \in I^m \colon f(\bar{x}) > \lambda\} = \mu\{\bar{x} \in I^m \colon g(\bar{x}) > \lambda\}, \quad \lambda > 0,$$

where μe denotes the Lebesgue measure of a set $e \subset I^m$.

For a non-negative measurable function f, its non-increasing rearrangement is defined by $f^*(t) = \inf\{\lambda > 0 \colon \mu\{\bar{x} \in I^m \colon f(\bar{x}) > \lambda\} < t\}$. It is known that the functions f and f^* are equimeasurable ([13, Ch. 2, Section 2]).

Let X be a Banach space of Lebesgue measurable functions on I^m with the norm $\|f\|_X$. The space X is called symmetric if the following conditions are satisfied:

1) if $|f(\bar{x})| \le |g(\bar{x})|$ almost everywhere on I^m and $g \in X$, then $f \in X$ and $\|f\|_X \le \|g\|_X$,

2) if $f \in X$ and the functions $|f(\bar{x})|$ and $|g(\bar{x})|$ are equimeasurable, then $g \in X$ and $\|f\|_X = \|g\|_X$ (see [13, Ch. 2, Sec. 4.1]).

The norm $\|\chi_e\|_X$ of the characteristic function $\chi_e(t)$ of a measurable set $e \subset I^m$ is called the fundamental function of the space X and is denoted by $\varphi(\mu e) = \|\chi_e\|_X$.

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It is known that the non-increasing rearrangement of the characteristic function χ_e of a measurable set $e \subset [0,1]^m$ coincides with the function $\chi_{[0,t]}$, where $t = \mu e$. Therefore, the fundamental function of a symmetric space X is the function $\varphi(t) = \|\chi_{[0,t]}\|_X$, defined on the interval $[0,1]$. It is a concave, non-decreasing, and continuous function on $[0,1]$, satisfying $\varphi(0) = 0$ (see [13, Ch. 2, Sec. 4.4]). Such functions are called Φ–functions.

For a given function $\varphi(t)$, $t \in [0,1]$, we define

$$\alpha_\varphi = \underline{\lim}_{t\to 0}\frac{\varphi(2t)}{\varphi(t)}, \quad \beta_\varphi = \overline{\lim}_{t\to 0}\frac{\varphi(2t)}{\varphi(t)}.$$

A symmetric space X with the fundamental function φ and norm $\|f\|_X$ is denoted by $X(\varphi)$ and its norm by $\|f\|_{X(\varphi)}$.

It is known that, for any symmetric space $X(\varphi)$, the inequalities $1 \le \alpha_\varphi \le \beta_\varphi \le 2$ hold.

An example of a symmetric space is the Lebesgue space $L_q(\mathbb{T}^m)$ with the norm

$$\|f\|_q = \left(\int\limits_{I^m} |f(2\pi\bar{x})|^q d\bar{x}\right)^{1/q}, \quad 1 \le q < \infty.$$

Here and throughout the paper, $\mathbb{T}^m = [0,2\pi)^m$, and all functions f are assumed to be 2π–periodic in each variable.

Let ψ be a continuous, concave, and non-decreasing function on $[0,1]$ such that $\psi(0) = 0$ and $0 < \tau < \infty$. The generalized Lorentz space $L_{\psi,\tau}(\mathbb{T}^m)$ is defined as the set of measurable functions $f(\bar{x}) = f(x_1,\dots,x_m)$ that are 2π–periodic in each variable and satisfy (see [21])

$$\|f\|^*_{\psi,\tau} = \left(\int\limits_0^1 f^{*^\tau}(t)\psi^\tau(t)\frac{dt}{t}\right)^{1/\tau} < \infty.$$

It is known that under the conditions $1 < \alpha_\psi, \beta_\psi < 2$, the space $L_{\psi,\tau}(\mathbb{T}^m)$ is a symmetric space with the fundamental function ψ.

Note that for $\psi(t) = t^{1/q}$, the space $L_{\psi,\tau}(\mathbb{T}^m)$ coincides with the Lorentz space $L_{q,\tau}(\mathbb{T}^m)$, $1 < q, \tau < \infty$, which consists of all functions f such that (see [23, Ch. 1, Sec. 3])

$$\|f\|_{q,\tau} = \left(\frac{\tau}{q}\int\limits_0^1 \left(\int\limits_0^t f^*(y)dy\right)^\tau t^{\tau(\frac{1}{q}-1)-1}dt\right)^{1/\tau} < \infty.$$

We consider the anisotropic symmetric space $X(\bar{\varphi})$ of 2π–periodic functions of m variables, equipped with the norm

$$\|f\|^*_{X(\bar{\varphi})} = \|\dots\|f^{*_1,\dots,*_m}\|_{X(\varphi_1)}\dots\|_{X(\varphi_m)},$$

where $f^{*_1,\dots,*_m}(t_1,\dots,t_m)$ denotes the non-increasing rearrangement of the function $|f(2\pi\bar{x})|$ with respect to each variable $x_j \in [0,1)$ while the remaining variables are fixed (see [17]), and $X(\varphi_j)$ is a symmetric space in the variable x_j with the fundamental function φ_j (see [10]).

The associate space of the symmetric space $X(\bar{\varphi})$ is defined as the space of all measurable functions g such that (see [10])

$$\sup_{\substack{f\in X(\bar\varphi),\\ \|f\|^*_{X(\bar\varphi)}\le 1}} \int_{I^m} f(2\pi\bar{x})g(2\pi\bar{x})d\bar{x} < \infty,$$

and is denoted by $X^{'}(\bar{\tilde{\varphi}})$; its norm is denoted by $\|g\|^*_{X'(\bar{\tilde{\varphi}})}$, where

$$\bar{\tilde{\varphi}}(t) = (\tilde{\varphi}_1(t),...,\tilde{\varphi}_m(t)),\ \tilde{\varphi}_j(t) = \frac{t}{\varphi_j(t)}$$

for $t \in (0,1]$, with $\tilde{\varphi}_j(0) = 0$, $j = 1,...,m$. It is known that

$$\left|\int_{I^m} f(2\pi\bar{x})g(2\pi\bar{x})d\bar{x}\right| \le \|g\|^*_{X'(\bar{\tilde{\varphi}})}\|f\|^*_{X(\bar\varphi)},\ f\in X(\bar\varphi), g\in X^{'}(\bar{\tilde{\varphi}}).$$

Let $\bar{x} = (x_1,...,x_m) \in \mathbb{I}^m = [0,1]^m$, and let $\psi_j(x_j)$, $x_j \in [0,1]$ be given Φ–functions and $\tau_j \in [1,+\infty)$, $j = 1,...,m$. We denote by $L^*_{\bar\psi,\bar\tau}(\mathbb{T}^m)$ the generalized Lorentz space equipped with the anisotropic norm of Lebesgue measurable functions $f(2\pi\bar{x})$ that are 2π–periodic in each variable and satisfy

$$\|f\|^*_{\bar\psi,\bar\tau} = \left[\int_0^1 \psi_m^{\tau_m}(t_m)\left[\ldots\left[\int_0^1 \psi_1^{\tau_1}(t_1)\left(f^{*_1,...,*_m}(t_1,...,t_m)\right)^{\tau_1}\frac{dt_1}{t_1}\right]^{\frac{\tau_2}{\tau_1}}\right]^{\frac{\tau_m}{\tau_{m-1}}}\frac{dt_m}{t_m}\right]^{\frac{1}{\tau_m}}$$

being finite. For $\psi_j(t) = t^{\frac{1}{q_j}}$, $j = 1,...,m$, the Lorentz space $L^*_{\bar\psi,\bar\tau}(\mathbb{T}^m)$ is denoted by $L^*_{\bar q,\bar\tau}(\mathbb{T}^m)$, and instead of $\|\bullet\|^*_{\bar\psi,\bar\tau}$ we write $\|\bullet\|^*_{\bar q,\bar\tau}$ (see [17]).

Let $\mathring{L}^*_{\bar\psi,\bar\tau}(\mathbb{T}^m)$ denote the set of functions $f \in L^*_{\bar\psi,\bar\tau}(\mathbb{T}^m)$ such that

$$\int_0^{2\pi} f(\bar{x})\,dx_j = 0,\ \ \forall j = 1,...,m.$$

We use the following notation. Let $a_{\bar n}(f)$ be the Fourier coefficients of a function $f \in L_1(\mathbb{T}^m)$ with respect to the multiple trigonometric system, and let

$$\delta_{\bar s}(f,\bar{x}) = \sum_{\bar n\in\rho(\bar s)} a_{\bar n}(f)\,e^{i\langle\bar n,\bar x\rangle},$$

where $\langle\bar y,\bar x\rangle = \sum\limits_{j=1}^m y_jx_j$, and

$$\rho(\bar s) = \left\{\bar k = (k_1,...,k_m)\in\mathbb{Z}^m : [2^{s_j-1}] \le |k_j| < 2^{s_j},\ s_j\in\mathbb{Z}_+,\ j = 1,...,m\right\}$$

where $[a]$ is the integer part of a, and $\mathbb{Z}_+ = \{0,1,2,\ldots\}$.

We consider the Nikol'skii–Besov class

$$S^{\bar r}_{X(\bar\varphi),\bar\theta}B = \left\{f\in\mathring{X}(\bar\varphi):\quad \|f\|^*_{X(\bar\varphi)} + \left\|\left\{\prod_{j=1}^m 2^{s_jr_j}\|\delta_{\bar s}(f)\|^*_{X(\bar\varphi)}\right\}_{\bar s\in\mathbb{Z}^m_+}\right\|_{l_{\bar\theta}} \le 1\right\},$$

where $\bar{\theta}=(\theta_1,...,\theta_m)$, $\bar{r}=(r_1,...,r_m)$, $1\le\theta_j\le+\infty$, $0<r_j<+\infty$, $j=1,...,m$.

