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REGULARITIES FOR A NEW CLASS OF SPACES BETWEEN DISTRIBUTIONS AND ULTRADISTRIBUTIONS

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Dedicated to Professor Mirjana Vuković on the occasion of her 70th birthday

ABSTRACT. We investigate regularity properties of generalized functions that lie between any space of ultradistribution in the sense of Beurling or Roumieu (re-introduced by Komatsu [7, 8]) and Schwartz's distributions. Their test function spaces are described by the means of sequences of the form $M_p = p^{\tau p^{\sigma}}$, $p \in \mathbf{N}$, $\tau > 0$, $\sigma > 1$. We study the properties of function associated to that sequence. In particular, we present sharp estimates for the asymptotic behavior of the associated function, and use that result to characterize the related wave front sets $\mathrm{WF}_{\tau,\sigma}$. Furthermore, we examine the Paley-Wiener type theorems for the test function spaces and the corresponding generalized functions with compact supports.

1. INTRODUCTION

The authors of this paper constructed and analyzed in [9, 10, 11], [14, 15, 16] a class of new generalized function spaces following Komatsu's classical construction of ultradistribution spaces, cf. [7]. The interest for such spaces can be found in the study of a class of strictly hyperbolic systems, see [2]. Moreover, common questions related to (ultra)distributions appear to be important in the setting of newly introduced spaces of generalized functions. This paper is a contribution to the analysis of new regularity properties in the realm of such generalized functions. More precisely, we continue with the analysis of wave front sets in this setting.

In the theory of ultradistributions, asymptotic properties of the associated function to a given positive and increasing sequence M_p, $p \in \mathbf{N}$, are essential for the proofs of *Paley-Wiener type* theorems. For example, it is well known (cf. [5, 8, 13]) that the associated function $T_\tau(k)$, $k > 0$, $\tau > 1$, to the Gevrey sequence $p!^\tau$, $p \in N$,

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satisfies

$$Ak^{1/\tau} - B < T_\tau(k) < Ak^{1/\tau}, \quad k > 0,$$

for suitable constants $A, B > 0$. These estimates imply the Paley-Wiener theorems for Gevrey functions (ultradistributions), which in its simplest form state the following: if φ is a compactly supported Gevrey function of Roumieu (resp. Beurling) type with index $\tau > 1$ (cf. [8, 13]) then

$$|\widehat{\varphi}(\xi)| \leq Ce^{-A|\xi|^{1/\tau}}, \quad \xi \in \mathbf{R}^d,$$

for some constants $A, C > 0$ (resp. for every $A > 0$ there exists $C > 0$).

Another application of such estimates is in microlocal analysis. In particular, Gevrey wave front sets $\mathrm{WF}_\tau(u)$, $\tau > 1$, (of Roumieu type) of a distribution u can be defined as follows (cf.[13]): $(x_0, \xi_0) \not\in \mathrm{WF}_\tau(u)$ if there exists conic neighborhood Γ of ξ_0, compact neighborhood K of x_0 and Gevrey function φ supported in K such that

$$|\widehat{\varphi u}(\xi)| \leq Ce^{-A|\xi|^{1/\tau}}, \quad \xi \in \Gamma,$$

for some $A, C > 0$.

In this paper we investigate the classes of *extended Gevrey functions* whose derivatives are controlled by the two-parameter sequences of the form $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, $\tau > 0$, $\sigma > 1$, and the corresponding wave front sets. Such classes are introduced and investigated in [9, 10, 11, 12, 14, 15, 16], and it turned out that they can be used in the study of a class of strictly hyperbolic equations and systems. In particular, the extended Gevrey class associated to the sequence $M_p^{1,2} = p^{p^2}$ is used in the analysis of the regularity of the corresponding Cauchy problem in [2].

The aim of this paper is two-fold. Firstly we give a review of some results related to a new class of spaces between distributions (strongly larger class) and ultradistributions. Such classes are strictly contained in any space of ultradistributions in the sense of Komatsu, [7] (see also Braun, Meise, Taylor, [1]).

The second aim is to present some of our most recent results related to the notion of wave front sets in that setting. The complete analysis is given in [12].

