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SOME RESULTS ON SASAKIAN AND $N(k)$-CONTACT METRIC MANIFOLDS

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ABSTRACT. The main objective of the present paper is to introduce a new curvature tensor, called the $*$-generalized quasi-conformal curvature tensor, in the setting of Sasakian and $N(k)$-contact metric manifolds. Subsequently, we investigate various semi-symmetric structures associated with this tensor and derive several interesting geometric characterizations and results.

1. INTRODUCTION

The symbols $\hat{S}$, ρ, R, and r denote the Ricci operator, Ricci tensor, Riemann curvature tensor, and scalar curvature, respectively. The generalized quasi-conformal curvature tensor was introduced and studied by Baishya and Chowdhury [2] in 2016 and was defined on a $(2n+1)$-dimensional Riemannian or semi-Riemannian manifold as follows:

$$\begin{aligned} G_q(X_1,X_2)X_3 = {} & R(X_1,X_2)X_3 + a[\rho(X_2,X_3)X_1 - \rho(X_1,X_3)X_2] \\ & + b[g(X_2,X_3)\hat{S}X_1 - g(X_1,X_3)\hat{S}X_2] \\ & - \frac{cr}{(2n+1)}(\frac{1}{2n} + a + b)[g(X_2,X_3)X_1 - g(X_1,X_3)X_2], \end{aligned}$$

where a, b, and c are constants. If we set $\alpha = a$, $\beta = b$, and $\gamma = -\frac{cr}{(2n+1)}\left(\frac{1}{2n} + a + b\right)$, then, for particular values of (α, β, γ), the generalized quasi-conformal curvature tensor (G_q-tensor) reduces to the following curvature tensors:

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Curvature Tensor	(α,β,γ)
Riemann R	$(0,0,0)$
Conharmonic H ([15])	$(-\frac{1}{2n-1},-\frac{1}{2n-1},0)$
Conformal C ([12])	$(-\frac{1}{2n-1},-\frac{1}{2n-1},\frac{r}{2n(2n-1)})$
Concircular $\tilde{C}$ ([27])	$(0,0,-\frac{r}{2n(2n+1)})$
Projective P ([27])	$(-\frac{1}{2n},0,0)$
m-*Projective* $\mathcal{M}$ ([19])	$(-\frac{1}{4n},0,0)$
$\mathcal{W}_1$ *-Curvature tensor* ([17])	$(\frac{1}{2n},0,0)$
$\mathcal{W}_2$ *-Curvature tensor* ([19])	$(0,-\frac{1}{2n},0)$
$\mathcal{W}_4$*-Curvature tensor* ([18])	$(0,0,-\frac{1}{2n})$

where

$$\begin{aligned} C(X_1,X_2) &= R(X_1,X_2)-\frac{1}{2n-1}[\big(X_1\wedge_g \hat{S}X_2\big)+\big(\hat{S}X_1\wedge_g X_2\big) \\ &+ \frac{r}{2n}(X_1\wedge_g X_2)], \qquad (1.1) \\ \tilde{C}(X_1,X_2) &= R(X_1,X_2)-\frac{r}{2n(2n+1)}(X_1\wedge_g X_2), \qquad (1.2) \\ P(X_1,X_2) &= R(X_1,X_2)-\frac{1}{2n}\big(X_1\wedge_\rho X_2\big), \qquad (1.3) \\ H(X_1,X_2) &= R(X_1,X_2)-\frac{1}{2n-1}\left[\big(X_1\wedge_g \hat{S}X_2\big)+\big(\hat{S}X_1\wedge_g X_2\big)\right], \qquad (1.4) \\ \mathcal{M}(X_1,X_2) &= R(X_1,X_2)-\frac{1}{4n}\left[\big(X_1\wedge_g \hat{S}X_2\big)+\big(\hat{S}X_1\wedge_g X_2\big)\right], \qquad (1.5) \\ \mathcal{W}_1(X_1,X_2) &= R(X_1,X_2)-\frac{1}{2n}\big(X_1\wedge_\rho X_2\big), \qquad (1.6) \\ \mathcal{W}_2(X_1,X_2) &= R(X_1,X_2)-\frac{1}{2n}[\big(\hat{S}X_1\wedge_g X_2\big)+\big(X_1\wedge_g \hat{S}X_2\big) \\ &- \big(X_1\wedge_\rho X_2\big)], \qquad (1.7) \\ \mathcal{W}_4(X_1,X_2) &= R(X_1,X_2)-\frac{1}{2n}(X_1\wedge_g X_2), \qquad (1.8) \end{aligned}$$

and

$$(X_1\wedge_B X_2)X_3 = B(X_2,X_3)X_1 - B(X_1,X_3)X_2. \qquad (1.9)$$

Recently, semisymmetric classes (see [3,4]) have been studied on $(LCS)_n$-manifolds and α-cosymplectic manifolds. The $*$-Ricci tensor ([20]) has attracted considerable attention from geometers in recent years (see [5, 13, 23, 24]). Hamada ([14]) defined the $*$-Ricci tensor of real hypersurfaces in non-flat complex space forms by

$$\rho^*(X_1,X_2) = g(\hat{S}^*X_1,X_2) = \frac{1}{2}trace\{\phi\circ R(X_1,\phi X_2)\},$$

where $\hat{S}^*$ and ρ^* denote the $*$-Ricci operator and the $*$-Ricci tensor, respectively. In 2019, Kaimakamis and Panagiotidou ([16]) introduced and studied the $*$-Weyl curvature tensor in the setting of real hypersurfaces in non-flat complex space forms, defining it as follows:

$$C^*(X_1,X_2) = R(X_1,X_2) - \frac{1}{2n-1}\left(\frac{r^*}{2n}(X_1\wedge_g X_2) + \left(X_1\wedge_g \hat{S}^*X_2\right) + \left(\hat{S}^*X_1\wedge_g X_2\right)\right),$$

where r^* denotes the $*$-scalar curvature. Subsequently, the $*$-Weyl curvature tensor was studied by Venkatesha ([24]) in the framework of Sasakian and (κ,μ)-contact metric manifolds. Furthermore, Oğuzhan et al. ([1,26]) introduced generalized symmetric connections, which may be regarded as a broader generalization of quarter-symmetric and semi-symmetric connections.

