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GRADED DIVISION RINGS

I. N. BALABA, A. V. MIKHALEV [1)]

Dedicated to Professor Mirjana Vuković n the occasion of her 70th birtday

ABSTRACT. In this paper the survey of the results concerning graded division rings is given. Even though that the graded division rings are not division rings in the usual sense the graded modules over graded division rings have properties similar to those of the linear spaces. We consider the properties of the graded linear spaces and graded rings of linear transformations, describe the isomorphisms and anti-isomorphism of graded rings of linear transformations.

1. INTRODUCTION

In the preface to the Russian edition of R. Baer‘s monograph [10] A. G. Kurosh wrote that in this book a new branch of algebra, projective algebra, is systematically presented. Projective algebra connected projective geometry with many sections of algebra: the theory of structures and division rings, general theory of associative rings and modules, the theory of classical groups.

Over the last decades, much attention has been given to algebraic objects with a group-graded structure. Special classes of rings and modules were associated with corresponding classes of graded rings and modules. In particular, instead of endomorphism rings of graded modules, it is natural to consider graded endomorphism rings.

A significant role in the theory of graded rings is played by the graded division rings, that is, graded rings in which every nonzero homogeneous element is invertible. Because graded modules over graded division rings have the properties similar to linear spaces, they are called graded linear spaces. We consider the properties of graded linear spaces and their endomorphism rings.

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A central problem in the study of operator rings of linear spaces and endomorphism rings of modules is the characterization of their isomorphisms. Isomorphisms of rings of linear transformations of vector spaces over division rings were characterized in [10]. In fact, the problem of characterizing isomorphisms of endomorphism rings of modules started from the Baer—Kaplansky theorem on the characterization of abelian p-groups by their endomorphism rings (the importance of the Baer—Kaplansky module approach in this problem was emphasized in [15]).

It is shown that each isomorphism of graded rings of linear transformations is induced by some semi-linear transformations of linear spaces [1], each anti-isomorphism of graded rings of linear transformations is induced by some anti-semilinear transformations. We also prove that any graded simple Artinian ring is a graded ring of linear transformations of some graded linear space.

Further authors obtained inducibility criteria for isomorphisms and anti-isomorphisms of graded endomorphism rings for the class of graded modules close to free ones (i.e., modules having a free cyclic direct summand) [4, 5, 6].

Various aspects of theory of graded division algebras have been investigated in [9, 19, 20]. In particular, in [9] a full classification, up to equivalence, of finite-dimensional graded division algebras over the field of real numbers with an abelian grading was given.

2. Preliminaries

Throughout the paper G is a multiplicative group with identity element e, the rings are associative G-graded with identity 1.

Recall certain definitions. For a graded ring $D = \bigoplus_{g\in G} D_g$ we denote by $h(D)$ the set of the homogeneous elements $\bigcup_{g\in G} D_g$. A nonzero element $a \in D_g$ is called the *homogeneous element of degree g*. An ideal I of D is called *graded* (or *homogeneous*) if $I = \bigoplus_{g\in G}(I \cap D_g)$. Similarly we define the set of the homogeneous elements $h(M)$ of a graded module M and a graded submodule $N \subseteq M$.

If $M = \bigoplus_{g\in G} M_g$ and $\sigma \in G$ then σ-*suspension* $M(\sigma)$ is the module M considered with grading $M(\sigma)_g = M_{\sigma g}$ for the right module and $M(\sigma)_g = M_{g\sigma}$ for the left module.

For the graded right D-modules M, N we denote by $\mathrm{HOM}_D(M,N)_g$ the set of the *homomorphisms of degree g*, that is, D-linear maps such that $f(M_h) \subseteq N_{gh}$ for all $h \in G$. In the consideration of left modules the expression $f \in HOM_D(M,N)g$ means that $(M_h)f \subseteq N_{hg}$.

It is clear that $\mathrm{HOM}_D(M,N) = \bigoplus_{g\in G} \mathrm{HOM}_D(M,N)_g$ is a G-graded abelian group, and $\mathrm{END}_D(M) = \mathrm{HOM}_D(M,M)$ is a G-graded ring, which will be called *the graded endomorphism ring* of the graded D-module M.

