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MIXED MATRIX PROBLEMS IN THE THEORY OF REPRESENTATIONS

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Dedicated to Professor Mirjana Vuković on the occasion of her 70th birthday

ABSTRACT. In this paper we consider flat mixed matrix problems, i.e. the problems of reducing a family of matrices by some family of admissible transformations. We present some examples of flat matrix problems of unbounded representation type which are considered over a discrete valuations ring with a skew field of fractions and its skew field extension of the second degree.

1. INTRODUCTION

Matrix problems, i.e. the problems of reducing a family of matrices by some family of admissible transformations, arise in many problems of representation theory of rings, algebras and species. The general definition of a matrix problem was given by Roiter [12] over a field and then was generalized by Drozd [4] to matrix problems over rings. This paper is devoted to the study of flat mixed matrix problems over a discrete valuation ring (DVR) with a skew field of fractions and its skew field extensions. Following to R.B.Warfield, Jr. [13] a ring is of bounded representation type if there is an upper bound on the number of generators required for indecomposable finitely presented right modules. Analogously we can define flat matrix problems of bounded representation type [10]. In this paper we present some examples of flat matrix problems of unbounded representation type which are considered over a DVR with a skew field of fractions and its skew field extension of the second degree.

Let $\mathcal{O}$ be a DVR (not necessary commutative) with a skew field of fractions D. Let D' be a skew field which is a finite extension of D. Assume that there exists a fixed field $K \subset D \subset D'$ such that $\dim_K D < \infty$ and $\dim_K D' < \infty$. Moreover, in this paper we assume that $K \subset C$ and $K \subset C'$, where C and C' are centers of D and

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D' correspondingly. Then $\dim_C D < \infty$ and $\dim_{C'} D' < \infty$. We denote $[D' : D]_r$ the dimension of D' as a right vector space over D and $[D' : D]_l$ the dimension of D' as a left vector space over D. Then under our assumptions $[D' : D]_r = [D' : D]_l < \infty$. We denote this common dimension as $[D' : D]$.

Since $O \cap K$ is a DVR, it can be shown that there exists a maximal order O' in a skew field D'. From results of [1] it follows that:

1) O' is a hereditary ring;

2) the Jacobson radical R' of the ring O' is a maximal two-sided ideal;

3) $\bigcap\limits_{i=1}^{\infty} (R')^i = 0$;

4) O' is a principal ideal ring.

It can be constructed the localization $\overline{O} = O'_{R'}$ by the maximal two-sided principal ideal R' in the sense of the paper [2], which is a DVR with the unique maximal ideal $\overline{R}$, moreover $O \subset O' \subset \overline{O}$.

In this paper we will consider skew field extensions of the second degree. Let $D \subset D'$ and $[D' : D] = 2$. If $\mathrm{char} D \neq 2$ then D' is the Galois field over D and there is an element $\alpha \in D' \setminus D$ such that $\alpha^2 \in D$ and $\alpha D = D\alpha$ [11].

Let O be a DVR with a classical skew field of fractions D. Consider the ring $\alpha O \alpha^{-1} \subset D$. Then O and $\alpha O \alpha^{-1}$ are hereditary principal ideal domains and therefore they are maximal orders in D [1]. Moreover, they are conjugated by inner automorphism of D, that is there exists an element $d \in D$ such that $O = d(\alpha O \alpha^{-1})d^{-1} = (d\alpha)O(d\alpha)^{-1}$. Denote $\alpha_1 = d\alpha$, then $\alpha_1 D \subset D\alpha_1$ and $\alpha_1^2 \in D$. Since O is a DVR we can choose $\alpha_1^2 \in O$. So in this case there is an element $\alpha \in D' \setminus D$ such that $\alpha^2 \in O$, $\alpha O = O\alpha$ and $\alpha D = D\alpha$.

Let $\mathrm{char} D = 2$. If C is a center of D and C' is a center of D' then two cases are possible:

1) $C' \not\subset C$

2) $C' \subset C \subset D \subset D'$.

In the first case there is an element $\alpha \in C' \setminus C$ such that $\alpha^2 \in C$ or $\alpha^2 + \alpha \in C$. Then $\alpha \in D' \setminus D$ and $\alpha d = d\alpha$ for any $d \in D$. Moreover, we can consider that $\alpha^2 \in O$ or $\alpha^2 + \alpha v \in O$, where $v \in O \cap C$.

In the second case $[C : C'] = [D' : D] = 2$ and two cases are possible:

a) the field C is a separable extension of the field C'. Then C is a Galois field over C' and there is an element $\alpha \in D' \setminus D$ such that $\alpha O = O\alpha$ and $\alpha^2 \in O$ [3].

b) the field C is a pure nonseparable extension of the field C'. Then there is an element $\alpha \in D' \setminus D$ such that $\alpha d + d\alpha \in D$ for any $d \in D$ and $\alpha^2 + \alpha \in D$ [3].

So we will distinguish two essentially different cases:

I. There is an element $\alpha \in D' \setminus D$ such that $\alpha^2 \in O$ and $\alpha O = O\alpha$.

II. There is an element $\alpha \in D' \setminus D$ such that $\alpha^2 + \alpha \in D$ and $\alpha d + d\alpha \in D$ for any $d \in D$.

We use the notions and results from [7], [8] and [9]. Throughout this paper all rings are assumed to be associative with $1 \neq 0$ and all modules are assumed to be unital.

2. Mixed matrix problems over discrete valuation rings and skew fields

In this section we consider some flat mixed matrix problems over DVRs $\mathcal{O}_i$ for $i = 1, \ldots, k$ and their common skew field of fractions D and its skew field extension D'. These matrix problems generalize a flat matrix problem considered by Zavadskij and Revitskaya [14]. Some examples of such flat matrix problems were considered in [5], [6].

Let $\mathcal{O}$ be a DVR with a classical skew field of fractions D and let D' be a skew field extension of D.

By left $\mathcal{O}$-elementary transformations of rows of a matrix $\mathbf{T}$ with entries in D we mean transformations of two types:

(a) multiplying a row on the left by an invertible element of $\mathcal{O}$;

(b) adding a row multiplied on the left by an element of $\mathcal{O}$ to another row.