In the case $X(\bar{\varphi})=L_p(\mathbb{T}^m)$, $1<p<\infty$, the class $S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B$ was introduced and studied in [6], [15], and [16].

For a fixed vector $\bar{\gamma}=(\gamma_1,\dots,\gamma_m)$, $\gamma_j>0$, $j=1,\dots,m$, define

$$Q_n^{\bar{\gamma}}=\cup_{\langle\bar{s},\bar{\gamma}\rangle<n}\rho(\bar{s}),\quad T(Q_n^{\bar{\gamma}})=\Big\{t(\bar{x})=\sum_{\bar{k}\in Q_n^{\bar{\gamma}}}b_{\bar{k}}e^{i\langle\bar{k},\bar{x}\rangle}\Big\}.$$

We use the following notation. Let $E_n^{(\bar{\gamma})}(f)_{X(\bar{\varphi})}$ denote the best approximation of a function $f\in X(\bar{\varphi})$ by trigonometric polynomials from $T(Q_n^{\bar{\gamma}})$, and let $S_n^{\bar{\gamma}}(f,\bar{x})=\sum_{\bar{k}\in Q_n^{\bar{\gamma}}}a_{\bar{k}}(f)\cdot e^{i\langle\bar{k},\bar{x}\rangle}$ denote the corresponding partial Fourier sum of f.

We denote by $C(p,q,y,\dots)$ positive constants that depend only on the parameters indicated in the parentheses; these constants are not necessarily the same in different formulas. The notation $A(y)\asymp B(y)$ means that there exist positive constants C_1,C_2 such that $C_1\cdot A(y)\le B(y)\le C_2\cdot A(y)$.

Exact order estimates for the best approximations of functions from various classes in the Lebesgue space $L_p(\mathbb{T}^m)$ are well known (see the survey articles [11], [20], [25], the monograph [24], and the references therein). Approximation problems in the spaces $L^*_{\bar{q},\bar{\tau}}(\mathbb{T}^m)$ were studied in [1], [2], and [7]– [9].

The main goal of the present paper is to determine the order of the quantity

$$E_n^{(\bar{\gamma})}(S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B)_{\bar{\psi},\bar{\tau}}:=\sup_{f\in S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B}E_n^{(\bar{\gamma})}(f)_{\bar{\psi},\bar{\tau}}.$$

The paper is organized as follows. In Section 2, we present several auxiliary results. In Section 3, we formulate and prove the main results. Theorem 3.2 establishes a sufficient condition for a function $f\in X(\bar{\varphi})$ to belong to the generalized Lorentz space $L^*_{\bar{\psi},\bar{\tau}}(\mathbb{T}^m)$. Theorem 3.4 serves as a converse to Theorem 3.2. Theorem 3.3 provides a condition for the embedding of the class $S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B$ into the space $L^*_{\bar{\psi},\bar{\tau}}(\mathbb{T}^m)$. Finally, Theorem 3.5 establishes estimates for the best approximations of functions $f\in S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B$ by trigonometric polynomials whose harmonic indices belong to a step hyperbolic cross.

2. Auxiliary results

In this section, we present some well-known results and prove a number of auxiliary lemmas. The following generalization of the discrete Hardy inequality is well known:

Lemma 2.1. *(see [12]). Let $0<\theta<+\infty$, and let a_k and $b_k, k=0,1,2,...$ be positive numbers.*

a) If $\sum\limits_{k=0}^{n}a_k\le C\cdot a_n$, then $\sum\limits_{n=0}^{\infty}a_n\Big(\sum\limits_{k=n}^{\infty}b_k\Big)^{\theta}\le C\sum\limits_{n=0}^{\infty}a_nb_n^{\theta}$.

b) If $\sum\limits_{k=n}^{\infty} a_k \le Ca_n,$ *then* $\sum\limits_{n=0}^{\infty} a_n \Big(\sum\limits_{k=0}^{n} b_k\Big)^{\theta} \le C \cdot \sum\limits_{n=0}^{\infty} a_n \cdot b_n^{\theta}.$

The proof of Lemma 2.1 can be found in [3].

Lemma 2.2. *(see [14], [22]) If* $1 < \alpha_\psi, \beta_\psi < 2$ *for the* Φ*–function* $\psi(x)$*,* $x \in [0,1]$*, then for any* $q > 0$ *the following relations hold*

$$\int\limits_0^x \frac{\psi^q(t)}{t} dt = O(\psi^q(x)), x \to +0,$$

$$\int\limits_x^1 [t\psi^q(t)]^{-1} dt = O(\psi^{-q}(x)), x \to +0.$$

Lemma 2.3. *(see [14], [22]) If* Φ*–functions* $\varphi(x)$ *and* $\psi(x)$*,* $x \in [0,1]$ *satisfy the conditions* $\alpha_\varphi > \beta_\psi > 1$*, then for*

$$g(x) = \begin{cases} \frac{\varphi(x)}{\psi(x)}, & \textit{if } x \in (0,1] \\ 0, & \textit{if } x = 0 \end{cases}$$

there exists a Φ*–function* $g_1(x)$ *such that* $g(x) \asymp g_1(x), x \in [0,1]$*, and* $\alpha_{g_1} > 1$*.*

Lemma 2.4. *If* Φ*–functions* $\varphi(x)$ *and* $\psi(x)$*,* $x \in [0,1]$ *satisfy the conditions* $1 < \alpha_\psi \le \beta_\psi < \alpha_\varphi \le \beta_\varphi < 2$*, and* $0 < \theta < \infty$*, then the following inequality holds*

$$\sum_{s=0}^{n} \Big(\frac{\psi(2^{-s})}{\varphi(2^{-s})}\Big)^{\theta} \le C\Big(\frac{\psi(2^{-n})}{\varphi(2^{-n})}\Big)^{\theta}, \ n \in \mathbb{N}.$$

Proof. We consider the function

$$g(x) = \begin{cases} \frac{\varphi(x)}{\psi(x)}, & \text{if } x \in (0,1] \\ 0, & \text{if } x = 0. \end{cases}$$

Since $1 < \alpha_\psi \le \beta_\psi < \alpha_\varphi$, it follows from Lemma 2.3 that there exists a Φ–function g_1 such that $g(x) \asymp g_1(x)$ as $x \to +0$, and $\alpha_{g_1} > 1$. Therefore,

$$\sum_{s=0}^{n} \Big(\frac{\psi(2^{-s})}{\varphi(2^{-s})}\Big)^{\theta} \le C \sum_{s=0}^{n} g_1^{-\theta}(2^{-s}). \tag{2.1}$$

Since $g_1 \uparrow$ on $(0,1]$, it follows that

$$\sum_{s=0}^{n} g_1^{-\theta}(2^{-s}) \le \ln 2 \sum_{s=0}^{n} \int_{2^{-s-1}}^{2^{-s}} g_1^{-\theta}(t)\frac{dt}{t} = \ln 2 \int_{2^{-n-1}}^{1} g_1^{-\theta}(t)\frac{dt}{t}. \tag{2.2}$$

Combining inequalities (2.1) and (2.2) with Lemma 2.2, we obtain the the desired conclusion. $\square$

Lemma 2.5. *Let a Φ–function ψ satisfy the conditions $1 < \alpha_\psi \le \beta_\psi < 2$, then the following inequality holds*

$$\sum_{s=n}^{\infty} \psi^{\theta}(2^{-s}) \le C\psi^{\theta}(2^{-n}), n \in \mathbb{N}.$$

Proof. Since $\frac{\psi(t)}{t} \downarrow$ on $(0,1]$, we have

$$\psi^{\theta}(2^{-s}) \le C \int_{2^{-s-1}}^{2^{-s}} \psi^{\theta}(t) \frac{dt}{t}.$$

Therefore,

$$\sum_{s=n}^{\infty} \psi^{\theta}(2^{-s}) \le C \sum_{s=n}^{\infty} \int_{2^{-s-1}}^{2^{-s}} \psi^{\theta}(t) \frac{dt}{t} = C \int_{0}^{2^{-n}} \psi^{\theta}(t) \frac{dt}{t}.$$

Thus, by Lemma 2.2, we obtain the assertion of the lemma. □

Remark 2.1. In Lemma 2.4 and Lemma 2.5 the coefficients C do not depend on n. The integral forms of these lemmas can be found in [14].

We now establish a multidimensional version of Lemma 2.1.