It is well known that if either G is finite or M is finitely generated, or both M and N have finite support then $\mathrm{HOM}_D(M,N) = \mathrm{Hom}_D(M,N)$, where $\mathrm{Hom}_D(M,N)$ is the set of all homomorphism from M_D to N_D [17, Corollary 2.4.4 – 2.4.6]. In general, the inclusion is proper.

A graded D-module F is said to be gr-free if it has a basis consisting of homogeneous elements, that is $F \cong \bigoplus_{j\in J} D(\sigma_j)$, where $\{\sigma_j, j \in J\}$ is a family of elements of G. Note that gr-free D-module is necessarily a free D-module when viewed as D-module by forgetting the grading. At the same time free graded D-module is not always gr-free.

The graded ring is called *graded division ring* if its every nonzero homogeneous element is invertible.

If $D = \bigoplus_{g\in G} D_g$ is G-graded division ring, then D_e is division ring and support $G' = \mathrm{Supp} D = \{g \in G \mid D_g \neq 0\}$ of D is subgroup of group G, D is strongly graded by G'. Moreover $D = D_e * G'$ is a crossed product of division ring D_e and a group G' (see [12]).

Example 2.1. A group ring $A = kG$, where k is a division ring with a canonical G-grading $A_g = kg$ is a graded division rings.

Example 2.2. The ring of Laurent polynomials $A = k[x, x^{-1}]$, where k is a division ring, has the standard $\mathbb{Z}$-grading $A_n = kx^n$ $(n \in \mathbb{Z})$. Clearly A is a $\mathbb{Z}$-graded division ring.

Further details on graded rings and modules may be found in [17]

3. GRADED LINEAR SPACES AND THEIR ENDOMORPHISM RINGS

The graded modules over graded division ring are called *graded linear spaces* because they have the properties similar to linear spaces.

The following lemma shows how homogeneous bases of graded linear spaces are constructed.

Lemma 3.1. *Let D be G-graded division ring, and $V = \bigoplus_{g\in G} V_g$ be right strongly graded D-module. If a family $\mathcal{B} = \{v_i \mid v_i \in V_{g_i}\}_{i\in I}$ is the homogeneous basis of V then a family $\{v_i d_i \mid v_i \in \mathcal{B},\ d_i \in D_{g_i^{-1}g} \setminus \{0\}\}_{i\in I}$ is a basis of right D_e-module V_g for every $V_g \neq 0$ $(g \in G)$. Conversely, if $\mathcal{B} = \{v_i\}_{i\in I}$ is a basis of right D_e-module V_g then $\{v_i\}_{i\in I}$ is the homogeneous basis of graded D-module V.*

Theorem 3.2. [1, 16] *All homogeneous bases of graded linear spaces V over graded division ring D have the same cardinal number.*

Remark that the elements of the homogeneous basis of a graded linear space V over graded division ring D are linearly independent over D and every graded subspace of a graded linear space has a complementary graded subspace.

Theorem 3.3. *A graded ring D with identity 1 is graded division ring if and only if every right (left) graded D-module is gr-free.*

Proof. Let D be a graded division ring and V be a nonzero right (left) graded D-module. Then any subset consisting of one homogeneous element of V is linearly independent over D. The set Θ of all linearly independent subsets of V satisfies the condition of Zorn's Lemma so there is maximum linearly independent homogeneous subset B in V It is easily check that B basis of V, hence V is gr-free.

Conversely, let every right graded D-module be gr-free, and $0 \neq a \in h(D)$. Since aD is gr-free $aD \cong D(g)$ for some $g \in G$. Therefore $ab = 1$ for some $b \in h(D)$ and D is graded division ring. □

A graded endomorphism ring $\mathrm{END}_D(V)$ of a graded linear D-spaces V is called the *graded ring of linear transformations*.