In a similar way we can define left D-elementary (D'-elementary) transformations of rows and, by symmetry, right $\mathcal{O}$-elementary and right D-elementary (D'-elementary) transformations of columns.

Elementary transformations (a) and (b) can be given by invertible elementary matrices. Multiplications on the left (right) of a matrix $\mathbf{T}$ by elementary matrices correspond to elementary row (column) transformations.

Let $\Delta = \{\mathcal{O}_i\}_{i=1,\ldots,k}$ be a family of discrete valuation rings $\mathcal{O}_i$ with a common skew field of fractions D and its skew field extension D'. Consider the following flat matrix problem over Δ and D'.

Main flat mixed matrix problem.

Let $\Delta = \{\mathcal{O}_i\}_{i=1,\ldots,k}$ be a family of discrete valuation rings $\mathcal{O}_i$ with a common skew field of fractions D and its skew field extension D'.

Let $\mathbf{T}$ be a block rectangular matrix with entries in D' partitioned into n horizontal strips $\{\mathbf{T}_i\}_{i=1,\ldots,n}$ and m vertical strips $\{\mathbf{T}^j\}_{j=1,\ldots,m}$ so that each block $\mathbf{T}_i^j$ is the intersection of the j-th vertical strip and the i-th horizontal strip, some of these matrices may be empty.

One has the following admissible transformations with the matrix $\mathbf{T}$:

1. Left F_{i_s}-elementary transformations of rows within the strip $\mathbf{T}_i$, where $F_{i_s} \in \Delta \cup D'$.

2. Right F_{j_t}-elementary transformations of rows within the strip $\mathbf{T}^j$, where $F_{j_t} \in \Delta \cup D'$.

The admissible transformations with the matrix $\mathbf{T}$ can be given in the form $\mathbf{T} \mapsto \mathbf{XTY}$, where $\mathbf{X} = \mathrm{diag}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ and $\mathbf{Y} = \mathrm{diag}(\mathbf{Y}_1, \ldots, \mathbf{Y}_m)$, and all $\mathbf{X}_i$ and

$\mathbf{Y}_j$ are square invertible matrices. Moreover, $\mathbf{X}_i \subset M_{m_i}(F_{i_s})$ and $\mathbf{Y}_j \subset M_{k_j}(F_{j_t})$, where $F_{i_s}, F_{j_t} \in \Delta \cup D'$.

Indecomposable matrices and equivalent matrices are defined in a natural way.

A flat matrix problem is said to be of **finite type** if the number of non-equivalent indecomposable matrices is finite.

Definition 2.1. [14] *The vector*

$$d = d(\mathbf{T}) = (d_1, d_2, \dots, d_n; d^1, d^2, \dots d^m), \tag{2.1}$$

where d_i is the number of rows of the i-th horizontal strip of $\mathbf{T}$ for $i = 1, \dots, n$ and d^j is the number of columns of the j-th vertical strip of $\mathbf{T}$ for $j = 1, \dots, m$, is called the **dimension vector** *of the partition matrix $\mathbf{T}$. We set*

$$\dim(\mathbf{T}) = \sum_{i=1}^{n} d_i + \sum_{j=1}^{m} d^j. \tag{2.2}$$

Definition 2.2. [14] *We say that a flat matrix problem is of* **bounded representation type** *if there is a constant C such that $\dim(\mathbf{X}) < C$ for all indecomposable matrices $\mathbf{X}$. Otherwise it is of* **unbounded representation type**.

This paper is devoted to study flat mixed matrix problems of unbounded representation type considered over a DVR with a skew field of fractions and its skew field extension of the second degree. These matrix problems arise, in particular, in considering representations of $(D, \mathcal{O})$-species and some classes of semidistributive, hereditary and semihereditary rings [10].

3. Mixed matrix problems of unbounded representation type

Matrix Problem I.

Let $\mathcal{O}$ be a DVR ring with a skew field of fractions D and the Jacobson radical $R = \pi\mathcal{O} = \mathcal{O}\pi$, and D' a skew field extension of D of degree 2.

Consider a block rectangular matrix $\mathbf{T}$ with entries in D' partitioned into two vertical and two horizontal strips

$$\mathbf{T} = \begin{array}{|c|c|} \hline \mathbf{A} & \mathbf{I} \\ \hline \mathbf{O} & \mathbf{B} \\ \hline \end{array} \tag{3.1}$$

where $\mathbf{A} \in M_{m\times n_1}(D')$, $\mathbf{B} \in M_{k\times m}(D')$, $\mathbf{I} \in M_m(D')$ is the identity matrix and $\mathbf{O} \in M_{k\times n_1}(D')$ is the zero matrix.

Assume that on the matrix $\mathbf{T}$ one has the following admissible transformations:

1. *Left D'-elementary transformations of rows within any horizontal strip of the matrix $\mathbf{T}$.*
2. *Right $\mathcal{O}$-elementary transformations of columns within the first vertical strip of the matrix $\mathbf{T}$.*
3. *Right D'-elementary transformations of columns within the second vertical strip of the matrix $\mathbf{T}$.*

Moreover, these transformations should not change the given matrix **I**.

Lemma 3.1. *The matrix problem I is of unbounded representation type.*

Proof. I. Consider the case I, i.e. there is an element $\alpha \in D' \setminus D$ such that $\alpha^2 \in O$ and $\alpha O = O\alpha$. Let $R = \pi O = O\pi$. We set $n_1 = 2n$, $m = n$, $k = 1$ and

$$\mathbf{A} = \left[\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & \alpha & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \alpha\pi^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & \alpha\pi^{2(n-1)} \end{array}\right], \tag{3.2}$$

$$\mathbf{B} = \left[\begin{array}{ccccc} \pi^{n-1} & \pi^{n-2} & \cdots & \pi & 1 \end{array}\right]. \tag{3.3}$$

We show that the matrix $\mathbf{T}$ of the form (3.1) with the pair of matrices (3.2), (3.3) is indecomposable for any $n > 0$. For this aim it is sufficient to show that the following matrix equalities:

$$\mathbf{S}_1\mathbf{A} = \mathbf{A}\mathbf{W}, \quad \mathbf{S}_2\mathbf{B} = \mathbf{B}\mathbf{S}_1 \tag{3.4}$$

where

$$\mathbf{W}^2 = \mathbf{W} \in M_{n_1}(O), \;\; \mathbf{S}_1^2 = \mathbf{S}_1 \in M_m(D'), \;\; \mathbf{S}_2^2 = \mathbf{S}_2 \in M_k(D'), \tag{3.5}$$

hold only if $\mathbf{W}$, $\mathbf{S}_1$, $\mathbf{S}_2$ are either all the identity matrices or all the zero matrices.