Motivated by these developments, we introduce the notion of the $*$-generalized quasi-conformal curvature tensor (hereafter denoted by the G_q^*-tensor), defined as follows:

$$\begin{aligned} G_q^*(X_1,X_2)X_3 &= R(X_1,X_2)X_3 + \alpha[g(\hat{S}^*X_2,X_3)X_1 - g(\hat{S}^*X_1,X_3)X_2] \\ &+ \beta[g(X_2,X_3)\hat{S}^*X_1 - g(X_1,X_3)\hat{S}^*X_2] \\ &+ \gamma[g(X_2,X_3)X_1 - g(X_1,X_3)X_2]. \end{aligned} \tag{1.10}$$

Similarly, from the G_q^*-tensor, by setting particular values of (α,β,γ) one can easily get the following curvature tensors:

Curvature Tensor	(α,β,γ)
$*$-*Conharmonic* H^*	$(-\frac{1}{2n-1},-\frac{1}{2n-1},0)$
$*$-*Conformal* C^* ([16])	$(-\frac{1}{2n-1},-\frac{1}{2n-1},\frac{r^*}{2n(2n-1)})$
$*$-*Concircular* $\tilde{C}^*$	$(0,0,-\frac{r^*}{2n(2n+1)})$
$*$-*Projective* P^*	$(-\frac{1}{2n},0,0)$
$*$-$\mathfrak{m}$-*Projective* $\mathcal{M}^*$	$(-\frac{1}{4n},0,0)$
$*$-$\mathcal{W}_1$ -*Curvature tensor*	$(\frac{1}{2n},0,0)$
$*$-$\mathcal{W}_2$ -*Curvature tensor*	$(0,-\frac{1}{2n},0)$

where

$$\begin{aligned} C^*(X_1,X_2) &= R(X_1,X_2) - \frac{1}{2n-1}[\left(X_1\wedge_g \hat{S}^*X_2\right) + \left(\hat{S}^*X_1\wedge_g X_2\right) \\ &\quad + \frac{r^*}{2n}(X_1\wedge_g X_2)], \\ \tilde{C}^*(X_1,X_2) &= R(X_1,X_2) - \frac{r^*}{2n(2n+1)}(X_1\wedge_g X_2), \end{aligned}$$

$$P^*(X_1,X_2) = R(X_1,X_2) - \frac{1}{2n}\left(X_1 \wedge_{\rho^*} X_2\right),$$
$$H^*(X_1,X_2) = R(X_1,X_2) - \frac{1}{2n-1}\left[\left(X_1 \wedge_g \hat{S}^* X_2\right) + \left(\hat{S}^* X_1 \wedge_g X_2\right)\right],$$
$$\mathcal{M}^*(X_1,X_2) = R(X_1,X_2) - \frac{1}{4n}\left[\left(X_1 \wedge_g \hat{S}^* X_2\right) + \left(\hat{S}^* X_1 \wedge_g X_2\right)\right],$$
$$\mathcal{W}_1^*(X_1,X_2) = R(X_1,X_2) - \frac{1}{2n}\left(X_1 \wedge_{\rho^*} X_2\right),$$
$$\mathcal{W}_2^*(X_1,X_2) = R(X_1,X_2) - \frac{1}{2n}[\left(\hat{S}^* X_1 \wedge_g X_2\right) + \left(X_1 \wedge_g \hat{S}^* X_2\right) - \left(X_1 \wedge_{\rho^*} X_2\right)].$$

For a $(0,2)$-tensor field $\mathcal{F}$ and a $(0,m)$-tensor field Υ, where $(m \geq 1)$, on M^{2n+1}, we define $R \circ \Upsilon$ and $Q(\mathcal{F}, \Upsilon)$ ([25]) as follows:

$$\begin{aligned}(R\circ\Upsilon)&(X_1,X_2,...,X_m;X,Y)\\ &= -\Upsilon(R(X,Y)X_1,X_2,...,X_m) - \Upsilon(X_1,R(X,Y)X_2,...,X_m) - ...\\ &\quad - \Upsilon(X_1,X_2,...,R(X,Y)X_m),\end{aligned} \tag{1.11}$$

and

$$\begin{aligned}Q(\mathcal{F},\Upsilon)&(X_1,X_2,...,X_m;X,Y)\\ &= -\Upsilon((X\wedge_{\mathcal{F}} Y)X_1,X_2,...,X_m) - \Upsilon(X_1,(X\wedge_{\mathcal{F}} Y)X_2,...,X_m) - ...\\ &\quad - \Upsilon(X_1,X_2,...,(X\wedge_{\mathcal{F}} Y)X_m),\end{aligned} \tag{1.12}$$

respectively.

The remainder of the paper is organized as follows: In section 2, we present some known results related to Sasakian and $N(k)$-contact metric manifolds. Thereafter, we investigate the semi-symmetric condition $R \cdot G_q^* = 0$ on a Sasakian manifold and show that the manifold reduces to either an η-Einstein manifold or an Einstein manifold and has constant scalar curvature (3.3). We also obtain the form of the Ricci operator on a Sasakian manifold satisfying the semi-symmetric conditions $G_q^* \circ \hat{S}^* = 0$ and $Q(\rho, \mathcal{W}_2^*) = 0$. Furthermore, we show that if a Sasakian manifold satisfies any one of the semi-symmetric conditions $Q(\rho, H^*) = 0$, $\quad Q(\rho, C^*) = 0$, $\quad Q(\rho, \tilde{C}^*) = 0$, or $Q(\rho, \mathcal{M}^*) = 0$, then it reduces to an Einstein manifold. On the other hand, if it satisfies either $Q(\rho, P^*) = 0$ or $Q(\rho, \mathcal{W}_1^*) = 0$, then it reduces to an η-Einstein manifold. Next, we study $N(k)$-contact metric manifolds admitting various semi-symmetric curvature restrictions. We prove that if a $N(k)$-contact metric manifold satisfies the semi-symmetry condition $\hat{S} \circ G_q^* = 0$, then either the manifold is locally isometric to the trivial sphere bundle $E^{n+1} \times S^n(4)$ or it is ξ-star generalized quasi-conformally flat. We also show that a $N(k)$-contact metric manifold satisfying the condition $G_q^* \circ$

$\rho = 0$ reduces either to a Sasakian manifold or to an η-Einstein manifold. Finally, we prove that a $N(k)$-contact metric manifold satisfying the curvature restriction $G_q^* \circ h = 0$ reduces to a Sasakian manifold.