It is known (see, for example, [17, Proposition 2.10.5]) that if V is a graded finitely generated right module with a basis of homogeneous elements $v_1, v_2, ..., v_n$, $v_i \in V_{g_i^{-1}}$ $(i = 1, \dots, n)$ then $\mathrm{END}_D(V)$ is isomorphic to a graded matrix ring $M_n(D)(g_1, \dots, g_n) = \bigoplus_{h \in G} M_n(D)_h(g_1, \dots, g_n)$, where

$$M_n(D)_h(g_1, \dots, g_n) = \begin{pmatrix} D_{g_1^{-1}hg_1} & D_{g_1^{-1}hg_2} & \cdots & D_{g_1^{-1}hg_n} \\ D_{g_2^{-1}hg_1} & D_{g_2^{-1}hg_2} & \cdots & D_{g_2^{-1}hg_n} \\ \vdots & \vdots & \ddots & \vdots \\ D_{g_n^{-1}hg_1} & D_{g_n^{-1}hg_2} & \cdots & D_{g_n^{-1}hg_n} \end{pmatrix}.$$

Theorem 3.4. *Every graded ring of linear transformations* $\mathrm{END}_D(V)$ *of a graded linear space* V_D *is gr-regular, that is every its homogeneous element is regular in the sense of von Neumann.*

Proof. Let $\varphi \in (\mathrm{END}_D(V))_g$ then $\varphi(V)$ is a graded subspace of V so $\varphi(V)$ is direct summand of V. From [11, Proposition 1] it follows that the φ is a regular element in $\mathrm{End}_D(V)$, i.e. there exist such element $\psi \in \mathrm{End}_D(V)$ such that $\varphi\psi\varphi = \varphi$. Define $\psi' : V \to V$ as $\psi'(v) = \psi(v)_{g^{-1}h}$ for any $v \in V_h$, here $\psi(v)_{g^{-1}h}$ is a homogeneous component of $\psi(v)$ of degree $g^{-1}h$. It is clear that ψ' is homomorphism of g^{-1} degree such that $\varphi\psi'\varphi = \varphi$. Therefore $\mathrm{END}_D(V)$ is a gr-regular ring. □

The partially ordered set $(P; \leq)$ and $(Q; \leq)$ are called *isomorphic* (*anti-isomorphic*) if there exists isomorphism (anti-isomorphism) of sets P and Q, i.e. a one-to-one map $\varphi : P \to Q$ such that $a \leq b$ takes place for $a, b \in P$ if and only if $\varphi(a) \leq \varphi(b)$ $(\varphi(a) \geq \varphi(b))$.

The following theorem is a graded analog of the "triangle Galois theory" [10, chV, §2].

Theorem 3.5. [1, Theorem 2.3.] *Let V be a G-graded right linear space over graded division ring D, $A = \mathrm{END}_D(V)$ be its graded ring of linear transformations. Then following statements hold:*

1) *there exists an anti-isomorphism between the lattice of the graded subspaces of V and the lattice of the left graded annihilators of the graded ring A;*

2) *there exists an isomorphism between the lattice of the graded subspaces of V and the lattice of the right graded annihilators of the graded ring A;*

3) *there exists an anti-isomorphism between the lattices of the right and of the left graded annihilators of the graded ring A.*

4. ISOMORPHISMS AND ANTI-ISOMORPHISMS OF GRADED RINGS OF LINEAR TRANSFORMATIONS

A map $\varphi : A \to B$ of two G-graded rings is called *an isomorphism* of graded rings if it is a ring isomorphism and $\varphi(A)_g \subseteq B_g$ for all $g \in G$. A map $\varphi : A \to B$ of two G-graded rings is called *an anti-isomorphism* of graded rings if it is a ring anti-isomorphism and $\varphi(A)_g \subseteq B_{g^{-1}}$ for all $g \in G$.

It is clear that the anti-isomorphism of graded rings A and B is the isomorphism of graded ring A and the opposite graded ring B^{op}.