Let $\mathbf{W} = (x_{ij})$, $\mathbf{S}_1 = (y_{kl})$, $\mathbf{S}_2 = (z)$, where $x_{ij} \in O$, and $y_{kl}, z \in D'$. In this case the matrix equalities (3.4) give the following system of linear equations for x_{ij}, y_{kl} and z:

$$y_{ij} = x_{ij} + \alpha\pi^{2(i-1)}x_{n+i,j} \;\; (i, j = 1, 2, ..., n) \tag{3.6}$$

$$y_{ij}\alpha\pi^{2(j-1)} = x_{i,n+j} + \alpha\pi^{2(i-2)}x_{n+i,n+j} \;\; (i, j = 1, 2, ..., n) \tag{3.7}$$

$$z\pi^{n-i} = \pi^{n-1}y_{1i} + \pi^{n-2}y_{1i} + \pi^{n-2}y_{2i} + \cdots + \pi y_{n-1,i} + y_{ni} \;\; (i = 1, ..., n) \tag{3.8}$$

Since $\mathbf{S}_2^2 = \mathbf{S}_2$, we have $z^2 = z$, and so $z = 0$ or $z = 1$.

1) If $z = 0$ then from (3.6) and (3.8) it follows that

$$(\pi^{n-1}x_{1i} + \pi^{n-2}x_{2i} + \cdots + \pi x_{n-1,i} + x_{ni}) +$$
$$(\pi^{n-1}\alpha x_{n+1,i} + \pi^{n-2}\alpha\pi^2 x_{n+2,i} + \pi\alpha\pi^{2(n-2)}x_{2n-1,i} + \alpha\pi^{2(n-1)}x_{2n,i}) = 0.$$

Taking into account that elements $1, \alpha$ are linear independent over D and $\alpha O = O\alpha$, $\pi \in D$, we have:

$$\pi^{n-1}x_{1i} + \pi^{n-2}x_{2i} + \cdots + \pi x_{n-1,i} + x_{ni} = 0 \tag{3.9}$$

$$\pi^{n-1}\varepsilon_1 x_{n+1,i} + \pi^n\varepsilon_2 x_{n+2,i} + \cdots + \pi^{2n-3}\varepsilon_{n-1}x_{2n-1,i} + \pi^{2n-2}x_{2n,i} = 0, \tag{3.10}$$

where $\varepsilon_i \in O^*$, $i = 1, ..., n+1$, which implies that $x_{n,i} \in R$, $x_{n+1,i} \in R$ $(i = 1, ..., n)$. Analogously from equalities (3.7) and (3.8) we obtain that $x_{n,n+i} \in R$, $x_{n+1,n+i} \in R$ $(i = 1, ..., n)$. This implies that

$$x_{n,i} \in R, \; x_{n+1,i} \in R \text{ for } \; (i = 1, ..., 2n). \tag{3.11}$$

Substituting (3.6) into (3.7) we get:

$$x_{ij}\alpha\pi^{(2(j-1)} + \alpha\pi^{2(i-1)}x_{n+i,j}\alpha\pi^{2(j-1)} = x_{i,n+j} + \alpha\pi^{2(i-1)}x_{n+i,n+j}$$

for $i, j = 1, \ldots, n$. Since $\alpha O = O\alpha$ and $\alpha^2 \in O$,

$$x_{ij}\pi^{2(j-1)} = \pi^{2(i-1)}x'_{n+i,n+j}, \tag{3.12}$$

$$x'_{ij}\pi^{2(j-1)} = \pi^{2(i-1)}x_{n+i,n+j}, \tag{3.13}$$

$$x_{i,n+j} = \alpha^2\pi^{2(i-1)}\pi^{2(j-1)}x'_{n+i,j}, \tag{3.14}$$

and $x'_{ij} = x_{ij}\varepsilon_{ij}$, where $\varepsilon_{ij} \in O^*$ $(i, j = 1, 2, \ldots, 2n)$.

Taking into account (3.11), we obtain from (3.12)–(3.14) that $x_{ij} \in R$ if $i > j$, $x_{n+i,n+j} \in R$ if $i < j$ and $x_{i,n+j} \in R$, for all $i, j = 1, .., n$. Moreover, substituting (3.12)–(3.13) into (3.9)–(3.10) we obtain that $x_{ii} \in R$ and $x_{n+i,n+i} \in R$ for all $i = 1, \ldots, n$. Thus, $x_{ij} \in R$ if $i \geq j$, $x_{n+i,n+j} \in R$ if $i \leq j$ and $x_{i,n+j} \in R$, for all $i, j = 1, .., n$. Therefore the matrix $\mathbf{W}$ has the following form:

$$\mathbf{W} = \left(\begin{array}{c|c} W_{11} & W_{12} \\ \hline W_{21} & W_{22} \end{array}\right),$$

where

$$W_{12} \in \begin{pmatrix} R & O & \cdots & O \\ R & R & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ R & R & \cdots & R \end{pmatrix}, \quad W_{22} \in \begin{pmatrix} R & R & \cdots & R \\ O & R & \cdots & R \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & R \end{pmatrix}$$

$W_{12} \in M_n(R)$, and $W_{21} \in M_n(O)$.

Since $\mathbf{W}^2 = \mathbf{W}$ and $\bigcap\limits_{i=1}^{\infty} R^i = 0$, we obtain that $\mathbf{W} = \mathbf{O}$. Then from (3.6) it follows that also $\mathbf{S}_1 = \mathbf{O}$.