2. Preliminaries

The present section gives preliminaries which will be helpful to establish our results. All manifolds are considered to be connected and smooth. According to Blair ([7]), an almost contact metric structure (ϕ, ξ, η, g) on a Riemannian manifold M^{2n+1} possesses:

- an 1-form η and a vector field ξ, the Reeb field, such that $\eta(\xi) = 1$;
- an endomorphism $\phi : TM \to TM$ such that $\phi^2 = -I + \eta \otimes \xi$;
- a Riemannian metric g such that $g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2)$ for any vector fields X_1, X_2 on M^{2n+1}.

From the above conditions, we get $\phi \circ \xi = 0$, $\eta \circ \phi = 0$. For an *almost contact metric structure* $(M^{2n+1}, \phi, \xi, \eta, g)$, its Kähler form $\omega \in \Omega^2(M)$ is given by $\omega(X_1, X_2) = g(\phi X_1, X_2)$. This implies that $\eta \wedge \omega^n$ is a volume form on M^{2n+1}. The *almost contact metric structure* (ϕ, ξ, η, g) is

- contact metric if $d\eta = \omega$;
- normal if $N_\phi + d\eta \otimes \xi = 0$;
- Sasakian if it is contact metric and normal.

A contact metric manifold is called Sasakian ([11], [6]) if any one of the following equivalent conditions hold

$$(\nabla_{X_1}\phi)X_2 = g(X_1, X_2)\xi - \eta(X_2)X_1, \tag{2.1}$$
$$R(X_1, X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2, \tag{2.2}$$

for any vector fields X_1, X_2 on M^{2n+1}. On a Sasakian manifold the following relations hold ([6])

$$\nabla_{X_1}\xi = -\phi X_1, \tag{2.3}$$
$$\hat{S}\xi = 2n\xi. \tag{2.4}$$

A contact metric manifold is called an η-Einstein manifold if for any smooth functions λ and μ on M^{2n+1}, the Ricci tensor ρ takes the form

$$\rho = \lambda g + \mu \eta \otimes \eta.$$

Lemma 2.1. *([13]) The relation between the $*$-Ricci tensor and the Ricci tensor on a Sasakian manifold M^{2n+1} is given by*

$$\rho^* = \rho - (2n-1)g - \eta \otimes \eta. \tag{2.5}$$

In view of (2.4), relation (2.5) gives

$$\hat{S}^*\xi = 0. \tag{2.6}$$

By virtue of (2.6), relation (1.10) gives

$$G_q^*(\xi,X_2)X_3 = R(\xi,X_2)X_3 + \alpha g(\hat{S}^*X_2,X_3)\xi - \beta\eta(X_3)\hat{S}^*X_2 + \gamma[g(X_2,X_3)\xi - \eta(X_3)X_2].$$

Taking the innerproduct with ξ and using (2.2), the above relation becomes

$$g(G_q^*(\xi,X_2)X_3,\xi) = \alpha[\rho(X_2,X_3) - (2n-1)g(X_2,X_3) - \eta(X_2)\eta(X_3)] + (1+\gamma)[g(X_2,X_3) - \eta(X_2)\eta(X_3)]. \tag{2.7}$$

In ([8]) Blair et al. introduced and studied a new type of contact metric manifolds known as a (k,μ)-contact metric manifolds. Thereafter, Boeckx ([10]) completely classified these manifolds. A contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ is said to be a (k,μ)-space if the Riemann curvature tensor admits

$$R(X_1,X_2)\xi = k\{\eta(X_2)X_1 - \eta(X_1)X_2\} + \mu\{\eta(X_2)hX_1 - \eta(X_1)hX_2\}, \tag{2.8}$$

for all vector fields X_1,X_2 on M^{2n+1} and for some real numbers (k,μ). This type of space arises through a D-homothetic deformation ([21]) to a contact metric manifold which satisfies $R(X_1,X_2)\xi = 0$. The class of (k,μ)-spaces covers Sasakian manifolds (for $k=1$) and the trivial sphere bundle $E^{n+1}\times S^n(4)$ (for $k=\mu=0$). There exist examples of non-Sasakian (k,μ)-contact metric manifolds. For instance, the unit tangent bundles of Riemannian manifolds of constant curvature $k\neq 1$. Since a D-homothetic deformation preserves (k,μ)-contact metric structures, one can construct a lot of examples of (k,μ)-contact structures (see [8]). We denote the symmetric tensor field operator $h=\frac{1}{2}\pounds_\xi\phi$ on M^{2n+1} such that

$$h\xi = 0, tr(h) = 0, tr(h\phi) = 0, h\phi + \phi h = 0,$$
$$\nabla_{X_1}\xi = -\phi X_1 - \phi h X_1.$$

For $\mu = 0$ in (2.8) the (k,μ)-contact metric manifolds reduces to $N(k)$-contact metric manifolds.