Lemma 2.6. *Let $b_{\bar{k}} = b_{k_1,\ldots,k_m}$ be positive numbers for $\bar{k} = (k_1, \ldots, k_m) \in \mathbb{Z}_+^m$, and let a_{k_j}, $k_j = 0,1,2,\ldots$, and $1 \le \theta_j < +\infty$, $j = 1, \ldots, m$.*

a) If

$$\sum_{k_j=0}^{n_j} a_{k_j} \le C a_{n_j},\ j = 1, \ldots, m,$$

then

$$A_{\theta_1,\ldots,\theta_m} = \left\{ \sum_{n_m=0}^{\infty} a_{n_m} \left[\sum_{n_{m-1}=0}^{\infty} a_{n_{m-1}} \right.\right.$$
$$\left.\left. \ldots \left[\sum_{n_1=0}^{\infty} a_{n_1} \left(\sum_{k_m=n_m}^{\infty} \ldots \sum_{k_1=n_1}^{\infty} b_{\bar{k}} \right)^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \ldots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}$$
$$\le C \left\{ \sum_{n_m=0}^{\infty} a_{n_m} \left[\sum_{n_{m-1}=0}^{\infty} a_{n_{m-1}} \ldots \left[\sum_{n_1=0}^{\infty} a_{n_1} b_{\bar{n}}^{\theta_1} \right]^{\frac{\theta_2}{\theta_1}} \ldots \right]^{\frac{\theta_m}{\theta_{m-1}}} \right\}^{\frac{1}{\theta_m}}.$$

b) If

$$\sum_{k_j=n_j}^{\infty} a_{k_j} \le C a_{n_j},\ j = 1, \ldots, m,$$

then

$$\left\{\sum_{n_m=0}^{\infty} a_{n_m}\left[\sum_{n_{m-1}=0}^{\infty} a_{n_{m-1}}\dots\left[\sum_{n_1=0}^{\infty} a_{n_1}\left(\sum_{k_m=0}^{n_m}\dots\sum_{k_1=0}^{n_1} b_{\bar k}\right)^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}}\dots\right]^{\frac{\theta_m}{\theta_{m-1}}}\right\}^{\frac{1}{\theta_m}}$$
$$\le C\left\{\sum_{n_m=0}^{\infty} a_{n_m}\left[\sum_{n_{m-1}=0}^{\infty} a_{n_{m-1}}\dots\left[\sum_{n_1=0}^{\infty} a_{n_1} b_{\bar n}^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}}\dots\right]^{\frac{\theta_m}{\theta_{m-1}}}\right\}^{\frac{1}{\theta_m}}.$$

Proof. Let us prove a). For $m=1$, the result is known (see Lemma 2.1). Let $m=2$. Since $1\le\theta_1<\infty$, it follows from the properties of the norm that

$$\sum_{n_2=0}^{\infty} a_{n_2}\left[\sum_{n_1=0}^{\infty} a_{n_1}\left(\sum_{k_2=n_2}^{\infty}\sum_{k_1=n_1}^{\infty} b_{\bar k}\right)^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}} \le \sum_{n_2=0}^{\infty} a_{n_2}\left[\sum_{k_2=n_2}^{\infty}\left(\sum_{n_1=0}^{\infty} a_{n_1}\left(\sum_{k_1=n_1}^{\infty} b_{\bar k}\right)^{\theta_1}\right)^{\frac{1}{\theta_1}}\right]^{\theta_2}.$$

Now, applying a) of Lemma 2.1 twice, we obtain

$$\sum_{n_2=0}^{\infty} a_{n_2}\left[\sum_{n_1=0}^{\infty} a_{n_1}\left(\sum_{k_2=n_2}^{\infty}\sum_{k_1=n_1}^{\infty} b_{\bar k}\right)^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}} \le \sum_{n_2=0}^{\infty} a_{n_2}\left[\sum_{n_1=0}^{\infty} a_{n_1} b_{\bar n}^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}}.$$

Suppose that the statement is true for $m-1$. Let us prove it for m. Since θ_j, $j=1,\dots,m-1$, it follows from the properties of the norm and the assumption that

$$A_{\theta_1,\dots,\theta_m}^{\theta_m} \le \sum_{n_m=0}^{\infty} a_{n_m}\left[\sum_{k_m=n_m}^{\infty}\left[\sum_{n_{m-1}=0}^{\infty} a_{n_{m-1}}\right.\right.$$
$$\left.\left.\dots\left[\sum_{n_1=0}^{\infty} a_{n_1}\left(\sum_{k_{m-1}=n_{m-1}}^{\infty}\dots\sum_{k_1=n_1}^{\infty} b_{\bar k}\right)^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}}\dots\right]^{\frac{1}{\theta_{m-1}}}\right]^{\theta_m}$$
$$\le C\sum_{n_m=0}^{\infty} a_{n_m}\left[\sum_{k_m=n_m}^{\infty}\left[\sum_{n_{m-1}=0}^{\infty} a_{n_{m-1}}\dots\left[\sum_{n_1=0}^{\infty} a_{n_1} b_{n_1,\dots,n_{m-1},k_m}^{\theta_1}\right]^{\frac{\theta_2}{\theta_1}}\dots\right]^{\frac{1}{\theta_{m-1}}}\right]^{\theta_m}.$$

Now, by applying a) of Lemma 2.1, from this we obtain the required inequality. b) can be proved similarly. □

Lemma 2.7. *Let $1<\alpha_{\varphi_j}\le\beta_{\varphi_j}\le 2$, $j=1,...,m$, and $E_j\subset[0,2\pi)$, $j=1,...,m$, be measurable sets. Then for any trigonometric polynomial*

$$T_{\bar n}(\bar x)=\sum_{k_1=-n_1}^{n_1}\dots\sum_{k_m=-n_m}^{n_m} b_{\bar k} e^{i\langle\bar x,\bar k\rangle}$$

the following inequality holds

$$\int_{E_m}\dots\int_{E_1}|T_{\bar n}(\bar x)|dx_1\dots dx_m \le C\prod_{j\in e}\frac{1}{\varphi_j(n_j^{-1})}\prod_{j\in e}|E_j|\prod_{j\in\bar e}\tilde\varphi_j(|E_j|)\|T_{\bar n}\|^*_{X(\bar\varphi)},$$

where $e \subset \{1,...,m\}$ and $\bar{e}$ is the complement set of e, $|E_j|$ is the Lebesgue measure of E_j.

Proof. Let e be an arbitrary subset of $\{1,...,m\}$. Let $E_e = \prod_{j\in e} E_j$, $d^e\bar{x} = \prod_{j\in e} dx_j$. It is known that for any trigonometric polynomial $T_{\bar{n}}(\bar{x})$ with fixed x_j, $j \in \bar{e}$, the following formula holds

$$T_{\bar{n}}(\bar{x}) = \frac{1}{(2\pi)^{|e|}} \int_{I^{|e|}} T_{\bar{n}}(\bar{y}^e, \bar{x}^{\bar{e}}) \cdot \prod_{j\in e} D_{n_j}(x_j - y_j) d^e\bar{y},$$

where $D_n(t)$ is the Dirichlet kernel of the trigonometric system, $\bar{y}^e$ is the vector with coordinates y_j if $j \in e$, and $\bar{x}^{\bar{e}}$ is the vector with coordinates x_j for $j \in \bar{e}$ ($\bar{e}$ is the set–theoretic complement of e), and $|e|$ is the number of elements of e. By the property of the Lebesgue integral, we have

$$\begin{aligned}
\int_{E_{\bar{e}}} |T_{\bar{n}}(\bar{x})| d^{\bar{e}}\bar{x} &= \frac{1}{(2\pi)^{|e|}} \int_{E_{\bar{e}}} \Big| \int_{I^{|e|}} T_{\bar{n}}(\bar{y}^e, \bar{x}^{\bar{e}}) \cdot \prod_{j\in e} D_{n_j}(x_j - y_j) d^e\bar{y} \Big| d^{\bar{e}}\bar{x} \\
&\le \frac{1}{(2\pi)^{|e|}} \int_{E_{\bar{e}}} \int_{I^{|e|}} |T_{\bar{n}}(\bar{y}^e, \bar{x}^{\bar{e}})| \cdot \prod_{j\in e} |D_{n_j}(x_j - y_j)| d^e\bar{y} d^{\bar{e}}\bar{x} \\
&= \frac{1}{(2\pi)^{|e|}} \int_{I^m} |T_{\bar{n}}(\bar{y}^e, \bar{x}^{\bar{e}})| \cdot \prod_{j\in e} |D_{n_j}(x_j - y_j)| \cdot \prod_{j\in \bar{e}} \chi_{E_j}(x_j) d^e\bar{y} d^{\bar{e}}\bar{x},
\end{aligned}$$

where χ_E is the characteristic function of E. By appling Hölder's inequality to the integrals on the right-hand side of the last relation, we obtain

$$\begin{aligned}
\int_{E_{\bar{e}}} |T_{\bar{n}}(\bar{x})| d^{\bar{e}}\bar{x} &\le \frac{1}{(2\pi)^{|e|}} \|T_{\bar{n}}\|^*_{X(\bar{\varphi})} \Big\| \prod_{j\in e} |D_{n_j}(x_j - \bullet)| \prod_{j\in \bar{e}} \chi_{E_j} \Big\|^*_{X(\tilde{\bar{\varphi}})} \\
&= \frac{1}{(2\pi)^{|e|}} \|T_{\bar{n}}\|^*_{X(\bar{\varphi})} \prod_{j\in e} \|D_{n_j}(x_j - \bullet)\|^*_{X(\tilde{\varphi}_j)} \prod_{j\in \bar{e}} \|\chi_{E_j}\|^*_{X(\tilde{\varphi}_j)}. \qquad (2.3)
\end{aligned}$$

Since

$$\|\chi_{E_j}\|^*_{X(\tilde{\varphi}_j)} = \tilde{\varphi}_j(|E_j|),$$

inequality (2.3) implies

$$\int_{E_{\bar{e}}} |T_{\bar{n}}(\bar{x})| d^{\bar{e}}\bar{x} \le \frac{1}{(2\pi)^{|e|}} \|T_{\bar{n}}\|^*_{X(\bar{\varphi})} \prod_{j\in \bar{e}} \tilde{\varphi}_j(|E_j|) \prod_{j\in e} \sup_{x_j\in[0,2\pi]} \|D_{n_j}(x_j - \bullet)\|^*_{\tilde{\varphi}_j}, \qquad (2.4)$$

where $d^{\bar{e}}\bar{x} = \prod\limits_{j\in e} dx_j$.