2) If $z = 1$ then we may consider the matrices $\mathbf{W}'$, $\mathbf{S}'_1$, $\mathbf{S}'_2$ which are connected with matrices $\mathbf{W}$, $\mathbf{S}_1$, $\mathbf{S}_2$ by the following:

$$\mathbf{W}' = \mathbf{I} - \mathbf{W}, \;\; \mathbf{S}'_1 = \mathbf{I} - \mathbf{S}_1, \;\; \mathbf{S}'_2 = \mathbf{I} - \mathbf{S}_2. \tag{3.15}$$

Substituting these relations to (3.4) – (3.5) we obtain:

$$\mathbf{S}'_1\mathbf{A} = \mathbf{A}\mathbf{W}', \quad \mathbf{S}'_2\mathbf{B} = \mathbf{B}\mathbf{S}'_1. \tag{3.16}$$

Taking into account conditions (3.5) we obtain the analogous conditions:

$$\mathbf{W}'^2 = \mathbf{W}', \;\; {\mathbf{S}'_1}^2 = \mathbf{S}'_1, \;\; {\mathbf{S}'_2}^2 = \mathbf{S}'_2.$$

Since $z = 1$, $\mathbf{S}'_2 = 0$ and so we have the previous case. Therefore $\mathbf{W}' = \mathbf{O}$, $\mathbf{S}'_1 = \mathbf{O}$ that implies $\mathbf{W}$, $\mathbf{S}_1$ and $\mathbf{S}_2$ are the identity matrices. So in this case the matrix $\mathbf{T}$ is indecomposable. Since $d(\mathbf{T}) = 4n + 1$, the matrix problem I is of unbounded representation type.

II. Consider now the case II, i.e. there is an element $\alpha \in D' \setminus D$ such that $\alpha^2 + \alpha \in D$ and $d\alpha + \alpha d \in D$ for any $d \in D$. Let $\mathcal{O}'$ be a maximal order in the skew field D' with the Jacobson radical R', and $\mathcal{O}'_{R'}$ the localization $\mathcal{O}'$ over the radical R'. Denote $\overline{\mathcal{O}} = \mathcal{O}'_{R'}$. Then $\overline{\mathcal{O}}$ is a local ring with the unique maximal ideal $\overline{R}$, and $\mathcal{O} \subset \mathcal{O}' \subset \overline{\mathcal{O}}$. We can consider that $\alpha \in \overline{\mathcal{O}}$. We choose an element $\pi \in \overline{R} \cap \mathcal{O}$ and $\pi \neq 0$.

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{W}$, $\mathbf{S}_1$, $\mathbf{S}_2$ be matrices as in the first case. Then we obtain system (3.6)-(3.8). Since $z^2 = z$, then $z = 0$ or $z = 1$.

1) If $z = 0$ then from equations (3.6), (3.8), taking into account that $1, \alpha$ are linear independent over D and $\alpha d = d\alpha + d_1$ for any $d \in D$, $\alpha \in \overline{\mathcal{O}}$, we get that

$$\pi^{n-1}x_{1i} + \pi^{n-2}x_{2i} + \cdots + \pi x_{n-1,i} + x_{ni} +$$
$$+\pi^{n-1}r_1 x_{n+1,i} + \pi^{n-2}r_2\pi^2 x_{n+2,i} + \cdots + \pi r_{n-1}\pi^{2(n-2)}x_{2n-1,i} = 0 \tag{3.17}$$

where $r_1, \ldots, r_{n-1} \in \mathcal{O}$.

$$\pi^{n-1}x_{n+1,i} + \pi^n x_{n+2,i} + \cdots + \pi^{2n-3}x_{2n-1,i} + \pi^{2n-2}x_{2n,i} = 0 \tag{3.18}$$

for $i = 1, \ldots, n$. Therefore $x_{n,i} \in \overline{R}$, $x_{n+1,i} \in \overline{R}$ for $i = 1, \ldots, n$.

Analogously from equations (3.7) and (3.8) we get that $x_{n,n+i} \in R'$, $x_{n+1,n+i} \in R'$ for $i = 1, \ldots, n$. Thus,

$$x_{n,i} \in R', \;\; x_{n+1,i} \in R' \;\text{ for }\; i = 1, \ldots, 2n \tag{3.19}$$

Substituting (3.6) into (3.7) and taking into account that $\alpha d = d\alpha + d_1$, $\alpha^2 + \alpha \in D$, $\alpha \in \mathcal{O}'$, we obtain:

$$x_{ij}\pi^{2(j-1)} + \pi^{2(i-1)}x_{n+i,j}\pi^{2(j-1)} + \pi^{2(i-1)}(\alpha x_{n+i,j} + x_{n+i,j}\alpha)\pi^{2(j-1)} =$$
$$= \pi^{2(i-1)}x_{n+i,n+j}, \tag{3.20}$$

$$(\alpha x_{i,j} + x_{i,j}\alpha)\pi^{2(j-1)} + r\pi^{2(i-1)}x_{n+i,j}\pi^{2(j-1)} = x_{i,n+j} \tag{3.21}$$

for $i, j = 1, \ldots, n$, where $r \in \mathcal{O}' \cap D$.

Then from conditions (3.19) it follows that $x_{ij} \in R'$ if $i > j$, and $x_{n+i,n+j} \in R'$ if $i < j$, for $i, j = 1, \ldots, n$. Moreover, after substituting (3.20), (3.21) into (3.17), (3.18) we get $x_{ii} \in R'$, $x_{n+i,n+j} \in R'$ for $i = 1, \ldots, n$. Thus we have the same conditions as in the first case. So the matrix problem I is of unbounded representation type.

Matrix Problem II.

Let $\mathcal{O}$ be a discrete valuation ring with a skew field of fractions D and the Jacobson radical $R = \pi\mathcal{O} = \mathcal{O}\pi$, and let D' be a skew field extension of D of degree 2.

Consider a block rectangular matrix $\mathbf{T}$ *partitioned into two vertical strips*

$$\mathbf{T} = \boxed{\begin{array}{c} \mathbf{A} \\ \hline \mathbf{B} \end{array}} \tag{3.22}$$

where $\mathbf{A} \in M_{m\times n_1}(D)$, $\mathbf{B} \in M_{k\times n_1}(D')$.