The paper is organized as follows: Subsection 1.1 contains notation and basic facts on the Lambert W function (cf. [3]) that will be used throughout the paper. The basic facts concerning classes of extended Gevrey functions and corrssponding wave front sets are introduced in Subsections 1.2 and 1.3, respectively. In Section 2 we introduce the function $T_{\tau,\sigma,h}(k)$ associated to the sequence $p^{\tau p^\sigma}$, $p \in \mathbf{N}$, and present the essential estimates in Theorem 2.1. These estimates are used in the proof of Proposition 2.1 to characterize the wave front sets from Subsection 1.3. In Section 3 we discuss different versions of the Paley-Wiener theorems for extended Gevrey functions with compact support, and prove the Theorem 3.2 as our final result.

1.1. **Preliminaries**

We denote by $\mathbf{N}$, $\mathbf{Z}_+$, $\mathbf{R}$, $\mathbf{C}$ the sets of nonnegative integers, positive integers, real numbers and complex numbers, respectively. For $x\in\mathbf{R}^d$ we put $\langle x\rangle=(1+|x|^2)^{1/2}$. The integer parts (the floor and the ceiling functions) of $x\in\mathbf{R}_+$ are denoted by $\lfloor x\rfloor:=\max\{m\in\mathbf{N}: m\le x\}$ and $\lceil x\rceil:=\min\{m\in\mathbf{N}: m\ge x\}$. For a multi-index $\alpha=(\alpha_1,\dots,\alpha_d)\in\mathbf{N}^d$ we write $\partial^\alpha=\partial^{\alpha_1}\dots\partial^{\alpha_d}$, $D^\alpha=(-i)^{|\alpha|}\partial^\alpha$, and $|\alpha|=|\alpha_1|+\dots|\alpha_d|$. We write $A(\theta)\lesssim B(\theta)$, $\theta\in U$, if there is a constant $c>0$ such that $A(\theta)\le cB(\theta)$ for all $\theta\in U$, and $A(\theta)\asymp B(\theta)$ if $A(\theta)\lesssim B(\theta)$ and $B(\theta)\lesssim A(\theta)$ for all $\theta\in U$. Here U is an open set in $\mathbb{R}^d$.

The Lebesgue spaces over an open set $U\subset\mathbf{R}^d$ are denoted by $L^p(U)$, $1\le p<\infty$, and the Fourier-Laplace transform is denoted by

$$\widehat{u}(\eta)=\int_{\mathbf{R}^d}u(x)e^{ix\cdot\eta}\,dx,\quad \eta\in\mathbf{C}^d,\quad u\in L^1(\mathbf{R}^d).$$

Let U be open set in $\mathbf{R}^d$ and K be compact set in U (we write $K\subset\subset U$). By $C^\infty(U)$, C_K^∞ we denote the spaces of smooth functions on U and its subspace of functions supported in K and their strong duals are $\mathcal{D}'(U)$ and $\mathcal{E}'_K$, respectively.

By $W(x)$, $x\ge 0$, we denote the principal (real) branch of the *Lambert W function*, commonly denoted by W_0 (see [3]). The Lambert W function is defined as the inverse function of ze^z, $z\in\mathbf{C}$, wherefrom it easily follows that W satisfies the following property:

$$x=W(x)e^{W(x)},\quad x\ge 0.$$

Moreover, W is continuous, increasing and concave on $[0,\infty)$, $W(0)=0$, $W(e)=1$, and $W(x)>0$ for $x>0$. It can be shown that the following estimate holds:

$$\ln x-\ln(\ln x)\le W(x)\le \ln x-\frac{1}{2}\ln(\ln x),\quad x\ge e, \tag{1.1}$$

with the equality if and only if $x=e$. For more details about the Lambert W function we refer to [3, 6].

1.2. **Test function spaces**

In this subsection we recall the definition of test function spaces $\mathcal{E}_{\tau,\sigma}(U)$ and $\mathcal{D}_{\tau,\sigma}(U)$. Such spaces contains the functions whose derivatives are controlled by sequences of the form $M_p^{\tau,\sigma}=p^{\tau p^\sigma}$, $p\in\mathbf{N}$, $\tau>0$ and $\sigma>1$. They are introduced and studied in [9, 10, 11, 14, 15, 16] as *extended Gevrey classes*.