In the one-dimensional case, the following estimate is known (see, for example, [18])

$$\sup_{x_j\in[0,2\pi)} \|D_{n_j}(x_j - \bullet)\|^*_{X(\tilde{\varphi}_j)} \le C n_j \tilde{\varphi}_j(n_j^{-1}),\ j = 1,\dots m. \qquad (2.5)$$

Inequalities (2.4) and (2.5) imply that

$$\int_{E_{\bar{e}}} |T_{\bar{n}}(\bar{x})| d^{\bar{e}}\bar{x} \le C\|T_{\bar{n}}\|^*_{X(\bar{\varphi})} \prod_{j\in\bar{e}} \tilde{\varphi}_j(|E_j|) \prod_{j\in e} n_j \tilde{\varphi}_j(n_j^{-1})$$

for fixed $x_j, j \in e$. By integrating with respect to the variables $x_j, j \in e$, on both sides of this inequality, we obtain

$$\int_{E_{\bar{e}}} \int_{E_e} |T_{\bar{n}}(\bar{x})| d^e\bar{x} d^{\bar{e}}\bar{x} \le C\|T_{\bar{n}}\|^*_{X(\bar{\varphi})} \prod_{j\in\bar{e}} \tilde{\varphi}_j(|E_j|) \prod_{j\in e} n_j \tilde{\varphi}_j(n_j^{-1}) \prod_{j\in e} |E_j|.$$

□

3. Main results

We set

$$G_e(\bar{n}) = \{\bar{s} = (s_1,...,s_m) \in \mathbb{N}^m : s_j \le n_j, j \in e; \quad s_j > n_j, j \notin e\},$$

where $e \subset \{1,...,m\}$;

$$U_{\bar{n}}(f,\bar{x}) = \sum_{e\subset\{1,...,m\}} \sum_{\bar{s}\in G_e(\bar{n})} \delta_{\bar{s}}(f,\bar{x}).$$

Let

$$\bar{f}(\bar{t}) = \sup_{|E_m|\ge t_m} \frac{1}{|E_m|} \int_{E_m} ... \sup_{|E_1|\ge t_1} \frac{1}{|E_1|} \int_{E_1} |f(x_1,...,x_m)| dx_1...dx_m,$$

$|E_j|$ is the Lebesgue measure of $E_j \subset [0,2\pi)$.

Theorem 3.1. *Let $\bar{\varphi} = (\varphi_1,...,\varphi_m)$, and let φ_j satisfy the conditions $1 < \alpha_{\varphi_j} \le \beta_{\varphi_j} < 2$, $j = 1,\dots,m$. Then, for each function $f \in X(\bar{\varphi})$, the following inequality holds*

$$\bar{f}(\bar{t}) \le C \left\{ \prod_{j=1}^m \frac{1}{\varphi_j(t_j)} \sum_{s_m=n_m+1}^{\infty} ... \sum_{s_1=n_1+1}^{\infty} \|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})} + \right.$$
$$\left. + \sum_{e\subset\{1,...,m\}} \prod_{j\notin e} \frac{1}{\varphi_j(t_j)} \sum_{\bar{s}\in G_e(\bar{n})} \prod_{j\in e} \frac{1}{\varphi_j(2^{-n_j})} \|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})} \right\},$$

where $\bar{t} = (t_1,...,t_m) \in (2^{-n_1-1}, 2^{-n_1}] \times ... \times (2^{-n_m-1}, 2^{-n_m}]$, $n_j = 1,2,...$; $j = 1,...,m$.

Proof. Let $E_j \subset [0,2\pi)$ be a Lebesgue measurable subset. Then, by the properties of the integral, we get

$$\int_{E_m} ... \int_{E_1} |f(x_1,...,x_m)| dx_1...dx_m \le \int_{E_m} ... \int_{E_1} |f(\bar{x}) - U_{\bar{n}}(f,\bar{x})| d\bar{x} +$$
$$+ \int_{E_m} ... \int_{E_1} |U_{\bar{n}}(f,\bar{x})| d\bar{x}. \quad (3.1)$$

Using Hölder's integral inequality, we obtain

$$\int_{E_m} \cdots \int_{E_1} |f(\bar{x}) - U_{\bar{n}}(f,\bar{x})| d\bar{x} \leq C \prod_{j=1}^{m} \frac{|E_j|}{\varphi_j(|E_j|)} \|f - U_{\bar{n}}(f)\|^*_{X(\bar{\varphi})}. \tag{3.2}$$

Let $\forall e \subset \{1,...,m\}$. Then, by applying Lemma 2.7 we obtain

$$\int_{E_m} \cdots \int_{E_1} \left| \sum_{\bar{s} \in G_e(\bar{n})} \delta_{\bar{s}}(f,\bar{x}) \right| d\bar{x} \leq$$

$$C \prod_{j \in e} |E_j| \prod_{j \notin e} \frac{|E_j|}{\varphi_j(|E_j|)} \sum_{\bar{s} \in G_e(\bar{n})} \prod_{j \in e} \frac{1}{\varphi_j(2^{-s_j})} \|\delta_{\bar{s}}\|^*_{X(\bar{\varphi})}. \tag{3.3}$$

Further, taking into account that $|E_j| \geq t_j$ and the properties of the function φ_j from inequalities (3.1)–(3.3), we obtain

$$\prod_{j=1}^{m} |E_j|^{-1} \int_{E_m} \cdots \int_{E_1} |f(\bar{x}) - U_{\bar{n}}(f,\bar{x})| d\bar{x}$$

$$\leq C \Bigg\{ \prod_{j=1}^{m} \frac{1}{\varphi_j(t_j)} \sum_{s_m = n_m+1}^{\infty} \cdots \sum_{s_1 = n_1+1}^{\infty} \|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})} +$$

$$+ \sum_{e \subset \{1,...,m\}} \prod_{j \notin e} \frac{1}{\varphi_j(t_j)} \sum_{\bar{s} \in G_e(\bar{n})} \prod_{j \in e} \frac{1}{\varphi_j(2^{-s_j})} \|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})} \Bigg\}.$$

This implies the assertion of the theorem. □

Theorem 3.2. *Let* $1 < \alpha_{\psi_j} \leq \beta_{\psi_j} < \alpha_{\varphi_j} \leq \beta_{\varphi_j} < 2$, $1 \leq \tau_j < +\infty$, $j = 1,...,m$. *If* $f \in X(\bar{\varphi})$, *and*

$$\left\{ \prod_{j=1}^{m} \frac{\psi_j(2^{-s_j})}{\varphi_j(2^{-s_j})} \|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})} \right\}_{\bar{s} \in \mathbb{Z}^m_+} \in l_{\bar{\tau}},$$

then $f \in L^*_{\bar{\psi},\bar{\tau}}(\mathbb{T}^m)$ *and the following inequality holds*

$$\|f\|^*_{\bar{\psi},\bar{\tau}} \leq C \left\| \left\{ \prod_{j=1}^{m} \frac{\psi_j(2^{-s_j})}{\varphi_j(2^{-s_j})} \|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})} \right\}_{\bar{s} \in \mathbb{Z}^m_+} \right\|_{l_{\bar{\tau}}}.$$

Proof. According to Lemma 2 in [10], the following inequality holds

$$f^{*_1,...,*_m}(t_1,...,t_m) \leq \bar{f}(t_1,...,t_m) := \sup_{|E_m| \geq t_m} \int_{E_m} dx_m ... \sup_{|E_1| \geq t_1} \int_{E_1} |f(x_1,...,x_m)| dx_1.$$

Therefore, $\|f\|^*_{\bar{\psi},\bar{\tau}} \leq C \|\bar{f}\|^*_{\bar{\psi},\bar{\tau}}$, $1 \leq \tau_j < +\infty, j = 1,...,m$. Taking into account the relation

$$\int_{2^{-n-1}}^{2^{-n}} \psi_j(t) \frac{dt}{t} \asymp \psi_j(2^{-n}) \tag{3.4}$$

and using Theorem 3.1, we have

$$\|f\|^*_{\bar\psi,\bar\tau} \le C\Bigg[\Bigg\{\sum_{n_m=0}^{\infty}\left(\frac{\psi_m(2^{-n_m})}{\varphi_m(2^{-n_m})}\right)^{\tau_m}\Bigg[\dots$$

$$\Bigg[\sum_{n_1=0}^{\infty}\left(\frac{\psi_1(2^{-n_1})}{\varphi_1(2^{-n_1})}\right)^{\tau_1}\left(\sum_{s_m=n_m+1}^{\infty}\dots\sum_{s_1=n_1+1}^{\infty}\|\delta_{\bar s}(f)\|^*_{X(\bar\varphi)}\right)^{\tau_1}\Bigg]^{\frac{\tau_2}{\tau_1}}\dots\Bigg]^{\frac{\tau_m}{\tau_{m-1}}}\Bigg\}^{\frac{1}{\tau_m}}$$

$$+\sum_{e\subset\{1,\dots,m\}}\Bigg\{\sum_{n_m=0}^{\infty}\int_{2^{-n_m-1}}^{2^{-n_m}}\psi_m^{\tau_m}(t_m)t_m^{-1}\Bigg[\dots\Bigg[\sum_{n_1=0}^{\infty}\int_{2^{-n_1-1}}^{2^{-n_1}}\psi_1^{\tau_1}(t_1)t_1^{-1}$$

$$\times\left(\prod_{j\notin e}\frac{1}{\varphi_j(t_j)}\sum_{\bar s\in G_e(\bar n)}\prod_{j\in e}\frac{1}{\varphi_j(2^{-s_j})}\|\delta_{\bar s}(f)\|^*_{X(\bar\varphi)}\right)^{\tau_1}dt_1\Bigg]^{\frac{\tau_2}{\tau_1}}\dots$$

$$\dots\Bigg]^{\frac{\tau_m}{\tau_{m-1}}}dt_m\Bigg\}^{\frac{1}{\tau_m}}\Bigg] = C\Bigg[J_1+\sum_{e\subset\{1,\dots,m\}}J_e\Bigg]. \quad (3.5)$$

According to Lemma 2.4 and Lemma 2.5, the numbers

$$a_{s_j} = \frac{\psi(2^{-s_j})}{\varphi(2^{-s_j})}$$

and $a_{s_j} = \psi(2^{-s_j})$, $j = 1,\dots,m$ satisfy the conditions $a)$ and $b)$ of Lemma 2.6, respectively.