Assume that on the matrix **T** *one has the following admissible transformations*:
1. *Left* D*-elementary transformations of rows within the first horizontal strip of the matrix* **T**.
2. *Left* D'*-elementary transformations of rows within the second horizontal strip of the matrix* **T**.
3. *Right* O*-elementary transformations of columns of the matrix* **T**.

Lemma 3.2. *The matrix problem II is of unbounded representation type.*

Proof. Analogously to the matrix problem I for given two matrices $\mathbf{A} \in M_{m\times n_1}(D)$, $\mathbf{B} \in M_{k\times n_1}(D')$ we have the following matrix equalities:

$$\mathbf{S}_1\mathbf{A} = \mathbf{A}\mathbf{W}, \quad \mathbf{S}_2\mathbf{B} = \mathbf{B}\mathbf{W}, \tag{3.23}$$

where

$$\mathbf{W}^2 = \mathbf{W} \in M_{n_1}(O), \;\; \mathbf{S}_1^2 = \mathbf{S}_1 \in M_m(D), \;\; \mathbf{S}_2^2 = \mathbf{S}_2 \in M_k(D'). \tag{3.24}$$

I. Consider the case I, i.e. there is an element $\alpha \in D' \setminus D$ such that $\alpha^2 \in D$ and $\alpha O = O\alpha$, $\alpha^2 \in O$. Let $0 \neq \pi \in R$, where R is the Jacobson radical of O.

We set $m = 1$, $n_1 = 2n+1$, $k = n$ and

$$\mathbf{A} = \left[\begin{array}{c|ccc|ccc} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right] \tag{3.25}$$

$$\mathbf{B} = \left[\begin{array}{c|cccc|cccc} \pi^{-n} & 1 & 0 & \cdots & 0 & \alpha & 0 & \cdots & 0 \\ \pi^{-(n-1)} & 0 & 1 & \cdots & 0 & 0 & \alpha\pi^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \pi^{-1} & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & \alpha\pi^{2(n-1)} \end{array}\right] \tag{3.26}$$

Let $\mathbf{W} = (x_{ij})$, $\mathbf{S}_1 = (z)$, $\mathbf{S}_2 = (y_{kl})$, where $x_{ij} \in O$, for $i, j = 1, \ldots, 2n+1$, $z \in D$, $y_{kl} \in D'$, for $k, l = 1, \ldots, n$. Taking into account the matrix equations (3.23) we obtain the system of linear equations for x_{ij}, z and y_{kl}:

$$x_{11} = z \tag{3.27}$$

$$x_{i1} = 0 \;\; (i = 2, \ldots, 2n+1), \tag{3.28}$$

$$\begin{aligned} x_{11}\pi^{-(n-i+1)} + x_{1,i+1} + x_{1,n+i+1}\alpha\pi^{2(i-1)} = \\ = \pi^{-n}y_{1i} + \pi^{-(n-1)}y_{2i} + \cdots + \pi^{-1}y_{ni} \quad (i = 1, \ldots, n) \end{aligned} \tag{3.29}$$

$$x_{i+1,i}\pi^{-(n-j+1)} + x_{i+1,j+1} + x_{i+1,n+j+1}\alpha\pi^{2(j-1)} = y_{ij} \tag{3.30}$$

$$x_{n+i+1,1}\pi^{-(n-j+1)} + x_{n+i+1,j+1} + x_{n+i+1,n+j+1}\alpha\pi^{2(j-1)} = \alpha\pi^{2(i-1)}y_{ij} \tag{3.31}$$

for $i, j = 1, \ldots, n$. Since $\mathbf{W}^2 = \mathbf{W}$, $\mathbf{S}_i^2 = \mathbf{S}_i$ $(i = 1, 2)$, we get $z^2 = z$, which implies that $z = 0$ or $z = 1$.

Let $z = 0$, then $x_{11} = 0$ and the equations (3.29)–(3.31) take the form:

$$x_{1,j+1} + x_{1,n+j+1}\alpha\pi^{2(j-1)} = \pi^{-n}y_{1j} + \pi^{-(n-1)}y_{2j} + \cdots + \pi^{-1}y_{nj} \tag{3.32}$$

$$x_{i+1,j+1} + x_{i+1,n+j+1}\alpha\pi^{2(j-1)} = y_{ij} \tag{3.33}$$

$$x_{n+i+1,j+1}+x_{n+i+1,n+j+1}\alpha\pi^{2(j-1)}=\alpha\pi^{2(i-1)}y_{ij} \tag{3.34}$$

for $i,j=1,\dots,n$. Substituting (3.33) into (3.32) and taking into account that $\alpha D=D\alpha$ and elements 1, α are linear independent over D we get:

$$x_{1,j+1}=\pi^{-n}x_{2,j+1}+\pi^{-(n-1)}x_{3,j+1}+\cdots+\pi^{-1}x_{n+1,j+1}) \tag{3.35}$$

$$x_{1,n+j+1}\pi^{2(j-1)}\varepsilon=\pi^{-n}x_{2,n+j+1}\pi^{2(j-1)}\varepsilon_1++\cdots+\pi^{-1}x_{n+1,n+j+1}\pi^{2(j-1)}\varepsilon_n \tag{3.36}$$

for $j=1,\dots,n$, where $\varepsilon,\varepsilon_i\in O^*$. This implies that $x_{2,j}\in R$ for $j=2,\dots,2n+1$. Analogously, substituting (3.34) into (3.32) we get:

$$\alpha^2\pi^{2(j-1)}x_{1,n+j+1}=\pi^{-n}x'_{n+2,j+1}+\cdots+\pi^{2n-3}x'_{2n+1,j+1} \tag{3.37}$$

$$x_{1,j+1}=\pi^{-n}x'_{n+2,n+j+1}+\pi^{-(n-3)}x'_{n+3,n+j+1}+\cdots+\pi^{2n-3}x'_{2n+1,n+j+1} \tag{3.38}$$