The following formulas are also valid for non-Sasakian (k,μ)-contact manifolds ([8]):

$$(\nabla_{X_1}h)X_2 - (\nabla_{X_2}h)X_1 = (1-k)\{2g(X_1,\phi X_2)\xi + \eta(X_1)\phi X_2 - \eta(X_2)\phi X_1\} + (1-\mu)\{\eta(X_1)\phi hX_2 - \eta(X_2)\phi hX_1\}, \tag{2.9}$$

$$\hat{S}X_1 = (2(n-1)-n\mu)X_1 + (2(n-1)+\mu)hX_1 + (2(1-n)+n(2k+\mu))\eta(X_1)\xi, \tag{2.10}$$

$$\hat{S}\xi = 2nk\xi, \tag{2.11}$$

$$(k-1)\phi^2 = h^2, \;\; k<1, \tag{2.12}$$

equality holds when $k=1$ (equivalently, $h=0$), i.e., M is Sasakian. For the non-Sasakian case, i.e., $k<1$, the (k,μ)-nullity condition determines the curvature of M completely. In view of this, Boeckx ([10]) proved that a non-Sasakian (k,μ)-contact metric manifold is locally homogeneous and hence analytic. Moreover, the constant scalar curvature r of such structures is given by

$$r = 2n\{2(n-1)+k-n\mu\}.$$

On a (k,μ)-contact metric manifolds, we have

$$(\nabla_\xi \hat{S})X_1 = \mu(2(n-1)+\mu)h\phi X_1, \tag{2.13}$$

for any vector field X_1 on M. On a non-Sasakian (k,μ)-contact metric manifolds, Ghosh and Patra ([13]) calculated the expression of the $*$-Ricci tensor,

$$\rho^*(X_1,X_2) = -(k+n\mu)(g(X_1,X_2)-\eta(X_1)\eta(X_2)), \tag{2.14}$$

which gives

$$r^* = -2n(k+n\mu). \tag{2.15}$$

Corollary 2.2. *In a (k,μ)-contact metric manifolds, the following are true*

$$\hat{S}^*\hat{S} = \hat{S}\hat{S}^*, \quad \hat{S}h = h\hat{S}. \tag{2.16}$$

3. Sasakian manifolds admitting the semisymmetry $R\cdot G_q^* = 0$

Suppose $(M^{2n+1},\phi,\xi,\eta,g)$ is a Sasakian manifold satisfying the relation $R\cdot G_q^*=0$. Then,

$$R(\xi,X_4)G_q^*(X_1,X_2)X_3 - G_q^*(R(\xi,X_4)X_1,X_2)X_3$$
$$-G_q^*(X_1,R(\xi,X_4)X_2)X_3 - G_q^*(X_1,X_2)R(\xi,X_4)X_3 = 0.$$

Taking the innerproduct with respect to ξ, we have

$$g(R(\xi,X_4)G_q^*(X_1,X_2)X_3,\xi) - g(G_q^*(R(\xi,X_4)X_1,X_2)X_3,\xi)$$
$$-g(G_q^*(X_1,R(\xi,X_4)X_2)X_3,\xi) - g(G_q^*(X_1,X_2)R(\xi,X_4)X_3,\xi) = 0.$$

Let $\{E_i\}_{i=1}^{2n+1}$ be the set of orthonormal vector fields on M^{2n+1}, then contracting over X_4 and X_1, the above equation yields

$$\sum_{i=1}^{2n+1} g(R(\xi,E_i)G_q^*(E_i,X_2)X_3,\xi) - \sum_{i=1}^{2n+1} g(G_q^*(R(\xi,E_i)E_i,X_2)X_3,\xi)$$
$$-\sum_{i=1}^{2n+1} g(G_q^*(E_i,R(\xi,E_i)X_2)X_3,\xi) - \sum_{i=1}^{2n+1} g(G_q^*(E_i,X_2)R(\xi,E_i)X_3,\xi) = 0. \tag{3.1}$$

Using (2.2), (2.7) and the properties of Sasakian metric, we bring out the following

$$\sum_{i=1}^{2n+1} g(R(\xi,E_i)G_q^*(E_i,X_2)X_3,\xi)$$
$$=\rho(X_2,X_3)+\{(2n-1)\alpha-\beta\}\{\rho(X_2,X_3)-(2n-1)g(X_2,X_3)-\eta(X_2)\eta(X_3)\}$$
$$+\{(r-4n^2)\beta+2n\gamma\}g(X_2,X_3)-(1+\gamma)\{g(X_2,X_3)-\eta(X_2)\eta(X_3)\},$$

$$\sum_{i=1}^{2n+1} g(G_q^*(R(\xi,E_i)E_i,X_2)X_3,\xi)=2n(\gamma+1)\{g(X_2,X_3)-\eta(X_2)\eta(X_3)\}$$
$$+2n\alpha\{\rho(X_2,X_3)-(2n-1)g(X_2,X_3)-\eta(X_2)\eta(X_3)\},$$

$$\sum_{i=1}^{2n+1} g(G_q^*(E_i,R(\xi,E_i)X_2)X_3,\xi)=-(\gamma+1)\{g(X_2,X_3)-\eta(X_2)\eta(X_3)\}$$
$$-\alpha\{\rho(X_2,X_3)-(2n-1)g(X_2,X_3)-\eta(X_2)\eta(X_3)\},$$

$$\sum_{i=1}^{2n+1} g(G_q^*(E_i,X_2)R(\xi,E_i)X_3,\xi)$$
$$=[2n-\alpha(2n-r)-2n\alpha(2n-1)+2n\gamma]\eta(X_2)\eta(X_3).$$

Next, using the above four relations in (3.1), we obtain

$$\rho(X_2,X_3)=\frac{\{\beta(r-4n^2)+(2n-1)\beta-1\}}{(\beta-1)}g(X_2,X_3)$$
$$+\frac{\{(4n^2-r)\alpha+\beta\}}{(\beta-1)}\eta(X_2)\eta(X_3). \tag{3.2}$$

Contracting the above equation, we have

$$r=\frac{4n^2(\alpha-2n\beta)-(2n+1)}{(\alpha-2n\beta-1)}. \tag{3.3}$$

Thus, we can state that:

Theorem 3.1. *Let (M^{2n+1},ϕ,ξ,η,g) be a Sasakian manifold admitting the semi-symmetric condition $R\cdot G_q^*=0$. Then, the manifold reduces to an η-Einstein manifold and is of constant scalar curvature (3.3).*

From relation (3.2), we can derive the following table by considering various particular choices of the parameters (α,β,γ):

$R\cdot G_q^* = 0$	(α,β,γ)	*Type*
$R\cdot H^* = 0$	$(-\frac{1}{2n-1}, -\frac{1}{2n-1}, 0)$	η-*Einstein*
$R\cdot C^* = 0$	$(-\frac{1}{2n-1}, -\frac{1}{2n-1}, \frac{r^*}{2n(2n-1)})$	η-*Einstein*
$R\cdot \tilde{C}^* = 0$	$(0,0,-\frac{r^*}{2n(2n+1)})$	*Einstein*
$R\cdot P^* = 0$	$(-\frac{1}{2n}, 0, 0)$	η-*Einstein*
$R\cdot \mathcal{M}^* = 0$	$(-\frac{1}{4n}, 0, 0)$	η-*Einstein*
$R\cdot \mathcal{W}_1^* = 0$	$(\frac{1}{2n}, 0, 0)$	η-*Einstein*
$R\cdot \mathcal{W}_2^* = 0$	$(0, -\frac{1}{2n}, 0)$	η-*Einstein*

In view of the above table, we can state:

Corollary 3.2. *Let (M^{2n+1},ϕ,ξ,η,g) be a Sasakian manifold that admits the semi-symmetric conditions $R\cdot H^* = 0, R\cdot C^* = 0, R\cdot P^* = 0, R\cdot \mathcal{M}^* = 0, R\cdot \mathcal{W}_1^* = 0$ and $R\cdot \mathcal{W}_2^* = 0$. Then the manifold reduces to an η-Einstein manifold.*

Corollary 3.3. *Let (M^{2n+1},ϕ,ξ,η,g) be a Sasakian manifold that admits the semi-symmetric condition $R\cdot \tilde{C}^* = 0$. Then the manifold reduces to an Einstein manifold.*

4. Sasakian metric admitting the structure $G_q^*\circ \hat{S}^* = 0$

Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a Sasakian manifold satisfying the relation $G_q^*\circ \hat{S}^* = 0$. Then

$$G_q^*(X_5,X_4)\circ \hat{S}^*X_1 = 0,$$

i.e.,

$$G_q^*(X_5,X_4)\hat{S}^*X_1 - \hat{S}^*(G_q^*(X_5,X_4)X_1) = 0.$$

Taking the inner product on both sides w.r.t. ξ, we have

$$\eta(G_q^*(X_5,X_4)\hat{S}^*X_1) - g(\hat{S}^*(G_q^*(X_5,X_4)X_1),\xi) = 0. \tag{4.1}$$

Again from (2.6), we get

$$g(\hat{S}^*(G_q^*(X_5,X_4)X_1),\xi) = 0. \tag{4.2}$$

Substituting (4.2) into (4.1), we get

$$\eta(G_q^*(X_5,X_4)\hat{S}^*X_1) = 0. \tag{4.3}$$

Now, recalling equation (1.10), we obtain

$$\begin{aligned} G_q^*(X_5,X_4)\hat{S}^*X_1 = {} & R(X_5,X_4)\hat{S}^*X_1 + \alpha[g(\hat{S}^*X_4,\hat{S}^*X_1)X_5 - g(\hat{S}^*X_5,\hat{S}^*X_1)X_4] \\ & + \beta[g(X_4,\hat{S}^*X_1)\hat{S}^*X_5 - g(X_5,\hat{S}^*X_1)\hat{S}^*X_4] \\ & + \gamma[g(X_4,\hat{S}^*X_1)X_5 - g(X_5,\hat{S}^*X_1)X_4]. \end{aligned} \tag{4.4}$$

Taking the inner product of both sides of (4.4) with respect to ξ, and using (2.2) and (2.5), we obtain

$$\begin{aligned}
&\eta(G_q^*(X_5,X_4)\hat{S}^*X_1)\\
&\quad=\{1+\gamma-(2n-1)\alpha\}[\rho^*(X_1,X_4)\eta(X_5)-\rho^*(X_1,X_5)\eta(X_4)]\\
&\quad\quad+\alpha[\rho^*(\hat{S}X_4,X_1)\eta(X_5)-\rho^*(\hat{S}X_5,X_1)\eta(X_4)].
\end{aligned}$$

In view of (4.3), the foregoing equation becomes

$$\begin{aligned}
&\{1+\gamma-(2n-1)\alpha\}[\rho^*(X_1,X_4)\eta(X_5)-\rho^*(X_1,X_5)\eta(X_4)]\\
&\quad=-\alpha[\rho^*(\hat{S}X_4,X_1)\eta(X_5)-\rho^*(\hat{S}X_5,X_1)\eta(X_4)].
\end{aligned}$$

Taking $X_4=\xi$ in the above equation, and using (2.5), we obtain

$$\{1+\gamma-(2n-1)\alpha\}\rho^*(X_1,X_5)+\alpha\rho^*(\hat{S}X_5,X_1)=0.$$

Next, using (2.5) in the above equation, we get

$$\begin{aligned}
&\{1+\gamma-2\alpha(2n-1)\}\rho(X_5,X_1)+\alpha\rho^2(X_1,X_5)\\
&\quad=(2n-1)\{1+\gamma-(2n-1)\alpha\}g(X_1,X_5)+(1+\gamma+\alpha)\eta(X_5)\eta(X_1).
\end{aligned}$$

Making a contraction in the above equation, we have

$$\begin{aligned}
\alpha\parallel\hat{S}\parallel^2&=(4n^2-1)\{1+\gamma-(2n-1)\alpha\}\\
&\quad+(1+\gamma+\alpha)-\{1+\gamma-2\alpha(2n-1)\}r.
\end{aligned}$$

This leads us to state:

Theorem 4.1. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a Sasakian manifold that satisfies the semi-symmetric condition $G_q^*\circ\hat{S}^*=0$. Then*

$$\begin{aligned}
\alpha\parallel\hat{S}\parallel^2&=(4n^2-1)\{1+\gamma-(2n-1)\alpha\}\\
&+(1+\gamma+\alpha)-\{1+\gamma-2\alpha(2n-1)\}r.
\end{aligned}$$

5. Sasakian metric with $Q(\rho,G_q^*)=0$

Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a Sasakian manifold that satisfies the curvature condition $Q(\rho,G_q^*)=0$. Then,

$$((X_1\wedge_\rho X_2)\circ G_q^*)(X_5,X_4)X_6=0,$$

i.e.,

$$\begin{aligned}
&(X_1\wedge_\rho X_2)G_q^*(X_5,X_4)X_6-G_q^*((X_1\wedge_\rho X_2)X_5,X_4)X_6\\
&-G_q^*(X_5,(X_1\wedge_\rho X_2)X_4)X_6-G_q^*(X_5,X_4)(X_1\wedge_\rho X_2)X_6=0,
\end{aligned}\tag{5.1}$$

for any vector fields X_1,X_2,X_5,X_4 and X_6 on M^{2n+1}.

Next, recalling relation (1.9), equation (5.1) yields

$$\begin{aligned}&\rho(X_2,G_q^*(X_5,X_4)X_6)X_1-\rho(X_1,G_q^*(X_5,X_4)X_6)X_2\\&-\rho(X_2,X_5)G_q^*(X_1,X_4)X_6+\rho(X_1,X_5)G_q^*(X_2,X_4)X_6\\&-\rho(X_2,X_4)G_q^*(X_5,X_1)X_6+\rho(X_1,X_4)G_q^*(X_5,X_2)X_6\\&=\rho(X_2,X_6)G_q^*(X_5,X_4)X_1-\rho(X_1,X_6)G_q^*(X_5,X_4)X_2.\end{aligned}\tag{5.2}$$

Putting $X_5=X_2=X_6=\xi$ in equation (5.2) and applying (1.10), (2.2), and (2.5), we obtain

$$\begin{aligned}&\beta\rho^2(X_4,X_1)+\{1+\gamma-(2n-1)\beta-2n\alpha\}\rho(X_4,X_1)\\&\quad=\{2n(1+\gamma)-2n\alpha(2n-1)\}g(X_4,X_1)+2n(\beta-\alpha)\eta(X_4)\eta(X_1).\end{aligned}\tag{5.3}$$

For the case $\alpha=\beta$, equation (5.3) implies that the Ricci tensor assumes either of the forms

$$\rho(X_4,X_1)=2ng(X_4,X_1),$$

or

$$\rho(X_4,X_1)=\frac{(2n-1)\alpha-(1+\gamma)}{\alpha}g(X_4,X_1).$$

In both cases, the Ricci tensor is proportional to the metric tensor g. Hence, M^{2n+1} is an Einstein manifold.

The conclusions of the above discussion are presented in the following table:

Restrictions	(α,β,γ)	*Type*
$Q(\rho,H^*)=0$	$(-\frac{1}{2n-1},-\frac{1}{2n-1},0)$	*Einstein*
$Q(\rho,C^*)=0$	$(-\frac{1}{2n-1},-\frac{1}{2n-1},\frac{r*}{2n(2n+1)})$	*Einstein*
$Q(\rho,\tilde{C}^*)=0$	$(0,0-\frac{r*}{2n(2n+1)})$	*Einstein*
$Q(\rho,P^*)=0$	$(-\frac{1}{2n},0,0)$	η-*Einstein*
$Q(\rho,\mathcal{M}^*)=0$	$(-\frac{1}{4n},-\frac{1}{4n},0)$	*Einstein*
$Q(\rho,\mathcal{W}_1^*)=0$	$(\frac{1}{n-1},0,0)$	η-*Einstein*
$Q(\rho,\mathcal{W}_2^*)=0$	$(0,-\frac{1}{n-1},0)$	$\beta\rho^2+\{1-(2n-1)\beta\}\rho$ $=2ng+2n\beta\eta\otimes\eta$.

Thus we can conclude the following:

Theorem 5.1. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a Sasakian manifold satisfying any one of the semi-symmetric conditions*

$$Q(\rho,H^*)=0,\quad Q(\rho,C^*)=0,\quad Q(\rho,\tilde{C}^*)=0,\quad Q(\rho,\mathcal{M}^*)=0.$$

Then, in each case, the manifold is Einstein.

Theorem 5.2. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a Sasakian manifold satisfying any one of the semi-symmetric conditions $Q(\rho,P^*)=0$, and $Q(\rho,\mathcal{W}_1^*)=0$. Then, in each case, the manifold is η-Einstein.*

Theorem 5.3. *Let* $(M^{2n+1},\phi,\xi,\eta,g)$ *be a Sasakian manifold satisfying the semi-symmetric condition* $Q(\rho,\mathcal{W}_2^*)=0$*. Then,*

$$\parallel \hat{S} \parallel^2 = (3n-2)r - 2n(n-1)(2n+1) + 2n.$$

6. $N(k)$-Contact Metric Manifolds with $\hat{S}\circ G_q^* = 0$

The k-nullity distribution ([22]) of a Riemannian manifold (M,g) for a real number k is a distribution

$$N(k): p \to N_p(k) = [X_3 \in T_pM : R(X_1,X_2)X_3 = k\{g(X_2,X_3)X_1 - g(X_1,X_3)X_2\}],$$

for any $X_1,X_2 \in T_pM$, where R is the Riemannian curvature tensor and T_pM is the tangent vector space of M^{2n+1} at any point $p \in M$. Thus if the characteristic vector field ξ belongs to the k-nullity distribution, then we obtain

$$R(X_1,X_2)\xi = k\{\eta(X_2)X_1 - \eta(X_1)X_2\}. \tag{6.1}$$

Therefore, a $N(k)$-contact metric manifold is a contact metric manifold whose characteristic vector field ξ satisfies (6.1). From (2.2) and (6.1) we see that a $N(k)$-contact metric manifold reduces to a Sasakian manifold for $k=1$.