Let $e = \{1,\dots,i\}, i \le m$. Then, using relation (3.5) and successively applying the triangle inequality and assertions $a)$ and $b)$ of Lemma 2.6, we have

$$J_e \le \Bigg\{\sum_{n_m=0}^{\infty}\left(\frac{\psi_m(2^{-n_m})}{\varphi_m(2^{-n_m})}\right)^{\tau_m}\Bigg[\dots$$

$$\dots\Bigg[\sum_{n_{i+1}=0}^{\infty}\left(\frac{\psi_{i+1}(2^{-n_{i+1}})}{\varphi_{i+1}(2^{-n_{i+1}})}\right)^{\tau_{i+1}}\Bigg[\sum_{n_i=0}^{\infty}\psi_i^{\tau_i}(2^{-n_i})\Bigg[\dots\Bigg[\sum_{n_1=0}^{\infty}\psi_1^{\tau_1}(2^{-n_1})$$

$$\times\Bigg(\sum_{s_m=n_m+1}^{\infty}\dots\sum_{s_{i+1}=n_{i+1}+1}^{\infty}\sum_{s_i=1}^{n_i}\dots\sum_{s_1=1}^{n_1}\prod_{j=1}^{i}\frac{1}{\varphi_j(2^{-s_j})}$$

$$\times\|\delta_{\bar s}(f)\|^*_{X(\bar\varphi)}\Bigg)^{\tau_1}\Bigg]^{\frac{\tau_2}{\tau_1}}\dots\Bigg]^{\frac{\tau_i}{\tau_{i-1}}}\Bigg]^{\frac{\tau_{i+1}}{\tau_i}}\dots\Bigg]^{\frac{\tau_m}{\tau_{m-1}}}\Bigg\}^{\frac{1}{\tau_m}}$$

$$\leq C\left\{\sum_{n_m=0}^{\infty}\left(\frac{\psi_m(2^{-n_m})}{\varphi_m(2^{-n_m})}\right)^{\tau_m}\left[\sum_{n_{m-1}=0}^{\infty}\left(\frac{\psi_{m-1}(2^{-n_{m-1}})}{\varphi_{m-1}(2^{-n_{m-1}})}\right)^{\tau_{m-1}}\dots\right.\right.$$

$$\left.\left.\dots\left[\sum_{n_1=0}^{\infty}\left(\frac{\psi_1(2^{-n_1})}{\varphi_1(2^{-n_1})}\right)^{\tau_1}\left(\|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})}\right)^{\tau_1}\right]^{\frac{\tau_2}{\tau_1}}\dots\right]^{\frac{\tau_m}{\tau_{m-1}}}\right\}^{\frac{1}{\tau_m}}. \quad (3.6)$$

Let $e \subset \{1,2,\dots,m\}$, $e \neq \emptyset$, and $j_0 = \min e$, $k_0 = \max e$. If $\{j_0, j_0+1,\dots,k_0\} \cap \bar{e} = \emptyset$, where $\bar{e}$ is the complement of e, then

$$\sum_{\bar{s}\in G_e(\bar{n})}\prod_{j\in e}\frac{1}{\varphi_j(2^{-s_j})}\|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})} =$$

$$\sum_{s_m=n_m+1}^{\infty}\dots\sum_{s_{k_0+1}=n_{k_0+1}+1}^{\infty}\sum_{s_{k_0}=1}^{n_{k_0}}\dots\sum_{s_{j_0}=1}^{n_{j_0}}\sum_{s_{j_0-1}=n_{j_0-1}+1}^{\infty}\dots\sum_{s_1=n_1+1}^{\infty}\prod_{j\in e}\frac{1}{\varphi_j(2^{-s_j})}\|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})}.$$

If $\{j_0, j_0+1,\dots,k_0\} \cap \bar{e} = \{l_0,\dots,l_1\}$, then

$$\sum_{\bar{s}\in G_e(\bar{n})}\prod_{j\in e}\frac{1}{\varphi_j(2^{-s_j})}\|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})} =$$

$$\sum_{s_m=n_m+1}^{\infty}\dots\sum_{s_{k_0+1}=n_{k_0+1}+1}^{\infty}\sum_{s_{k_0}=1}^{n_{k_0}}\dots\sum_{s_{l_1-1}=0}^{n_{l_1-1}}\sum_{s_{l_1}=n_{l_1}+1}^{\infty}\dots\sum_{s_{l_0}=n_{l_0}+1}^{\infty}\sum_{s_{l_0+1}=1}^{n_{l_0+1}}\dots\sum_{s_{j_0}=1}^{n_{j_0}}$$

$$\sum_{s_{j_0-1}=n_{j_0-1}+1}^{\infty}\dots\sum_{s_1=n_1+1}^{\infty}\prod_{j\in e}\frac{1}{\varphi_j(2^{-s_j})}\|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})}.$$

Now, using these equalities and successively applying the assertions $a)$ and $b)$ of Lemma 2.6, we obtain that inequality (3.6) holds for an arbitrary non-empty subset of e. Further, applying assertion $a)$ of Lemma 2.6, we obtain

$$J_1 \leq C\left\{\sum_{n_m=0}^{\infty}\left(\frac{\psi_m(2^{-n_m})}{\varphi_m(2^{-n_m})}\right)^{\tau_m}\right.$$

$$\left.\times\left[\dots\left[\sum_{n_1=0}^{\infty}\left(\frac{\psi_1(2^{-n_1})}{\varphi_1(2^{-n_1})}\right)^{\tau_1}\|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})}\right]^{\frac{\tau_2}{\tau_1}}\dots\right]^{\frac{\tau_m}{\tau_{m-1}}}\right\}^{\frac{1}{\tau_m}}. \quad (3.7)$$

Now, the assertion of the theorem follows from inequalities (3.5)–(3.7) . $\square$

Remark 3.1. In the case $X(\bar{\varphi}) = L^*_{\bar{\varphi},\bar{\tau}}(\mathbb{T}^m)$, Theorem 3.2 was announced in [4] without proof.

Theorem 3.3. *Let $1 \le \theta_j \le +\infty$, $1 \le \tau_j < +\infty$, $j = 1,\dots,m$, and suppose functions φ_j and ψ_j satisfy the conditions $1 < \alpha_{\psi_j} \le \beta_{\psi_j} < \alpha_{\varphi_j} \le \beta_{\varphi_j} < 2$, $j = 1,\dots,m$. If*

$$\left[\sum_{s_j=0}^{\infty}\left(\frac{\psi_j(2^{-s_j})}{\varphi_j(2^{-s_j})}2^{-s_j\tau_j}\right)^{\varepsilon_j}\right]^{\frac{1}{\varepsilon_j}} < +\infty, \tag{3.8}$$

where $\varepsilon_j = \tau_j\beta_j'$, $\beta_j' = \frac{\beta_j}{\beta_j-1}$, $j = 1,\dots,m$, if $\beta_j = \frac{\theta_j}{\tau_j} > 1$ and $\varepsilon_j = +\infty$, if $\theta_j \le \tau_j$, $j = 1,\dots,m$, then $S^{\bar r}_{X(\bar\varphi),\bar\theta}B \subset L^*_{\bar\psi,\bar\tau}(\mathbb{T}^m)$ and

$$\|f\|^*_{\bar\psi,\bar\tau} \le C\|f\|_{S^{\bar r}_{X(\bar\varphi),\bar\theta}B}.$$

Proof. If $\tau_j < \theta_j$, $j = 1,\dots,m$, then applying Hölder's inequality with $\beta_j = \frac{\theta_j}{\tau_j}$, $\frac{1}{\beta_j} + \frac{1}{\beta_j'} = 1$, $j = 1,\dots,m$, we obtain

$$\begin{aligned}\sigma_{\bar\tau,\bar\theta,\bar\varphi}(f) &= \left\|\left\{\prod_{j=1}^{m}\frac{\psi_j(2^{-s_j})}{\varphi_j(2^{-s_j})}\|\delta_{\bar s}(f)\|^*_{X(\bar\varphi)}\right\}_{\bar s\in\mathbb{Z}^m_+}\right\|_{l_{\bar\tau}} \\ &\le \left\|\left\{\prod_{j=1}^{m}\frac{\psi_j(2^{-s_j})}{\varphi_j(2^{-s_j})}2^{-s_jr_j}\right\}_{\bar s\in\mathbb{Z}^m_+}\right\|_{l_{\bar\varepsilon}}\left\|\left\{\prod_{j=1}^{m}2^{s_jr_j}\|\delta_{\bar s}(f)\|^*_{X(\bar\varphi)}\right\}_{\bar s\in\mathbb{Z}^m_+}\right\|_{l_{\bar\theta}},\end{aligned} \tag{3.9}$$

where $\bar\varepsilon = (\varepsilon_1,\dots,\varepsilon_m)$, $\varepsilon_j = \tau_j\beta_j'$, $j = 1,\dots,m$.