for $j=1,\dots,n$, where $x'_{n+i,j+1}=x_{n+i,j+1}\varepsilon_{ij}$, $x'_{n+i,n+j+1}=x_{n+i,n+j+1}\tilde{\varepsilon}_{ij}$, ε_{ij}, $\tilde{\varepsilon}_{ij}\in O^*$, for $i,j=1,\dots,n$.
Then from equations (3.37), (3.38) we get that $x_{n+2,j}\in R$ for $j=2,\dots,2n+1$.
Substituting (3.32) into (3.34) we get:

$$x_{n+i+1,j+1}=\alpha^2\pi^{2(i-1)}\pi^{2(j-1)}x_{i+1,n+j+1}\varepsilon_{ij}, \tag{3.39}$$

$$x_{n+i+1,n+j+1}\pi^{2(j-1)}=x_{i+1,j+1}\varepsilon_{ij}\pi^{2(i-1)}, \tag{3.40}$$

$$\pi^{2(j-1)}x_{n+i+1,n+j+1}\tilde{\varepsilon}_{ij}=\pi^{2(i-1)}x_{i+1,j+1}, \tag{3.41}$$

where $\varepsilon_{ij},\tilde{\varepsilon}_{ij}\in O^*$. From these equations it follows that: $x_{n+i+1,j+1}\in R$; $x_{i+1,j+1}\in R$ if $i<j$; $x_{n+i+1,n+j+1}\in R$ if $i>j$ for $i,j=1,\dots,n$. Moreover, substituting (3.40),(3.41) into (3.35), (3.38) we get that $x_{i+1,i+1}\in R$, $x_{n+i+1,n+i+1}\in R$ for $i=1,\dots,n$. Thus the matrix $\mathbf{W}$ has the same form as in the previous lemma and therefore we analogously obtain that $\mathbf{W}$, $\mathbf{S}_1$ and $\mathbf{S}_2$ are the zero matrices.

Analogously to lemma 3.1. it can be considered the case when $z=1$. So the matrix $\mathbf{T}$ is indecomposable. The case II can be considered as in lemma 3.1. Since $d(\mathbf{T})=4n+2$, the matrix problem II is of unbounded representation type.

Matrix Problem III.

Let O' be a discrete valuation ring with a skew field of fractions D' and the Jacobson radical R', and let D be a skew subfield of D' of degree 2.

Consider a block rectangular matrix $\mathbf{T}$ *with entries in D' partitioned into two vertical and two horizontal strips*

$$\mathbf{T}=\begin{array}{|c|c|}\hline \mathbf{A} & \mathbf{I} \\ \hline \mathbf{O} & \mathbf{B} \\ \hline \end{array} \tag{3.42}$$

where $\mathbf{A}\in M_{m\times n}(D')$, $\mathbf{B}\in M_{k\times m}(D)$, $\mathbf{I}\in M_m(D')$ is the identity matrix and $\mathbf{O}\in M_{k\times n}(D')$ is the zero matrix.

Assume that on the matrix $\mathbf{T}$ *one has the following admissible transformations*:
1. *Left D-elementary transformations of rows within any horizontal strip of the matrix* $\mathbf{T}$.

2. *Right $\mathcal{O}'$-elementary transformations of columns within the first vertical strip of the matrix* **T**.
3. *Right D-elementary transformations of columns within the second vertical strip of the matrix* **T**.

Moreover, these transformations should not change the given matrix **I**.

Lemma 3.3. *The matrix problem III is of unbounded representation type.*

Proof. To prove this lemma it is sufficient to consider the following system of matrix equations

$$\mathbf{AW} = \mathbf{S}_1\mathbf{A}, \quad \mathbf{BS}_1 = \mathbf{S}_2\mathbf{B}, \tag{3.43}$$

where

$$\mathbf{W}^2 = \mathbf{W} \in M_{n_1}(\mathcal{O}'), \quad \mathbf{S}_1^2 = \mathbf{S}_1 \in M_m(D), \quad \mathbf{S}_2^2 = \mathbf{S}_2 \in M_k(D) \tag{3.44}$$

Suppose that an element $\theta \in D' \setminus D$ is chosen in such a way that in the case I: $\theta D = D\theta$, $\theta^2 \in D$, and in the case II: $\theta d + d\theta \in D$ for any $d \in D$ and $\theta^2 + \theta \in D$.

Let R' be the Jacobson radical of the ring $\mathcal{O}'$ and $0 \neq \pi \in R' \cap D$. We set $n_1 = 2n$, $m = 2n$, $k = 1$ and

$$\mathbf{A} = \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & \pi & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \pi^3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & \pi^{2n-1} \\ \hline \theta & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \theta & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta & 0 & 0 & \cdots & 0 \end{array}\right), \tag{3.45}$$

$$\mathbf{B} = \left(\begin{array}{cccc|ccc} \pi^{n-1} & \cdots & \pi & 1 & 0 & \cdots & 0 \end{array}\right). \tag{3.46}$$

We show that in this case the system of matrix equations (3.43)–(3.44) has only two solutions: 1) all matrices **W**, $\mathbf{S}_1$, $\mathbf{S}_2$ are the identity matrices; 2) all matrices **W**, $\mathbf{S}_1$, $\mathbf{S}_2$ are the zero matrices.

Let $\mathbf{W} = (x_{ij})$, $\mathbf{S}_1 = (y_{kl})$, $\mathbf{S}_2 = (z)$, where $x_{ij} \in \mathcal{O}'$, $y_{kl}, z \in D$. Then we obtain the following system of linear equations for x_{ij}, y_{kl}, z:

$$y_{ij} + y_{i,n+j}\theta = x_{ij} + \pi^{2i-1}x_{n+i,j} \tag{3.47}$$

$$y_{ij}\pi^{2j-1} = x_{i,n+j} + \pi^{2i-1}x_{n+i,n+j} \tag{3.48}$$

$$y_{n+i,j} + y_{n+i,n+j}\theta = \theta x_{ij} \tag{3.49}$$

$$y_{n+i,j}\pi^{2j-1} = \theta x_{i,n+j} \tag{3.50}$$

$$z\pi^{n-j} = \sum_{i=1}^{n} \pi^{n-i} y_{ij} \tag{3.51}$$

$$0=\sum_{i=1}^{n}\pi^{n-i}y_{i,n+j} \tag{3.52}$$

for $i,j=1,\dots,n$.