Let $\tau,h>0$, $\sigma>1$ and $K\subset\subset\mathbf{R}^d$ with a smooth boundary. By $\mathcal{E}_{\tau,\sigma,h}(K)$ we denote the Banach space of functions $\phi\in C^\infty(K)$ such that

$$\|\phi\|_{\mathcal{E}_{\tau,\sigma,h}(K)}=\sup_{\alpha\in\mathbf{N}^d}\sup_{x\in K}\frac{|\partial^\alpha\phi(x)|}{h^{|\alpha|^\sigma}M_{|\alpha|}^{\tau,\sigma}}<\infty,$$

and $\mathcal{D}^K_{\tau,\sigma,h}$ is its subspace of functions supported in K.

Then obviously,

$$\mathcal{E}_{\tau_1,\sigma_1,h_1}(K) \hookrightarrow \mathcal{E}_{\tau_2,\sigma_2,h_2}(K), \quad 0 < h_1 \le h_2,\ 0 < \tau_1 \le \tau_2,\ 1 < \sigma_1 \le \sigma_2,$$

where $\hookrightarrow$ denotes the strict and dense inclusion.

For an open set U in $\mathbf{R}^d$ we define families of spaces by introducing the following projective and inductive limit topologies,

$$\mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{K\subset\subset U} \varinjlim_{h\to\infty} \mathcal{E}_{\tau,\sigma,h}(K),$$

$$\mathcal{E}_{(\tau,\sigma)}(U) = \varprojlim_{K\subset\subset U} \varprojlim_{h\to 0} \mathcal{E}_{\tau,\sigma,h}(K),$$

$$\mathcal{D}_{\{\tau,\sigma\}}(U) = \varinjlim_{K\subset\subset U} \mathcal{D}^K_{\{\tau,\sigma\}} = \varinjlim_{K\subset\subset U} \big(\varinjlim_{h\to\infty} \mathcal{D}^K_{\tau,\sigma,h}\big),$$

$$\mathcal{D}_{(\tau,\sigma)}(U) = \varinjlim_{K\subset\subset U} \mathcal{D}^K_{(\tau,\sigma)} = \varinjlim_{K\subset\subset U} \big(\varprojlim_{h\to 0} \mathcal{D}^K_{\tau,\sigma,h}\big).$$

We will write τ,σ for $\{\tau,\sigma\}$ or (τ,σ).

Notice that $\mathcal{E}_{\{\tau,1\}}(U) = \mathcal{E}_{\{\tau\}}(U)$, $\tau > 1$, is the Gevrey class, and $\mathcal{D}_{\{\tau,1\}}(U) = \mathcal{D}_{\{\tau\}}(U)$ is its subspace of compactly supported functions. Moreover, $\mathcal{E}_{\{1,1\}}(U) = \mathcal{E}_{\{1\}}(U)$ is the space of analytic functions on U.

The sequence $M_p = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, $\tau > 0$, $\sigma > 1$, satisfies the non-quasianalyticity condition

$$\sum_{p=1}^{\infty} \frac{M^{\tau,\sigma}_{p-1}}{M^{\tau,\sigma}_p} < \infty$$

(cf. [9]), and therefore the spaces $\mathcal{D}_{(\tau,\sigma)}(U)$ and $\mathcal{D}_{\{\tau,\sigma\}}(U)$ are non-trivial. Moreover, there exist compactly supported functions in $\mathcal{D}_{\{\tau,\sigma\}}(U)$ which do not belong to $\mathcal{D}_{\{t\}}(U)$ for any $t > 1$, see [9, Lemma 2.2] for the construction.

The spaces $\mathcal{E}_{\tau,\sigma}(U)$, $\mathcal{D}^K_{\tau,\sigma}$ and $\mathcal{D}_{\tau,\sigma}(U)$ are nuclear, closed under pointwise multiplication and finite order differentiation. The basic embedding properties are given in the following Proposition. We refer to [10] for its proof.