If $\theta_j \le \tau_j$, $j = 1,\dots,m$, then by Jensen's inequality (see [16, Lemma 3.3.3]), we have

$$\sigma_{\bar\tau,\bar\theta,\bar\varphi}(f) \le \left\|\left\{\prod_{j=1}^{m}2^{-s_jr_j}\|\delta_{\bar s}(f)\|^*_{X(\bar\varphi)}\right\}_{\bar s\in\mathbb{Z}^m_+}\right\|_{l_{\bar\theta}}\prod_{j=1}^{m}\sup_{s_j\in\mathbb{Z}_+}\frac{\psi_j(2^{-s_j})}{\varphi_j(2^{-s_j})}2^{-s_jr_j}. \tag{3.10}$$

By (3.8) and Theorem 3.2, inequalities (3.9) and (3.10) imply the assertion of the theorem. □

Theorem 3.4. *Let ψ_j satisfy the conditions $1 < 2^{1/\lambda_j} < \alpha_{\psi_j} \le \beta_{\psi_j} < 2$, $1 < \tau_j < +\infty$, $j = 1,\dots,m$. If $f \in L^*_{\bar\psi,\bar\tau}(\mathbb{T}^m)$ and*

$$f(\bar x) \sim \sum_{\bar s\in\mathbb{Z}^m_+} b_{\bar s}\sum_{\bar k\in\rho(\bar s)} e^{i\langle\bar k,\bar x\rangle},$$

then the following inequality holds

$$\begin{aligned}\|f\|^*_{\bar\psi,\bar\tau} \ge C\Bigg\{&\sum_{s_m=1}^{\infty}2^{s_m\frac{\tau_m}{\lambda_m}}\psi_m^{\tau_m}(2^{-s_m}) \\ &\times\left[\dots\left[\sum_{s_1=1}^{\infty}2^{s_1\frac{\tau_1}{\lambda_1}}\psi_1^{\tau_1}(2^{-s_1})\left(\|\delta_{\bar s}(f)\|^*_{\bar\lambda,\bar\tau}\right)^{\tau_1}\right]^{\frac{\tau_2}{\tau_1}}\dots\right]^{\frac{\tau_m}{\tau_{m-1}}}\Bigg\}^{\frac{1}{\tau_m}},\end{aligned}$$

where $b_{\bar{s}}$ are real numbers.

Proof. Let $S_{2^\nu,\dots,2^\nu}(f,\bar{x})$ be the rectangular partial sum of the Fourier series of a function $f \in L^*_{\bar{\psi},\bar{\tau}}(I^m)$. It is known that (see [10])

$$\|f\|^*_{\bar{\psi},\bar{\theta}} \asymp \sup_{\substack{g\in L^*_{\bar{\tilde{\psi}}',\bar{\tau}'} \\ \|g\|^*_{\bar{\tilde{\psi}},\bar{\tau}'}\le 1}} \int_{I^m} f(2\pi\bar{x})g(2\pi\bar{x})d\bar{x}, \tag{3.11}$$

where $\bar{\tau}' = (\tau_1',...,\tau_m')$, $\frac{1}{\tau_j}+\frac{1}{\tau_j'} = 1$, $j = 1,...,m$, and $\bar{\tilde{\psi}}(t) = (\tilde{\psi}_1(t),...,\tilde{\psi}_m(t))$, $\tilde{\psi}_j(t) = \frac{t}{\psi_j(t)}$, for $t \in (0,1]$ and $\tilde{\psi}_j(0) = 0$, $j = 1,...,m$. We introduce the notation

$$\sigma_\nu(f)_{\bar{\tau}_j} = \left\{\sum_{s_j=1}^{\nu-1} 2^{\frac{s_j\tau_j}{\lambda_j}}\psi_j^{\tau_j}(2^{-s_j}) \times \left[\dots\left[\sum_{s_1=1}^{\nu-1} 2^{\frac{s_1\tau_1}{\lambda_1}}\psi_1^{\tau_1}(2^{-s_1})\left(\|\delta_{\bar{s}}(f)\|^*_{\bar{\lambda},\bar{\tau}}\right)^{\tau_1}\right]^{\frac{\tau_2}{\tau_1}}\dots\right]^{\frac{\tau_j}{\tau_{j-1}}}\right\}^{\frac{1}{\tau_j}}.$$

We consider the trigonometric polynomial

$$g_\nu(\bar{x}) = \sum_{s_m=1}^{\nu-1}\dots\sum_{s_1=1}^{\nu-1} b_{\bar{s},\nu}\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle},$$

where

$$b_{\bar{s},\nu} = \left\|\left\{\prod_{j=1}^m 2^{\frac{s_j}{\lambda_j}}\psi_j(2^{-s_j})\|\delta_{\bar{s}}(f)\|^*_{\bar{\lambda},\bar{\tau}}\right\}_{\bar{s}=\bar{1}}^{\bar{\nu}}\right\|_{l_{\bar{\tau}}}^{-\frac{\tau_m}{\tau_m'}} \prod_{j=1}^{m-1}(\sigma_\nu(f)_{\bar{\tau}_j})^{\tau_{j+1}-\tau_j}$$
$$\times\prod_{j=1}^m (2^{s_j}\psi_j(2^{-s_j}))^{\tau_j}\left(\prod_{j=1}^m 2^{s_j}\right)^{-1}\prod_{j=2}^m 2^{s_j(1-\frac{1}{\lambda_j})(\tau_1-\tau_j)}|b_{\bar{s}}(f)|^{\tau_1-1} sign(b_{\bar{s}}(f)),$$

and $\bar{\tau}_j = (\tau_1,\dots,\tau_j)$. Then, taking into account the orthogonality of the trigonometric system, we have

$$\begin{aligned}\int_{I^m} f(2\pi\bar{x})g(2\pi\bar{x})d\bar{x} &= \int_{I^m} S_{2^\nu,\dots,2^\nu}(f,2\pi\bar{x})g_\nu(2\pi\bar{x})d\bar{x}\\ &= \sum_{s_m=1}^{\nu-1}\dots\sum_{s_1=1}^{\nu-1}\int_{I^m}\delta_{\bar{s}}(f,2\pi\bar{x})g_\nu(2\pi\bar{x})d\bar{x}.\end{aligned} \tag{3.12}$$

We prove that $\|g_\nu\|^*_{\bar{\tilde{\psi}},\bar{\tau}'} \le C_0$, where C_0 is a positive constant independent of ν. Taking into account the relation (see (2.5))

$$\Big\|\sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle\bar{k},\bar{x}\rangle}\Big\|^*_{\bar{\psi},\bar{\tau}} \asymp \prod_{j=1}^m 2^{s_j}\psi_j(2^{-s_j}), \tag{3.13}$$

for $1<\tau_j<+\infty,\ 1<\alpha_{\psi_j}\le\beta_{\psi_j}<2$, it is easy to verify the following inequality

$$\begin{aligned}\left(\|\delta_{\bar s}(g_\nu)\|^*_{\bar\lambda',\bar\tau'}\right)^{\tau_1'} &= \left(|b_{s,\nu}|\Big\|\sum_{\bar k\in\rho(\bar s)} e^{i\langle\bar k,\bar x\rangle}\Big\|^*_{\bar\lambda',\bar\tau'}\right)^{\tau_1'}\\ &\le C\prod_{j=2}^m\left(2^{\frac{s_j}{\lambda_j}}\psi_j(2^{-s_j})\right)^{\tau_j\tau_1'}\left(2^{\frac{s_1}{\lambda_1}}\psi_1(2^{-s_1})\right)^{(\tau_1+\tau_1')}\left(\|\delta_{\bar s}(f)\|^*_{\bar\lambda,\bar\tau}\right)^{\tau_1}\\ &\times\left(\prod_{j=1}^m(\sigma_\nu(f)_{\bar\tau_j})^{\tau_{j+1}-\tau_j}\left\|\left\{\prod_{j=1}^m 2^{\frac{s_j}{\lambda_j}}\psi_j(2^{-s_j})\|\delta_{\bar s}(f)\|^*_{\bar\lambda,\bar\tau}\right\}_{\bar s=\bar 1}^{\bar\nu}\right\|_{l_{\bar\tau}}^{-\frac{\tau_m}{\tau_m'}}\right)^{\tau_1'}, \quad (3.14)\end{aligned}$$

where $\bar\nu=(\nu,\dots,\nu)$. Further, by using inequality (3.14), we obtain

$$\begin{aligned}J(g_\nu) := &\left\{\sum_{s_m=1}^{\nu-1}\left(2^{\frac{s_m}{\lambda_m'}}\tilde\psi_m(2^{-s_m})\right)^{\tau_m'}\right.\\ &\left.\times\left[\dots\left[\sum_{s_1=1}^{\nu-1}\left(2^{\frac{s_1}{\lambda_1'}}\tilde\psi_1(2^{-s_1})\right)^{\tau_1'}\left(\|\delta_{\bar s}(f)\|^*_{\bar\lambda',\bar\tau'}\right)^{\tau_1'}\right]^{\frac{\tau_2'}{\tau_1'}}\dots\right]^{\frac{\tau_m'}{\tau_{m-1}'}}\right\}^{\frac{1}{\tau_m'}}\\ &\le C\left\|\left\{\prod_{j=1}^m 2^{\frac{s_j}{\lambda_j}}\psi_j(2^{-s_j})\|\delta_{\bar s}(f)\|^*_{\bar\lambda,\bar\tau}\right\}_{\bar s\in\mathbb{Z}_+^m}\right\|_{l_{\bar\tau}}^{-\frac{\tau_m}{\tau_m'}}\\ &\times\left\{\sum_{s_m=1}^{\nu-1}\left(2^{\frac{s_m}{\lambda_m'}}\tilde\psi_m(2^{-s_m})\right)^{\tau_m'}\left[\dots\left[\sum_{s_1=1}^{\nu-1}\left(2^{\frac{s_1}{\lambda_1'}}\tilde\psi_1(2^{-s_1})\right)^{\tau_1'}\prod_{j=2}^m\left(2^{\frac{s_j}{\lambda_j}}\psi_j(2^{-s_j})\right)^{\tau_j\tau_1'}\right.\right.\right.\\ &\left.\left.\left.\times\left(2^{\frac{s_1}{\lambda_1}}\psi_1(2^{-s_1})\right)^{(\tau_1+\tau_1')}\left(\|\delta_{\bar s}(f)\|^*_{\bar\lambda,\bar\tau}\right)^{\tau_1}\left(\prod_{j=1}^{m-1}(\sigma_\nu(f)_{\bar\tau_j})^{\tau_{j+1}-\tau_j}\right)^{\tau_1'}\right]^{\frac{\tau_2'}{\tau_1'}}\dots\right]^{\frac{\tau_m'}{\tau_{m-1}'}}\right\}^{\frac{1}{\tau_m'}}. \quad (3.15)\end{aligned}$$