Suppose that $\theta\in O'^{*}$. Consider the case I, i.e. $\theta d=\tilde{d}\theta$, $\theta^2\in D$ where $d,\tilde{d}\in D$. Then from (3.47)–(3.50) it follows that $y_{ij}\in O'\cap D$ for $i,j=1,\dots,n$.

Since $\mathbf{S}_2^2=\mathbf{S}^2$, we have $z^2=z$, that implies $(z=0)\vee(z=1)$. Assume $z=0$, then equation (3.51) takes the form:

$$\sum_{i=1}^{n}\pi^{n-i}y_{ij}=0\quad (j=1,\dots,n). \tag{3.53}$$

Substituting (3.50) into (3.48) we get:

$$\theta y_{ij}\pi^{2j-1}=y_{n+i,j}\pi^{2j-1}+\theta\pi^{2i-1}x_{n+i,n+j}$$

which implies that

$$y_{ij},\ y_{n+i,j}\in\pi^{2(i-j)}O' \tag{3.54}$$

$$x_{n+i,n+j}\in\pi^{2(j-1)}O' \tag{3.55}$$

for $i,j=1,\dots,n$. Substituting (3.49) into (3.47) we get:

$$\theta y_{ij}+\theta y_{i,n+j}\theta=y_{n+i,j}+y_{n+i,n+j}\theta+\theta\pi^{2i-1}x_{n+i,j}.$$

Taking into account that $\theta d=\tilde{d}\theta$, $\theta^2\in D$ and conditions (3.54), we obtain:

$$y_{n+i,n+j},\ y_{i,n+j}\in\pi^{2(i-j)}O'\ \ (i,j=1,\dots,n).$$

Therefore from (3.49) and (3.50) it follows that $x_{ij}\in\pi^{2(i-j)}O'$ and $x_{i,n+j}\in\pi^{\max\{2i-1,2j-1\}}O'$ for $i,j=1,\dots,n$. Multiplying the equality (3.47) on the left side by π^{n-i} and summing by i we have:

$$\sum_{i=1}^{n}\pi^{n-i}y_{ij}+\sum_{i=1}^{n}\pi^{n-i}y_{i,n+j}\theta=\sum_{i=1}^{n}\pi^{n-i}x_{ij}+\sum_{i=1}^{n}\pi^{n-i}\pi^{2i-1}x_{n+i,j}.$$

Taking into account (3.52) and (3.53) we obtain:

$$\sum_{i=1}^{n}\pi^{n-i}x_{ij}+\sum_{i=1}^{n}\pi^{n-i}\pi^{2i-1}x_{n+i,j}=0.$$

Since $x_{ij}\in\pi^{2(i-j)}O'$, from the last equality it follows that $x_{ii}\in\pi O'\subset R'$ for $i=1,\dots,n$. Multiplying the equality (3.48) on the left side by π^{n-i} and summing by i we obtain:

$$(\sum_{i=1}^{n}\pi^{n-i}y_{ij})\pi^{2j-1}=\sum_{i=1}^{n}\pi^{n-i}x_{i,n+j}+\sum_{i=1}^{n}\pi^{n-i}\pi^{2i-1}x_{n+i,n+j},$$

and taking into account (3.53):

$$\sum_{i=1}^{n}\pi^{n-i}x_{i,n+j}+\sum_{i=1}^{n}\pi^{n-i}\pi^{2i-1}x_{n+i,n+j}=0.$$

Since $x_{i,n+j} \in \pi^{\max\{2i-1,2j-1\}}O'$ and $x_{n+i,n+j} \in \pi^{2(j-i)}O'$, from the last equality it follows that $x_{n+i,n+j} \in \pi O' \subset R'$ for $i = 1,\dots,n$. Therefore we have the following conditions:

$$x_{ij} \in R' \text{ if } i \geq j; \quad x_{i,n+j} \in R'; \;\; x_{n+i,n+j} \in R' \text{ if } i \leq j$$

for $i, j = 1,\dots,n$.

So in this case the matrix $\mathbf{W}$ has the following form:

$$\mathbf{W} = \left(\begin{array}{c|c} W_{11} & W_{12} \\ \hline W_{21} & W_{22} \end{array}\right),$$

where

$$W_{12} \in \begin{pmatrix} R' & O' & \cdots & O' \\ R' & R' & \cdots & O' \\ \vdots & \vdots & \ddots & \vdots \\ R' & R' & \cdots & R' \end{pmatrix}, \quad W_{22} \in \begin{pmatrix} R' & R' & \cdots & R' \\ O' & R' & \cdots & R' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \cdots & R' \end{pmatrix},$$

$W_{12} \in M_n(R')$, and $W_{21} \in M_n(O')$.

Since $\mathbf{W}^2 = \mathbf{W}$ and $\bigcap\limits_{i=1}^{\infty} R'^i = 0$, we obtain that $\mathbf{W} = \mathbf{O}$.

Taking into account that the elements $1, \theta$ are linear independent over D from (3.47) and (3.49) it follows that $y_{ij} = 0$ for $i, j = 1,\dots,2n$, i.e. $\mathbf{S}_1 = \mathbf{O}$. So in this case $\mathbf{W}$, $\mathbf{S}_1$ and $\mathbf{S}_2$ are the zero matrices.

Similar to previous lemmas we can show that in the case $z = 1$ all matrices $\mathbf{W}$, $\mathbf{S}_1$ and $\mathbf{S}_2$ are the identity matrices. Analogously we can consider the case II.

Since $d(\mathbf{T}) = 6n+1$, the matrix problem III is of unbounded representation type.

Matrix Problem IV.

Let O' be a discrete valuation ring with a skew field of fractions D' and the Jacobson radical R', and D a skew subfield of D' of degree 2.

Consider a block rectangular matrix $\mathbf{T}$ *partitioned into two vertical strips*

$$\mathbf{T} = \begin{array}{|c|} \hline \mathbf{A} \\ \hline \mathbf{B} \\ \hline \end{array} \tag{3.56}$$

where $\mathbf{A} \in M_{m\times n}(D')$, $\mathbf{B} \in M_{k\times n}(D')$.