In this section, we consider the $N(k)$-contact metric manifold that satisfies the curvature condition $\hat{S}\circ G_q^*=0$. That is,

$$\hat{S}(G_q^*(X_1,X_2)X_3) - G_q^*(\hat{S}X_1,X_2)X_3 - G_q^*(X_1,\hat{S}X_2)X_3 - G_q^*(X_1,X_2)\hat{S}X_3 = 0,$$

for any vector fields X_1,X_2 and X_3 in M^{2n+1}.

Taking $X_3=\xi$ in the foregoing equation, and using (2.11) we obtain

$$\hat{S}(G_q^*(X_1,X_2)\xi) - G_q^*(\hat{S}X_1,X_2)\xi - G_q^*(X_1,\hat{S}X_2)\xi - 2nkG_q^*(X_1,X_2)\xi = 0. \tag{6.2}$$

Now, in view of (2.8), equation (1.10) reduces to

$$G_q^*(X_1,X_2)\xi = R(X_1,X_2)\xi + (\gamma - k\beta)\{\eta(X_2)X_1 - \eta(X_1)X_2\}. \tag{6.3}$$

Next, using (6.3), (2.14), and (2.16) in (6.2), we obtain

$$kG_q^*(X_1,X_2)\xi = 0. \tag{6.4}$$

This implies that either $k=0$ or $G_q^*(X_1,X_2)\xi=0$.

Now, $k=0$ implies that the contact metric manifold M^{2n+1} is locally isometric to the Riemannian product of a flat $(n+1)$- dimensional Riemannian manifold and a n-dimensional manifold of positive curvature 4, i.e., $E^{n+1}\times S^n(4)$ ([9]).

Therefore, we can conclude the following:

Theorem 6.1. *Suppose (M^{2n+1},ϕ,ξ,η,g) is a $N(k)$-contact metric manifold that admits the semi-symmetric condition $\hat{S}\circ G^*_q = 0$. Then, either the manifold is locally isometric to the trivial sphere bundle $E^{n+1}\times S^n(4)$ or is ξ-star generalized quasi conformally flat.*

7. $N(k)$-Contact Metric Manifolds with $G^*_q \circ \rho = 0$

Let a $N(k)$-contact metric manifold admits the curvature condition $G^*_q\circ\rho = 0$. Therefore,

$$\rho(G^*_q(X_1,X_2))X_5,X_4)+\rho(X_5,G^*_q(X_1,X_2)X_4) = 0. \tag{7.1}$$

In view of (1.10), and (2.8), we obtain

$$\begin{aligned}\rho(G^*_q(\xi,X_2))X_5,X_4) = {}&(k+\gamma)[g(X_2,X_5)\rho(\xi,X_4)-\eta(X_5)\rho(X_2,X_4)]\\ &+\alpha\rho^*(X_2,X_5)\rho(\xi,X_4)-\beta\eta(X_5)\rho(\hat{S}^*X_2,X_4).\end{aligned} \tag{7.2}$$

Again, using (7.2) in relation (7.1), we obtain

$$\begin{aligned}&(k+\gamma)[g(X_2,X_5)\rho(\xi,X_4)-\eta(X_5)\rho(X_2,X_4)+g(X_2,X_4)\rho(\xi,X_5)\\ &-\eta(X_4)\rho(X_2,X_5)]+\alpha[\rho^*(X_2,X_5)\rho(\xi,X_4)+\rho^*(X_2,X_4)\rho(\xi,X_5)]\\ &-\beta[\eta(X_5)\rho(\hat{S}^*X_2,X_4)+\eta(X_4)\rho(\hat{S}^*X_2,X_5)] = 0.\end{aligned} \tag{7.3}$$

Taking $X_5 = \xi$ in (7.3), and using (2.14), we get

$$\begin{aligned}&(k\beta-k-\gamma)\rho(X_2,X_4)+[(k+\gamma)2n-2nk\alpha]g(X_2,X_4)\\ &+2nk(\alpha-\beta)\eta(X_2)\eta(X_4) = 0.\end{aligned} \tag{7.4}$$

Now, replacing X_4 by hX_4 in (7.4), and using (2.12), we obtain

$$(k\beta-k-\gamma)\rho(hX_4,X_2)+[(k+\gamma)2n-2nk\alpha]g(hX_4,X_2) = 0. \tag{7.5}$$

Again, replacing X_4 by hX_4 in (7.5), and using (2.12), we have

$$\begin{aligned}(k-1)[(k\beta-k-\gamma)\rho(X_2,X_4)+((k+\gamma)2n-2nk\alpha)g(X_2,X_4)&\\ +2nk(\alpha-\beta)\eta(X_2)\eta(X_4)] = 0.&\end{aligned} \tag{7.6}$$

This implies that either $k = 1$ or $(k\beta-k-\gamma)\rho(X_2,X_4)+((k+\gamma)2n-2nk\alpha)g(X_2,X_4)+2nk(\alpha-\beta)\eta(X_2)\eta(X_4) = 0$.

Now a $N(k)$-contact metric manifold is a Sasakian manifold if and only if $k = 1$.