Proposition 1.1. *Let* $\sigma_1 \ge 1$. *Then for every* $\sigma_2 > \sigma_1$ *and* $\tau > 0$,

$$\varinjlim_{\tau\to\infty} \mathcal{E}_{\tau,\sigma_1}(U) \hookrightarrow \varprojlim_{\tau\to 0^+} \mathcal{E}_{\tau,\sigma_2}(U).$$

Moreover, if $0 < \tau_1 < \tau_2$, *then*

$$\mathcal{E}_{\{\tau_1,\sigma\}}(U) \hookrightarrow \mathcal{E}_{(\tau_2,\sigma)}(U) \hookrightarrow \mathcal{E}_{\{\tau_2,\sigma\}}(U), \quad \sigma \ge 1,$$

and

$$\varinjlim_{\tau\to\infty} \mathcal{E}_{\{\tau,\sigma\}}(U) = \varinjlim_{\tau\to\infty} \mathcal{E}_{(\tau,\sigma)}(U),$$

$$\varprojlim_{\tau\to 0^+} \mathcal{E}_{\{\tau,\sigma\}}(U) = \varprojlim_{\tau\to 0^+} \mathcal{E}_{(\tau,\sigma)}(U), \quad \sigma \ge 1.$$

As an immediate consequence of Proposition 1.1 we obtain

$$\varinjlim_{t\to\infty} \mathcal{E}_{\{t\}}(U) \hookrightarrow \mathcal{E}_{\tau,\sigma}(U) \hookrightarrow C^\infty(U), \quad \tau>0,\ \sigma>1,$$

and in that sense the regularity described by $\mathcal{E}_{\tau,\sigma}(U)$ extends the Gevrey regularity.

We end the subsection with ultradifferentiability property of classes $\mathcal{E}_{\tau,\sigma}(U)$.

Theorem 1.1. *Let $\tau>0$, $\sigma>1$, U open in $\mathbf{R}^d$ and let*

$$P(x,\partial)=\sum_{|\alpha|=0}^{\infty} a_\alpha(x)\partial^\alpha$$

be the infinite order differential operator of class (τ,σ) (resp. $\{\tau,\sigma\}$) on U, i.e., for every $K\subset\subset U$ there exists constant $L>0$ such that for any $h>0$ there exists $A>0$ (resp. for every $K\subset\subset U$ there exists $h>0$ such that for any $L>0$ there exists $A>0$) such that,

$$\sup_{x\in K}|\partial^\beta a_\alpha(x)| \le A h^{|\beta|^\sigma}|\beta|^{\tau|\beta|^\sigma}\frac{L^{|\alpha|^\sigma}}{|\alpha|^{\tau 2^{\sigma-1}|\alpha|^\sigma}}, \quad \alpha,\beta\in\mathbf{N}^d.$$

Then

$$P(x,\partial):\quad \mathcal{E}_{\tau,\sigma}(U)\longrightarrow \mathcal{E}_{\tau 2^{\sigma-1},\sigma}(U)$$

is a continuous linear mapping, and the same holds for

$$P(x,\partial):\quad \varinjlim_{\tau\to\infty}\mathcal{E}_{\tau,\sigma}(U)\longrightarrow \varinjlim_{\tau\to\infty}\mathcal{E}_{\tau,\sigma}(U).$$

We refer to [10] for the proof of Theorem 1.1, and to [9, 11, 14, 15, 16] for more details on $\mathcal{E}_{\tau,\sigma}(U)$ and $\mathcal{D}_{\tau,\sigma}(U)$ and their dual spaces of generalized functions.

1.3. **Wave front sets** $\mathrm{WF}_{\tau,\sigma}$

In this section we define wave front sets $\mathrm{WF}_{\{\tau,\sigma\}}(u)$ and $\mathrm{WF}_{(\tau,\sigma)}(u)$ of Roumieu and Beurling type, respectively, and discuss their basic properties. We write $\mathrm{WF}_{\tau,\sigma}(u)$ for $\mathrm{WF}_{\{\tau,\sigma\}}(u)$ or $\mathrm{WF}_{(\tau,\sigma)}(u)$.