Since $\frac{1}{\lambda_j'}=1-\frac{1}{\lambda_j}$, $\tau_j\tau_j'=\tau_1+\tau_1'$ and $\tilde\psi_j(t)=\frac{t}{\psi_j(t)}$, $j=1,\dots,m$, we have

$$\begin{aligned}\left(2^{\frac{s_j}{\lambda_j'}}\tilde\psi_j(2^{-s_j})\right)^{\tau_j'}\left(2^{\frac{s_j}{\lambda_j}}\psi_j(2^{-s_j})\right)^{\tau_j\tau_j'} &= \left(\frac{2^{-\frac{s_j}{\lambda_j}}}{\psi_j(2^{-s_j})}\right)^{\tau_j'}\left(2^{\frac{s_j}{\lambda_j}}\psi_j(2^{-s_j})\right)^{\tau_j+\tau_j'}\\ &= \left(2^{\frac{s_j}{\lambda_j}}\psi_j(2^{-s_j})\right)^{\tau_j},\ j=1,\dots,m.\end{aligned}$$

Now, considering this equality and the definition of the numbers $\sigma_\nu(f)_{\bar{\tau}_j}$, inequality (3.15) continues

$$J(g_\nu) \le C \left\| \left\{ \prod_{j=1}^{m} 2^{\frac{s_j}{\lambda_j}} \psi_j(2^{-s_j}) \|\delta_{\bar{s}}(f)\|^{*}_{\bar{\lambda},\bar{\tau}} \right\}_{\bar{s}\in\mathbb{Z}^m_+} \right\|_{l_{\bar{\tau}}}^{-\frac{\tau_m}{\tau'_m}}$$
$$\times \left\{ \sum_{s_m=1}^{\nu-1} \left(2^{\frac{s_m}{\lambda_m}} \psi_m(2^{-s_m}) \right)^{\tau_m} \left[\dots \left[\sum_{s_1=1}^{\nu-1} \left(2^{\frac{s_1}{\lambda_1}} \psi_1(2^{-s_1}) \right)^{\tau_1} \left(\|\delta_{\bar{s}}(f)\|^{*}_{\bar{\lambda},\bar{\tau}} \right)^{\tau_1} \right.\right.\right.$$

$$\left.\left.\left. \times \left(\prod_{j=1}^{m-1} (\sigma_\nu(f)_{\bar{\tau}_j})^{\tau_{j+1}-\tau_j} \right)^{\tau'_1} \right]^{\frac{\tau'_2}{\tau'_1}} \dots \right]^{\frac{\tau'_m}{\tau'_{m-1}}} \right\}^{\frac{1}{\tau'_m}} = \left\{ \sum_{s_m=1}^{\nu-1} \left(2^{\frac{s_m}{\lambda_m}} \psi_m(2^{-s_m}) \right)^{\tau_m} \right.$$
$$\left. \times \left[\dots \left[\sum_{s_1=1}^{\nu-1} \left(2^{\frac{s_1}{\lambda_1}} \psi_1(2^{-s_1}) \right)^{\tau_1} \left(\|\delta_{\bar{s}}(f)\|^{*}_{\bar{\lambda},\bar{\tau}} \right)^{\tau_1} \right]^{\frac{\tau_2}{\tau_1}} \dots \right]^{\frac{\tau_m}{\tau_{m-1}}} \right\}^{\frac{1}{\tau_m}}$$
$$\times C \left\| \left\{ \prod_{j=1}^{m} 2^{\frac{s_j}{\lambda_j}} \psi_j(2^{-s_j}) \|\delta_{\bar{s}}(f)\|^{*}_{\bar{\lambda},\bar{\tau}} \right\}_{\bar{s}\in\mathbb{Z}^m_+} \right\|_{l_{\bar{\tau}}}^{-\frac{\tau_m}{\tau'_m}} \le C. \quad (3.16)$$

Since $1 < 2^{\frac{1}{\lambda_j}} < \alpha_{\psi_j}$, we have $\beta_{\bar{\psi}_j} < 2^{\frac{1}{\lambda'_j}}$, $j = 1, ..., m$. Therefore, according to Theorem 3.2 and inequality (3.16), we obtain

$$\|g_\nu\|^{*}_{\bar{\bar{\psi}},\bar{\tau}'} \le CJ(g_\nu) \le C_0.$$

Hence, $\varphi_\nu = \frac{1}{C_0} g_\nu \in L^{*}_{\bar{\bar{\psi}},\bar{\tau}'}(\mathbb{T}^m)$, and $\|\varphi_\nu\|^{*}_{\bar{\bar{\psi}},\bar{\tau}'} \le 1$. Further, according to the orthogonality property, we have

$$\int_{I^m} \delta_{\bar{s}}(f,\bar{x}) g_\nu(\bar{x}) d\bar{x} = \sum_{l_m=1}^{\nu-1} \dots \sum_{l_1=1}^{\nu-1} b_{\bar{s}}(f) b_{\bar{l},\nu}$$
$$\times \int_{I^m} \left| \sum_{\bar{k}\in\rho(\bar{s})} e^{i\langle \bar{k},\bar{x}\rangle} \right|^2 d\bar{x} = (2\pi)^d \cdot b_{\bar{s}}(f) b_{\bar{s},\nu} \cdot \prod_{j=1}^{m} 2^{s_j-1}. \quad (3.17)$$

From the definition of the numbers $b_{\bar{s},\nu}$, it follows that

$$\prod_{j=1}^{m} 2^{s_j} b_{\bar{s}}(f) b_{\bar{s},\nu} = \prod_{j=1}^{m} 2^{s_j} \left\| \left\{ \prod_{j=1}^{m} 2^{\frac{s_j}{\lambda_j}} \psi_j(2^{-s_j}) \|\delta_{\bar{s}}(f)\|^{*}_{\bar{\lambda},\bar{\tau}} \right\}^{\bar{\nu}}_{\bar{s}=\bar{1}} \right\|_{l_{\bar{\tau}}}^{-\frac{\tau_m}{\tau'_m}}$$
$$\times \prod_{j=1}^{m-1} (\sigma_\nu(f)_{\bar{\tau}_j})^{\tau_{j+1}-\tau_j} \prod_{j=1}^{m} 2^{s_j\theta_j} \psi_j^{\tau_j}(2^{-s_j}) \left(\prod_{j=1}^{m} 2^{s_j} \right)^{-1}$$

$$\times \prod_{j=2}^{m} 2^{s_j(1-\frac{1}{\lambda_j})(\tau_1-\tau_j)} |b_{\bar{s}}(f)|^{\tau_1} = \prod_{j=1}^{m-1} (\sigma_\nu(f)_{\bar{\tau}_j})^{\tau_{j+1}-\tau_j}$$

$$\times \left\| \left\{ \prod_{j=1}^{m} 2^{\frac{s_j}{\lambda_j}} \psi_j(2^{-s_j}) \|\delta_{\bar{s}}(f)\|^*_{\bar{\lambda},\bar{\tau}} \right\}_{\bar{s}=\bar{1}}^{\bar{\nu}} \right\|_{l_{\bar{\tau}}}^{-\frac{\tau_m}{\tau'_m}} \prod_{j=1}^{m} 2^{s_j \tau_j} \psi_j^{\tau_j}(2^{-s_j})$$

$$\times \prod_{j=2}^{m} 2^{-s_j(1-\frac{1}{\lambda_j})\tau_j} 2^{-s_1(1-\frac{1}{\lambda_1})\tau_1} \left(\prod_{j=1}^{m} 2^{s_j(1-\frac{1}{\lambda_j})} |b_{\bar{s}}(f)| \right)^{\tau_1} \geq$$

$$\geq C \left\| \left\{ \prod_{j=1}^{m} 2^{\frac{s_j}{\lambda_j}} \psi_j(2^{-s_j}) \|\delta_{\bar{s}}(f)\|^*_{\bar{\lambda},\bar{\tau}} \right\}_{\bar{s}=\bar{1}}^{\bar{\nu}} \right\|_{l_{\bar{\tau}}}^{-\frac{\tau_m}{\tau'_m}} \prod_{j=1}^{m-1} (\sigma_\nu(f)_{\bar{\tau}_j})^{\tau_{j+1}-\tau_j}$$

$$\times \prod_{j=1}^{m} \left(2^{\frac{s_j}{\lambda_j}} \psi_j(2^{-s_j}) \right)^{\tau_j} \left(\|\delta_{\bar{s}}(f)\|^*_{\bar{\lambda},\bar{\tau}} \right)^{\tau_1}. \quad (3.18)$$