Definition 4.1. *Let V and W be the right graded linear spaces over graded division rings D and E correspondingly. A semi-linear σ-isomorphism ($\sigma \in G$) of graded linear spaces V_D and W_E is a pair (β,α), where $\alpha : D \to E$ is a ring isomorphism, $\beta : V \to W$ is an isomorphism of Abelian groups, if $(V_g)^\beta \subseteq W_{g\sigma}$, $(D_g)^\alpha \subseteq E_{\sigma^{-1}g\sigma}$ and $(vd)^\beta = v^\beta d^\alpha$ for all $v \in V,\ d \in D$.*

Theorem 4.2. [1, Theorem 3.1] *Let $A = \mathrm{END}_D(V)$ and $B = \mathrm{END}_E(W)$ be the graded ring of linear transformations of graded linear spaces V_D and W_E. Then $\phi : A \to B$ is an isomorphism of graded rings if and only if there exist an element $\sigma \in G$ and a semi-linear σ-isomorphism (β,α) of graded linear spaces V_D and W_E such that $f^\phi = \beta f \beta^{-1}$ for any $f \in A$.*

As a corollary, we obtain Rasin‘s theorem on inducibility endomorphisms of finite-dimensional superspaces semilinear isomorphism [18].

Definition 4.3. *Let ${}_DV$ be a left graded linear D-space and W_E be a right graded linear E-space. An anti-semilinear σ-isomorphism ($\sigma \in G$) of graded linear spaces ${}_DV$ and W_E is a pair (α,β), where $\alpha : D \to E$ is a ring anti-isomorphism, $\beta : V \to W$ is an isomorphism of Abelian groups, if $(V_g)^\beta \subseteq W_{g^{-1}\sigma}$, $(D_g)^\alpha \subseteq E_{\sigma^{-1}g^{-1}\sigma}$ and $(dv)^\beta = v^\beta d^\alpha$ for all $v \in V,\ d \in D$.*

If V_D is a right graded linear D-space, then the dual module $V^* = HOM(V_D, D_D)$ is a left graded linear D-space and there exists a canonical homomorphism $\omega_V : V \to V^{**}$ such that $(f)\omega_V(v) = f(v)$ for all $v \in V, f \in V^*$.

The linear space V_D is said to be *torsionless* if ω_V is a monomorphism, and *reflexive* if ω_V is an isomorphism.

If V_D is a torsionless module, then there exists a unique monomorphism $\varphi : \mathrm{END}(V_D) \to END({}_DV^*)$ such that $f(\eta v) = (f\eta^*)v$ for all $v \in V,\ f \in V^*$ (here $\eta^* = \varphi(\eta)$).

Since a graded linear space V_D is reflexive if and only if V_D is finite-dimensional from [2, Theorem 2] we get the following theorem.

Theorem 4.4. *Let $_DV$ and W_E be graded linear spaces over graded division rings D and E correspondingly. Then $\phi : \mathrm{END}_D(V) \to \mathrm{END}_E(W)$ is an anti-isomorphism of graded rings if and only if $_DV$ and W_E are finite-dimensional spaces and there exist an element $\sigma \in G$ and anti-semilinear σ-isomorphism (α,β) of graded linear spaces V_D^* and W_E such that $(f\eta^*)^\beta = \eta^\phi f^\beta$ for any $f \in V^\star$, $\eta \in \mathrm{END}_D(V)$.*

5. Good grading on matrix rings

A grading of the matrix rings $M_n(k)$ over a field k is called *good* (or *elementary*) if all matrix units E_{ij} are homogeneous elements (see [7, 13]). All these gradings can be obtained as graded rings of linear transformations of some graded linear space over field k.

Remark that there exist gradings of the matrix rings which are not graded endomorphism ring of the graded modules. An important place in the theory of graded rings is the problem of description all gradings. In [13] a description of all $\mathbb{Z}_2$-gradations $M_2(k)$ is given. In [7] has been completely described all abelian gradings on a matrix algebra over an algebraically closed field of characteristic zero, and in [8] this result was generalized to the case of non-abelian gradings.