Let $u\in\mathcal{D}'(U)$, $\tau>0$, $\sigma>1$ and $(x_0,\xi_0)\in U\times\mathbf{R}^d\backslash\{0\}$. Recall (cf. [10]), $(x_0,\xi_0)\not\in\mathrm{WF}_{\{\tau,\sigma\}}(u)$ (resp. $(x_0,\xi_0)\not\in\mathrm{WF}_{(\tau,\sigma)}(u)$) if there exists a conic neighborhood Γ of ξ_0, compact neighborhood K of x_0, and bounded sequence $\{u_N\}_{N\in\mathbf{N}}$ in $\mathcal{E}'_K$ such that $u_N=u$ on some neighborhood of x_0 and for some $A,h>0$ (resp. for every $h>0$ there exists $A>0$) so that

$$|\widehat{u}_N(\xi)|\le A\frac{h^N N!^{\tau/\sigma}}{|\xi|^{\lfloor N^{1/\sigma}\rfloor}}, \quad N\in\mathbf{N},\xi\in\Gamma. \tag{1.2}$$

The condition (1.2) can be replaced by an equivalent condition when instead of N we use another positive, increasing sequence a_N such that $a_N\to\infty$, $N\to\infty$. This change of variable with respect to $N\in\mathbf{N}$ we call *enumeration*. For example, enumeration $N\to N^\sigma$ and Stirling's formula applied to (1.2) give an equivalent estimate of the form

$$|\widehat{u}_N(\xi)|\le A_1\frac{h_1^{N^\sigma}N^{\tau N^\sigma}}{|\xi|^N}, \quad N\in\mathbf{N},\xi\in\Gamma,$$

for some constants $A_1, h_1 > 0$ (resp. for every $h_1 > 0$ there exists $A_1 > 0$). We refer to [10] for more details on enumeration.

For the analysis of $\mathrm{WF}_{\{\tau,\sigma\}}(u)$ (cf. [10, 11]) we have used the sequences of cutoff function in a similar way as it is done in [4] in the context of analytic wave front set $\mathrm{WF}_A = \mathrm{WF}_{\{1,1\}}$. Notice also that $\mathrm{WF}_{\{\tau,1\}} = \mathrm{WF}_\tau(u)$ is the Gevrey wave front set investigated in [13].

The following characterization of $\mathrm{WF}_{\tau,\sigma}(u)$ states that a bounded sequence of cut-off functions $\{u_N\}_{N\in\mathbf{N}} \subset \mathcal{E}'(U)$ can be replaced by a single function from $\mathcal{D}_{\tau,\sigma}(U)$. We refer to [11] for its proof.

Theorem 1.2. *Let $u \in \mathcal{D}'(U)$, $\tau > 0$, $\sigma > 1$, and let $(x_0,\xi_0) \in U \times \mathbf{R}^d \backslash \{0\}$. Then $(x_0,\xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$ (resp. $(x_0,\xi_0) \notin \mathrm{WF}_{(\tau,\sigma)}(u)$) if and only if there exists a conic neighborhood Γ of ξ_0, a compact neighborhood K of x_0 and $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$ (resp. $\phi \in \mathcal{D}^K_{(\tau,\sigma)}$) such that $\phi = 1$ on a neighborhood of x_0, and holds*

$$|\widehat{\phi u}(\xi)| \leq A \frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \Gamma,$$

for some $A, h > 0$ (resp. for every $h > 0$ there exists $A > 0$).

The singular support of a distribution with respect to classes $\mathcal{E}_{\tau,\sigma}(U)$ can be defined in the usual way as follows.

Definition 1.1. *Let $x_0 \in \mathbf{R}^d$ and $u \in \mathcal{D}'(U)$. Then $x_0 \notin \mathrm{singsupp}_{\tau,\sigma}(u)$ if and only if there exists an open neighborhood Ω of x_0 such that $u \in \mathcal{E}_{\tau,\sigma}(\Omega)$.*

We note that the local regularity described by the classes $\mathcal{E}_{\tau,\sigma}$ coincides with the wave front sets $\mathrm{WF}_{\tau,\sigma}$. In particular, the following Theorem holds (see [10] for Roumieu, and [15] for Beurling case).

Theorem 1.3. *Let $\tau > 0$ and $\sigma > 1$, $u \in \mathcal{D}'(U)$. Let $\pi_1 : U \times \mathbf{R}^d \backslash \{0\} \to U$ be the standard projection given with $\pi_1(x,\xi) = x$. Then*

$$\mathrm{singsupp}_{\tau,\sigma}(u) = \pi_1(\mathrm{WF}_{\tau,\sigma}(u)).$$

Hence, for $u \in \mathcal{D}'(U)$, $\tau > 0$ and $\sigma > 1$ we have

$$\mathrm{WF}(u) \subseteq \mathrm{WF}_{\tau,\sigma}(u) \subseteq \bigcap_{\tau>1} \mathrm{WF}_\tau(u) \subseteq \mathrm{WF}_A(u), \quad u \in \mathcal{D}'(U),$$

where $\mathrm{WF}(u)$ is the standard wave front sets $u \in \mathcal{D}'(U)$, cf. [4].