Now, taking into account (3.17) and (3.18), we obtain

$$\sup_{\substack{g \in L^*_{\bar{\psi},\bar{\tau}'} \\ \|g\|^*_{\bar{\psi},\bar{\tau}'} \leq 1}} \sum_{s_m=1}^{\nu-1} \cdots \sum_{s_1=1}^{\nu-1} \int_{I^m} \delta_{\bar{s}}(f,\bar{x}) g(\bar{x}) d\bar{x} \geq \sum_{s_m=1}^{\nu-1} \cdots \sum_{s_1=1}^{\nu-1} \int_{I^m} \delta_{\bar{s}}(f,\bar{x}) \varphi_\nu(\bar{x}) d\bar{x}$$

$$= C \sum_{s_m=1}^{\nu-1} \cdots \sum_{s_1=1}^{\nu-1} \int_{I^m} \delta_{\bar{s}}(f,\bar{x}) g(\bar{x}) d\bar{x} \geq \sum_{s_m=1}^{\nu-1} \cdots \sum_{s_1=1}^{\nu-1} \int_{I^m} \delta_{\bar{s}}(f,\bar{x}) g_\nu(\bar{x}) d\bar{x}$$

$$\geq C \left\| \left\{ \prod_{j=1}^{m} 2^{\frac{s_j}{\tau_j}} \psi_j(2^{-s_j}) \|\delta_{\bar{s}}(f)\|^*_{\bar{\tau},\bar{\theta}} \right\}_{\bar{s}=\bar{1}}^{\bar{\nu}} \right\|_{l_{\bar{\theta}}}^{-\frac{\theta_m}{\theta'_m}} \prod_{j=1}^{m-1} (\sigma_\nu(f)_{\bar{\theta}_j})^{\theta_{j+1}-\theta_j}$$

$$\times \sum_{s_m=1}^{\nu-1} \cdots \sum_{s_1=1}^{\nu-1} \prod_{j=1}^{m} \left(2^{\frac{s_j}{\tau_j}} \psi_j(2^{-s_j}) \right)^{\theta_j} \left(\|\delta_{\bar{s}}(f)\|^*_{\bar{\lambda},\bar{\tau}} \right)^{\tau_1}$$

$$= C \left\| \left\{ \prod_{j=1}^{m} 2^{\frac{s_j}{\lambda_j}} \psi_j(2^{-s_j}) \|\delta_{\bar{s}}(f)\|^*_{\bar{\lambda},\bar{\tau}} \right\}_{\bar{s}=\bar{1}}^{\bar{\nu}} \right\|_{l_{\bar{\tau}}}. \quad (3.19)$$

It follows from inequalities (3.12) and (3.19) that

$$\|f\|^*_{\bar{\psi},\bar{\tau}} \geq C \left\| \left\{ \prod_{j=1}^{m} 2^{\frac{s_j}{\lambda_j}} \psi_j(2^{-s_j}) \|\delta_{\bar{s}}(f)\|^*_{\bar{\lambda},\bar{\tau}} \right\}_{\bar{s} \in \mathbb{Z}_+^m} \right\|_{l_{\bar{\tau}}}.$$

□

Remark 3.2. In the case $\psi_j(t) = t^{\frac{1}{q_j}}$, $\varphi_j(t) = t^{\frac{1}{p_j}}$, $j = 1, \ldots, m$, Theorem 3.2 and Theorem 3.4 were proved in [2].

Further, for brevity, we set $\mu_j(s) = \frac{\psi_j(2^{-s})}{\varphi_j(2^{-s})}$.

Theorem 3.5. *Let* $1 \le \theta_j < +\infty$, $1 \le \tau_j < +\infty$, $j = 1,...,m$, *and let* φ_j, ψ_j *satisfy the conditions* $1 < \alpha_{\psi_j} \le \beta_{\psi_j} < \alpha_{\varphi_j} \le \beta_{\varphi_j} < 2$, $j = 1,...,m$, *and* (3.8) .

1) If $1 \le \tau_j < \theta_j < +\infty$, $j = 1,...,m$,, *then*

$$E_n^{\bar{\gamma}}(S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B)_{\bar{\psi},\bar{\tau}} \le C \left\| \left\{ \prod_{j=1}^{m} 2^{-s_j r_j} \mu_j(s_j) \right\}_{\bar{s} \in Y^m(\bar{\gamma},n)} \right\|_{\bar{\varepsilon}},$$

where $\bar{\varepsilon} = (\varepsilon_1,...,\varepsilon_m)$, $\varepsilon_j = \tau_j \beta'_j$, $\frac{1}{\beta_j} + \frac{1}{\beta'_j} = 1$, $\beta_j = \frac{\theta_j}{\tau_j}$.

2) If $1 \le \theta_j \le \tau_j < +\infty$, $j = 1,...,m$,, *then*

$$E_n^{\bar{\gamma}}(S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B)_{\bar{\psi},\bar{\tau}} \le C \sup \Big\{ \prod_{j=1}^{m} 2^{-s_j r_j} \mu_j(s_j) : \quad \bar{s} \in \mathbb{Z}^m_+, \quad \langle \bar{s}, \bar{\gamma} \rangle \ge n \Big\}.$$

Proof. Let $f \in S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B$. Then by Theorem 3.4, we obtain

$$\|f - S_n^{(\bar{\gamma})}(f)\|^*_{\bar{\psi},\bar{\tau}} \le C \left\| \left\{ \prod_{j=1}^{m} \mu_j(s_j) \|\delta_{\bar{s}}(f - S_n^{(\bar{\gamma})}(f))\|^*_{X(\bar{\varphi})} \right\}_{\bar{s} \in \mathbb{Z}^m_+} \right\|_{l_{\bar{\tau}}}. \tag{3.20}$$

Since $\delta_{\bar{s}}(f - S_n^{(\bar{\gamma})}(f)) = 0$, $\bar{s} \notin Y^m(\bar{\gamma},n)$ and $\delta_{\bar{s}}(f - S_n^{(\bar{\gamma})}(f)) = \delta_{\bar{s}}(f)$, $\bar{s} \in Y^m(\bar{\gamma},n)$, it follows from (3.20) that

$$\|f - S_n^{(\bar{\gamma})}(f)\|^*_{\bar{\psi},\bar{\tau}} \le C \left\| \left\{ \prod_{j=1}^{m} \mu_j(s_j) \|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})} \right\}_{\bar{s} \in Y^m(\bar{\gamma},n)} \right\|_{l_{\bar{\tau}}}. \tag{3.21}$$

We set

$$b_{\bar{s}}(n) = \prod_{j=1}^{m} \frac{\psi_j(2^{-s_j})}{\varphi_j(2^{-s_j})}, \text{ if } \bar{s} \in Y^m(\bar{\gamma},n)$$

and $b_{\bar{s}}(n) = 0$ if $\bar{s} \notin Y^m(\bar{\gamma},n)$.

We prove 1). Since $1 \le \tau_j < \theta_j < +\infty$, $j = 1,...,m$, we obtain, by applying Hölder's inequality with $\beta_j = \frac{\tau_j}{\theta_j}$, $\frac{1}{\beta_j} + \frac{1}{\beta'_j} = 1$, $j = 1,...,m$, that

$$\sigma_1(f,n) = \left\| \left\{ \prod_{j=1}^{m} \mu_j(s_j) \|\delta_{\bar{s}}(f)\|^*_{X(\bar{\varphi})} \right\}_{\bar{s} \in Y^m(\bar{\gamma},n)} \right\|_{l_{\bar{\tau}}} \le C \left\| \left\{ b_{\bar{s}}(n) \right\}_{\bar{s} \in \mathbb{Z}^m_+} \right\|_{l_{\bar{\varepsilon}}}, \tag{3.22}$$

where $\bar{\varepsilon} = (\varepsilon_1,...,\varepsilon_m)$, $\varepsilon_j = \tau_j \beta'_j$, $\quad j = 1,...,m$.

Let us prove 2). If $\theta_j \le \tau_j < +\infty$, $j = 1,...,m$,, then according to Jensen's inequality (see [16, Ch. 3, Sec. 3]), we obtain

$$\sigma_1(f,n) \le \|f\|_{S^{\bar{r}}_{X(\bar{\varphi}),\bar{\theta}}B} \sup_{\bar{s} \in Y^m(\bar{\gamma},n)} \prod_{j=1}^{m} \mu_j(s_j) 2^{-s_j r_j}. \tag{3.23}$$

Inequalities (3.21)–(3.23) imply the statements 1) and 2) of Theorem 3.5. $\square$

Remark 3.3. In the case $\varphi_j(t) = t^{1/p}$, $p_j = \tau_j^{(1)} = p$, and $\psi_j(t) = t^{1/q}$, $q_j = \tau_j^{(2)} = q$, $1 \le \theta_j = \theta \le \infty$ for $j = 1, ..., m$, Theorem 3.5 implies the previously known results of Ya.S. Bugrov, E.M. Galeev, V.N. Temlyakov, and A.S. Romanyuk (see, for example, the bibliography in [11], [19], [24]).

Further, from Theorem 3.5 with $\varphi_j(t) = t^{1/p_j}$, $\psi_j(t) = t^{1/q_j}$, $1 < p_j, q_j < \infty$, $j = 1, ..., m$, and $\gamma_j' = \gamma_j = 1$ for $j = 1, ..., \nu$, and $\gamma_j' < \gamma_j$ for $j = \nu+1, ..., m$, Theorem 2 in [1] follows (see also [2]), and for $1 \le \gamma_j' \le \gamma_j$ for $j = 1, ..., m$, Theorem 1 in [7] follows (see also [9]).

Remark 3.4. In the case $X(\bar{\varphi}) = L^*_{\bar{\varphi},\bar{\tau}}(\mathbb{T}^m)$, Theorem 3.4 and Theorem 3.5 were proved in [4]. The results of this article were previously published by the author in [5].

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