Following [13], G-grading of the matrix rings $M_n(A)$ over arbitrary graded rings A is called *good*, if there exist a finitely generated gr-free right A-module F such that $M_n(A) \approx \mathrm{END}_A(F)$ (as graded rings).

The following theorem gives a description of gr-simple gr-Artinian rings and refines the results of [17, Theorem 2.10.10] and [14, Proposition 1.3].

Theorem 5.1. *Let $A = \bigoplus_{g\in G} A_g$ be gr-simple gr-Artinian ring. Then A is isomorphic the matrix ring with good grading over some graded division ring D. Besides if $A \cong M_n(D)(\overline{g}) \cong M_m(E)(\overline{h})$, then $n = m$ and there exit $\sigma \in G$ and ring isomorphism $\beta : D \to E$ such that $\beta(D_g) = E_{\sigma^{-1}g\sigma}$.*

Proof. Since A is gr-Artinian ring then its graded Jacobson radical $\mathcal{J}_{gr}(A) = 0$. As A is gr-simple ring then it is gr-primitive. Using [16, Proposition 2.8] we have $A \cong \mathrm{END}_D(V)$ for some graded division ring D and finite generated graded D-module V_D. Thus, $A \cong M_n(D)(\overline{g})$.

Let $A \cong M_m(E)(\overline{h})$ then $A \cong \mathrm{END}_E(W)$ for some finite generated graded W_E over graded division ring E. From Theorem 4.2 it follows that there exist $\sigma \in G$ semi-linear σ-isomorphism linear spaces V_D and W_E that induces an isomorphism of graded rings $\varphi : \mathrm{END}_D(V) \to \mathrm{END}_E(W)$. Thus, $n = m$ and there exist ring isomorphism $\beta : D \to E$ such that $\beta(D_g) = E_{\sigma^{-1}g\sigma}$. □

Since a matrix ring over division ring is simple Artinian ring it is gr-simple gr-Artinian ring too.

Applying Theorem 6.1 we get

Corollary 5.2. *Let $M_n(K)$ be a matrix ring over division ring K graded by a group G. Then there exist natural number m, that is a divisor of n, graded division ring D and $\overline{g} = (g_1, \ldots, g_m) \in G^m$ such that $M_n(K) \cong M_m(D)(\overline{g})$.*

6. Semisimple Graded Rings

A graded ring A is called right gr-semisimple if it is a direct sum of minimal right graded ideals. This means that A_A is semisimple object of the category of right graded modules.

It is khown, A is right gr-semisimple ring if and only if it is left gr-semisimple [17, Propositin 2.9.5] so we will called such rings gr-semisimple.

The following theorem is the the graded version of Wedderburn—Artin theorem

Theorem 6.1. [3, Theorem 4] *For a graded ring A following statements are equivalent:*

1) *A is gr-semisimple ring;*

2) *A is gr-Artinian right (left) with zero graded Jacobson radical $\mathcal{J}_{gr}(A)$;*

3) *for every right (left) graded ideal L in A there exist homogeneous idempotent $e \in A$ such that $L = eA$ ($L = Ae$);*

4) *A is gr-Artinian right (left) and gr-regular ring;*

5) *A is gr-Noether rings right (left) and gr-regular ring;*

6) *A is gr-Artinian right (left) and gr-semiprime ring;*

7) *A is direct sum of matrix rings over graded division ring with good grading.*

Notice that the condition of gr-semisimplicity is weaker than the condition of semisimplicity. The semisimple graded ring is gr-semisimple but a group rings AG is semisimple if and only if ring A is a semisimple ring and G is a finite group with order invertible in A [21, Theorem 12.2].

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Irina N. Balaba
Tula State Lev Tolstoy Pedagogical University
Faculty of Mathematics,
Physics and Information Technologies
125 Prospect Lenina, 300026 Tula, Russia
e-mail: *ibalaba@mail.ru*

Alexander V. Mikhalev
Lomonosov Moscow State University
Faculty of Mechanics and Mathematics
High Algebra Department
1 Leninskiye Gory, Main Building,
119991, Moscow, GSP-1, Russia