One on the main properties of wave front sets is microlocal hypoelipticity. For the Roumieu wave front $\mathrm{WF}_{\{\tau,\sigma\}}(u)$ it can be stated as follows.

Theorem 1.4. [11] *Let $u \in \mathcal{D}'(U)$, $\tau > 0$ and $\sigma > 1$ and let $P(x,D) = \sum_{|\alpha|\leq m} a_\alpha(x) D^\alpha$ be partial differential operator of order m such that $a_\alpha(x) \in \mathcal{E}_{\{\tau,\sigma\}}(U)$, $|\alpha| \leq m$.*

Then if $P(x,D)u = f$ *in* $\mathcal{D}'(U)$*, it holds*

$$\mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(f) \subseteq \mathrm{WF}_{\{2^{\sigma-1}\tau,\sigma\}}(u) \subseteq \mathrm{WF}_{\{\tau,\sigma\}}(f) \cup \mathrm{Char}(P(x,D)).$$

In particular,

$$\mathrm{WF}_{0,\infty}(f) \subseteq \mathrm{WF}_{0,\infty}(u) \subseteq \mathrm{WF}_{0,\infty}(f) \cup \mathrm{Char}(P(x,D)),$$

where $\mathrm{WF}_{0,\infty}(u) = \bigcup_{\sigma>1} \bigcap_{\tau>0} \mathrm{WF}_{\{\tau,\sigma\}}(u)$ *and* $\mathrm{Char}(P(x,D))$ *is characteristic set of operator* $P(x,D)$*.*

Recall, if $P(x,D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ is a differential operator of order m on U and $a_\alpha \in C^\infty(U)$, $|\alpha| \le m$, then the characteristic set of $P = P(x,D)$ on U is given by

$$\mathrm{Char}(P) = \bigcup_{\overline{x} \in U} \mathrm{Char}_{\overline{x}}(P),$$

where

$$\mathrm{Char}_{\overline{x}}(P) = \{(\overline{x},\xi) \in U \times \mathbf{R}^d \backslash \{0\} \,|\, P_m(\overline{x},\xi) = 0\}$$

is its characteristic variety at $\overline{x} \in U$. Here $P_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha \in C^\infty(U \times \mathbf{R}^d \backslash \{0\})$ denotes the principal symbol of $P(x,D)$.

By the homogeneity of the principal symbol, it follows that $\mathrm{Char}(P)$ is a closed conical subset of $U \times \mathbf{R}^d \backslash \{0\}$.

2. The associated function

Let there be given $h > 0$, $\tau > 0$ and $\sigma > 1$. Then the function

$$T_{\tau,\sigma,h}(k) = \sup_{p \in \mathbf{N}} \ln_+ \frac{h^{p^\sigma} k^p}{M_p^{\tau,\sigma}}, \quad k > 0,$$

is called the (extended) associated function to the sequence $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbf{N}$, where $\ln_+ A = \max\{0, \ln A\}$, for $A > 0$.

For any given $\sigma > 1$ and $\tau > 0$ we put

$$\mathfrak{R}(h,k) := h^{-\frac{\sigma-1}{\tau}} e^{\frac{\sigma-1}{\sigma}} \frac{\sigma - 1}{\tau\sigma} \ln k, \quad h > 0,\ k > e. \tag{2.1}$$

The precise asymptotic behavior of $T_{\tau,\sigma,h}(k)$ when $1 < \sigma < 2$ is given in terms of the Lambert function W as follows

Theorem 2.1. *Let there be given* $h > 0$*,* $\tau > 0$ *and* $1 < \sigma < 2$*, and let* $T_{\tau,\sigma,h}$ *be the associated function to the sequence* $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$*,* $p \in \mathbf{N}$*. Then the following estimates hold:*

$$\tilde{A}_{\tau,\sigma,h} \exp\Big\{ \Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(h,k)) \ln^{\frac{\sigma}{\sigma-1}} k \Big\} \le e^{T_{\tau,\sigma,h}(k)}$$

$$\le A_{\tau,\sigma,h} \exp\Big\{ \Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(h,k)) \ln^{\frac{\sigma}{\sigma-1}} k \Big\}, \quad k > e, \tag{2.2}$$

for some constants $A_{\tau,\sigma,h}, \tilde{A}_{\tau,\sigma,h} > 0$*, and* $\mathfrak{R}(h,k)$ *given by* (2.1).