Assume that on the matrix $\mathbf{T}$ *one has the following admissible transformations*:

1. *Left D-elementary transformations of rows within each horizontal strip of the matrix* $\mathbf{T}$.

2. *Right O'-elementary transformations of columns of the matrix* $\mathbf{T}$.

Lemma 3.4. *The matrix problem IV is of unbounded representation type.*

Proof. Analogously to matrix problem II for two given matrices $\mathbf{A} \in M_{m\times n_1}(D)$ and $\mathbf{B} \in M_{k\times n_1}(D)$ we have the following matrix equalities:

$$\mathbf{AW} = \mathbf{S}_1\mathbf{A}, \quad \mathbf{BW} = \mathbf{S}_2\mathbf{B}, \tag{3.57}$$

where

$$\mathbf{T}^2 = \mathbf{T} \in M_{n_1}(\mathcal{O}'), \quad \mathbf{S}_1^2 = \mathbf{S}_1 \in M_{n_1}(D), \;\; \mathbf{S}_2^2 = \mathbf{S}_2 \in M_k(D) \tag{3.58}$$

Suppose that an element $\theta \in D' \setminus D$ is chosen in such a way that in the case I: $\theta D = D\theta$, $\theta^2 \in D$, and in the case II: $\theta d + d\theta \in D$ for any $d \in D$ and $\theta^2 + \theta \in D$.

Let R' be the Jacobson radical of the ring $\mathcal{O}'$ and $0 \neq \pi \in R' \cap D$. We set $n_1 = 2n$, $m = 2n$, $k = 1$ and matrices $\mathbf{A}$ and $\mathbf{B}$ are of the form (3.45) and (3.46). We show that in this case the matrix $\mathbf{T}$ is indecomposable. Let $\mathbf{W} = (x_{ij})$, $\mathbf{S}_1 = (y_{kl})$, $\mathbf{S}_2 = (z)$, where $x_{ij} \in \mathcal{O}'$, and $y_{kl}, z \in D$. Then we obtain the following system of linear equations for x_{ij}, y_{kl}, z

$$y_{ij} + y_{i,n+j}\theta = x_{ij} + \pi^{2i-1}x_{n+i,j} \tag{3.59}$$

$$y_{ij}\pi^{2j-1} = x_{i,n+j} + \pi^{2i-1}x_{n+i,n+j} \tag{3.60}$$

$$y_{n+i,j} + y_{n+i,n+j}\theta = \theta x_{ij} \tag{3.61}$$

$$y_{n+i,j}\pi^{2j-1} = \theta x_{i,n+j} \tag{3.62}$$

$$z\pi^{n-j} = \sum_{i=1}^{n} \pi^{n-i}x_{ij} \tag{3.63}$$

$$0 = \sum_{i=1}^{n} \pi^{n-i}x_{i,n+j} \tag{3.64}$$

for $i, j = 1, \ldots, n$.

Assume $\theta \in \mathcal{O}'^*$. Then $y_{ij} \in \mathcal{O}'^* \cap D$ $(i, j = 1, \ldots, 2n)$.

Consider the case I, i.e. $\theta d = \tilde{d}\theta$, $\theta^2 \in D$ where $d, \tilde{d} \in D$. Since $\mathbf{S}_2^2 = \mathbf{S}_2$, we have $z^2 = z$, that implies that $(z = 0) \vee (z = 1)$.

Let $z = 0$, then

$$\sum_{i=1}^{n} \pi^{n-i}x_{ij} = 0. \tag{3.65}$$

Then analogously to the previous lemma we obtain the following conditions:

$$y_{ij},\, y_{n+i,j} \in \pi^{2(j-1)}\mathcal{O}' \;\text{ for } i \geq j \tag{3.66}$$

$$x_{n+i,n+j} \in \pi^{2(j-1)}\mathcal{O}' \;\text{ for } i \leq j \tag{3.67}$$

$$y_{n+i,n+j},\, y_{i,n+j} \in \pi^{2(i-j)}\mathcal{O}' \;\text{ for } i \geq j \tag{3.68}$$

$$x_{i,j} \in \pi^{2(i-j)}\mathcal{O}' \;\text{ for } i \geq j \tag{3.69}$$

$$x_{i,n+j} \in \pi^{\max\{2i-1, 2j-1\}}\mathcal{O}' \tag{3.70}$$

for $i, j = 1, \ldots, n$.

From equation (3.62) we obtain:

$$\sum_{i=1}^{n} \pi^{n-i}y_{ij}\pi^{2j-1} = \sum_{i=1}^{n} \pi^{n-i}x_{i,n+j} + \sum_{i=1}^{n} \pi^{n-i}\pi^{2i-1}x_{n+i,n+j}$$

and taking into account (3.64):

$$\sum_{i=1}^{n}\pi^{n-i+2j-1}y_{ij}=\sum_{i=1}^{n}\pi^{n+i-1}x_{n+i,n+j}.$$

Comparing these equalities with (3.66), (3.67) we have that $x_{n+i,n+i}\in\pi O'\subset R'$ for $i=1,\dots,n$. From equations (3.59) and (3.60) we get:

$$\sum_{i=1}^{n}\pi^{n-i}x_{i,n+j}\pi^{1-2j}+\sum_{i=1}^{n}\pi^{n-i}\theta^{-1}y_{n+i,n+j}\theta=\sum_{i=1}^{n}\pi^{n-i}x_{i,j},$$

and taking into account (3.61) and (3.62) we obtain that $\sum\limits_{i=1}^{n}\pi^{n-i}\theta^{-1}y_{n+i,n+j}\theta=0$, which implies that $y_{n+i,n+i}\in\pi O'\subset R'$.

Since $y_{n+i,i}\in\pi^{2(i-1)}O'\subset R'$, from (3.59) it follows that $x_{ii}\in\pi O'\subset R'$. Thus, we have: $x_{ij}\in R'$ if $i\geq j$; $x_{i,n+j}\in R'$; $x_{n+i,n+j}\in R'$ if $i\leq j$ for $i,j=1,\dots,n$. The further reasoning is exactly the same as in the previous lemma.

So the matrix problem IV is of unbounded representation type.

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