Therefore we can conclude the following:

Theorem 7.1. *Suppose (M^{2n+1},ϕ,ξ,η,g) is a $N(k)$-contact metric manifold that admits the condition $G^*_q\circ\rho = 0$. Then the manifold reduces either to the Sasakian manifold or to the η-Einstein manifold.*

8. $N(k)$-Contact Metric Manifolds with $G_q^* \circ h = 0$

In this section we consider that a $N(k)$-contact metric manifold satisfies $G_q^* \circ h = 0$. Therefore

$$G_q^*(X_1, X_2)hX_3 - h(G_q^*(X_1, X_2)X_3) = 0. \tag{8.1}$$

In view of (1.10) and (2.8), we have

$$G_q^*(\xi, X_2)hX_3 = (k+\gamma)g(X_2, hX_3)\xi + \alpha\rho^*(X_2, hX_3)\xi,$$

and

$$h(G_q^*(\xi, X_2)X_3) = \{k(\beta - 1) - \gamma\}\eta(X_3)hX_2.$$

Substituting the above two relations into equation (8.1), we get

$$\{(k+\gamma) - \alpha k\}g(hX_2, X_3) = 0.$$

Replacing X_2 by hX_2 in the above equation, and using (2.12), we have

$$(k-1)\{(k+\gamma) - \alpha k\}[g(X_2, X_3) - \eta(X_3)\eta(X_2)] = 0.$$

After making a contraction, we obtain

$$(k-1)\{(k+\gamma) - \alpha k\} = 0.$$

This implies that $k = 1$ provided that $(k+\gamma) \neq \alpha k$. Now a $N(k)$-contact metric manifold is a Sasakian manifold if and only if $k = 1$.
Therefore we can state the following:

Theorem 8.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $N(k)$-contact metric manifold satisfying the curvature restriction*

$$G_q^* \circ h = 0.$$

Then M^{2n+1} reduces to a Sasakian manifold, provided $(\alpha - 1)k \neq \gamma$.

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References

[1] O. Bahadir, M. A. Choudhary, and S. Pandey, *LP-Sasakian manifolds with generalized symmetric metric connection*, Novi Sad J. Math, **51**(2) (2021), 75–87.
[2] K. K. Baishya and P. R. Chowdhury, *On generalized quasi-conformal $N(k,\mu)$-manifolds*, Commun. Korean Math. Soc., **31**(1) (2016), 163–176.
[3] M. R. Bakshi and K. K. Baishya, *Certain types of $(LCS)_n$-manifold and the case of the Riemannian soliton*, Differ. Geom. Dyn. Syst., **22** (2020), 11–25.
[4] M. R. Bakshi and K. K. Baishya, *Four classes of Riemann solitons on α-cosymplectic manifolds*, Afr. Mat., **32** (2021), 577–588.

[5] M. R. Bakshi, T. Barman, and K. K. Baishya, *The study of $*$-Ricci tensor on Lorentzian Para Sasakian manifolds*, Honam Mathematical J., **46**(1) (2024), 70–81.

[6] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds* Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, xii+260 pp. ISBN: 0-8176-4261-7, 2002.

[7] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics, Vol. 509. Springer-Verlag, Berlin-New York, 1976 vi+146 pp.

[8] D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math., **91** (1995), 189–214.

[9] D. E. Blair, *Two remarks on contact metric structures*, Tohoku Math. J., **29** (1977), 319-324.

[10] E. Boeckx, *A full classification of contact metric (k,μ)-spaces*, Illinois J. Math., **44**(1) (2000), 212–219.

[11] C. P. Boyer and K. Galicki, *Sasakian geometry*, Oxford Mathematical Monographs. Oxford University Press, Oxford, xii+613 pp. ISBN:978-0-19-856495-9, (2008).

[12] L. P. Eisenhart, *Riemannian Geometry*, 2d printing. Princeton University Press, Princeton, N. J., vii+306 pp. 1949.

[13] A. Ghosh and D, S. Patra, *$*$-Ricci soliton within the frame-work of Sasakian and (κ,μ)-contact manifold*, Int. J. Geom. Methods Mod. Phys., **15**(7) (2018), 1850120, 21 pp.

[14] T. Hamada, *Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor*, Tokyo J. Math., **25**(2) (2002), 473–483.

[15] Y. Ishii, (1957) *On conharmonic transformations*, Tensor(N.S.), **7** (1957), 73–80.

[16] G. Kaimakamis and K. Panagiotidou, *On a new type of tensor on real hypersurfaces in nonflat complex space forms*, Symmetry, **11**(4) (2019), 559. https://doi.org/10.3390/sym11040559.

[17] G. P. Pokhariyal, *Relativistic significance of curvature tensors*, Internat. J. Math. Math. Sci., **5**(1) (1982), 133–139.

[18] G. P. Pokhariyal and R. S. Mishra, *Curvature tensors and their relativistics significance*, Yokohama Math. J., **18** (1970), 105–108.

[19] G. P. Pokhariyal and R. S. Mishra, *Curvature tensors and their relativistic significance. II.*, Yokohama Math. J., **19**(2) (1971), 97–103.

[20] S. Tachibana, *On almost-analytic vectors in almost-Kählerian manifolds*, Tohoku Math. J., **11**(2) (1959), 247–265.

[21] S. Tanno, *The topology of contact Riemannian manifolds*, Illinois J. Math., **12** (1968), 700–717.

[22] S. Tanno, *Ricci curvatures of contact Riemannian manifolds*, Tohoku Math. J., 40(1988), 441–448.

[23] V. Venkatesha, D. M. Naik, and H. A. Kumara, *$*$-Ricci solitons and gradient almost $*$-Ricci solitons on Kenmotsu manifolds*, Math. Slovaca, **69**(6) (2019), 1447–1458.

[24] V. Venkatesha and H. A. Kumara, *$*$-Weyl curvature tensor within the framework of Sasakian and (κ,μ)-contact manifolds*, Tamkang J. Math., **52**(3) (2021), 383–395.

[25] L. Verstraelen, *Comments on pseudo-symmetry in the sense of Ryszard Deszcz*, Geometry and topology of submanifolds, VI (Leuven, 1993/Brussels, 1993),(1994), 199–209, World Sci. Publ., River Edge, NJ.

[26] S. K. Yadav, O. Bahadir, and S.K. Chaubey, *Almost Yamabe solitons on LP-Sasakian manifolds with generalized symmetric metric connection of type (α,β)*, Balkan J. of Geom. and App., **25**(2) (2020), 124–139.

[27] K. Yano and S. Bochner, *Curvature and Betti numbers*, Annals of Mathematics Studies, **32** (1953), Princeton University Press, Princeton, N. J., ix+190 pp.

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