Sketch of the proof. Let $h,\tau > 0$ and $1 < \sigma < 2$ be fixed. The idea of the proof is to show that there exists constants $H_{\tau,\sigma,h}, C_{\tau,\sigma,h} > 0$ such that

$$\Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(h,k))\ln^{\frac{\sigma}{\sigma-1}} k - H_{\tau,\sigma,h} \leq \sup_{p\in\mathbf{N}} \ln \frac{h^{p^\sigma}k^p}{p^{\tau p^\sigma}} \leq \Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(h,k))\ln^{\frac{\sigma}{\sigma-1}} k, \quad (2.3)$$

when $k \geq C_{\tau,\sigma,h} > e$. We consider separately the right hand side and the left hand side of (2.3). The complete proof is given in [12]. □

Remark 2.1. We note that when the conditions of Theorem 2.1 are satisfied with $\sigma \geq 2$ instead, then the following estimates hold:

$$\tilde{A}_{\tau,\sigma,h} \exp\Big\{(2^{\sigma-1}\tau)^{-\frac{1}{\sigma-1}}\Big(\frac{\sigma-1}{\sigma}\Big)^{\frac{\sigma}{\sigma-1}} W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(h,k))\ln^{\frac{\sigma}{\sigma-1}} k\Big\} \leq e^{T_{\tau,\sigma,h}(k)} \leq A_{\tau,\sigma,h} \exp\Big\{\Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(h,k))\ln^{\frac{\sigma}{\sigma-1}} k\Big\}, \quad k > e, \quad (2.4)$$

see [12] for the proof.

Remark 2.2. Let $\tau, h > 0$, $1 < \sigma < 2$ and $H := h^{-\frac{\sigma-1}{\tau}} e^{\frac{\sigma-1}{\sigma}} \dfrac{\sigma-1}{\tau\sigma}$. Then (1.1) implies

$$W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(h,k))\ln^{\frac{\sigma}{\sigma-1}} k = W^{-\frac{1}{\sigma-1}}(H\ln k)\ln^{\frac{\sigma}{\sigma-1}} k \asymp \frac{\ln^{\frac{\sigma}{\sigma-1}} k}{\ln^{\frac{1}{\sigma-1}}(H\ln k)}, \quad (2.5)$$

when $k \to \infty$. It follows that for every $M > 0$ there exists $B > 0$ such that

$$W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(h,k))\ln^{\frac{\sigma}{\sigma-1}} k > M\ln k, \quad k > B.$$

This estimate is useful when comparing the new regularity properties with the usual Gevrey regularity, cf. [12].

Theorem 2.1 can be used for characterization of wave front sets from Section 1.3. Recall, by Theorem 1.2, for a given $u \in \mathcal{D}'(U)$, $(x_0,\xi_0) \not\in \mathrm{WF}_{\{\tau,\sigma\}}(u)$ (resp. $(x_0,\xi_0) \not\in \mathrm{WF}_{(\tau,\sigma)}(u)$) if and only if there exists a conic neighborhood Γ of ξ_0, a compact neighborhood K of x_0 and $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$ (resp. $\phi \in \mathcal{D}^K_{(\tau,\sigma)}$) such that $\phi = 1$ on a neighborhood of x_0, and

$$|\widehat{\phi u}(\xi)| \leq A\frac{h^{N^\sigma} N^{\tau N^\sigma}}{|\xi|^N}, \quad N \in \mathbf{N}, \xi \in \Gamma, \quad (2.6)$$

for some $A, h > 0$ (resp. for every $h > 0$ there exists $A > 0$).

Proposition 2.1. *Let $u \in \mathcal{D}'(U)$, $\tau > 0$, $1 < \sigma < 2$. Then $(x_0,\xi_0) \not\in \mathrm{WF}_{\{\tau,\sigma\}}(u)$ (resp. $(x_0,\xi_0) \not\in \mathrm{WF}_{(\tau,\sigma)}(u)$) if and only if there exists a conic neighborhood Γ of ξ_0, a compact neighborhood K of x_0 and $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$ (resp. $\phi \in \mathcal{D}^K_{(\tau,\sigma)}$) such that*

$\phi = 1$ *on a neighborhood of* x_0*, and for some* $A, H > 0$ *(resp. for every* $H > 0$ *there exists* $A > 0$*) such that*

$$|\widehat{\phi u}(\xi)| \le A \exp\Big\{ -\Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}(H\ln(e+|\xi|))\ln^{\frac{\sigma}{\sigma-1}}(e+|\xi|)\Big\}, \qquad (2.7)$$

for $\quad N \in \mathbf{N}, \xi \in \Gamma.$

Proof. Since the proofs Roumieu and Beurling case are similar, we give the proof only for $\mathrm{WF}_{\{\tau,\sigma\}}(u)$.

Let $(x_0, \xi_0) \notin \mathrm{WF}_{\{\tau,\sigma\}}(u)$. This means that there is a conic neighborhood Γ of ξ_0, a compact neighborhood K of x_0 and a function $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$ such that $\phi = 1$ on a neighborhood of x_0 and such that (2.6) holds for some $A, h > 0$.

Then, by the following simple inequality

$$|\xi|^N \le (e+|\xi|)^N \le (2e)^N |\xi|^N, \quad N \in \mathbf{N}, \xi \in \mathbf{R}^d. \qquad (2.8)$$

it follows that there exist $A_1, h_1 > 0$ such that

$$\begin{aligned} |\widehat{\phi u}(\xi)| &\le \inf_{N\in\mathbf{N}} A_1 \frac{h_1^{N^\sigma} N^{\tau N^\sigma}}{(e+|\xi|)^N} \\ &= A_1 \Big(\sup_{N\in\mathbf{N}} \frac{(1/h_1)^{N^\sigma}(e+|\xi|)^N}{N^{\tau N^\sigma}}\Big)^{-1} = e^{-T_{\tau,\sigma,1/h_1}(e+|\xi|)} \\ &\le A_2 \exp\Big\{ -\Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}(\mathfrak{R}(1/h_1,k))\ln^{\frac{\sigma}{\sigma-1}}(e+|\xi|)\Big\}, \quad \xi \in \Gamma \end{aligned}$$

and the last inequality follows from the left hand side of (2.2). Now (2.7) follows for $H = h_1^{\frac{\sigma-1}{\tau}} e^{\frac{\sigma-1}{\sigma}} \dfrac{\sigma-1}{\tau\sigma}$.

Conversely, let Γ be the conic neighborhood of ξ_0, let K be the compact neighborhood of x_0 and let $\phi \in \mathcal{D}^K_{\{\tau,\sigma\}}$ such that $\phi = 1$ on a neighborhood of x_0, and such that (2.7) holds for some $A, H > 0$. Then the right hand side of (2.2) implies that for $h = H^{-\frac{\tau}{\sigma-1}} e^{\frac{\tau}{\sigma}} \Big(\dfrac{\sigma-1}{\tau\sigma}\Big)^{\frac{\tau}{\sigma-1}}$ we have

$$\begin{aligned} &\exp\Big\{ -\Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}(H\ln(e+|\xi|))\ln^{\frac{\sigma}{\sigma-1}}(e+|\xi|)\Big\} \\ &\qquad \le e^{-T_{\tau,\sigma,h}}(e+|\xi|) \le \frac{(1/h)^N N^{\tau N^\sigma}}{(e+|\xi|)^N} \quad N \in \mathbf{N}, \xi \in \Gamma, \qquad (2.9) \end{aligned}$$

and (2.6) easily follows after applying inequality (2.8). □

3. Paley-Wiener theorems

In this section we discuss some Paley-Wiener theorems for extended Gevrey functions with compact support of Roumieu and Beurling type. In particular, by the use of Theorem 2.1 we can prove the following.

Theorem 3.1. *Let $\tau > 0$, $1 < \sigma < 2$, U be open set in $\mathbf{R}^d$ and $K \subset\subset U$. If $\varphi \in \mathcal{D}^K_{\tau,\sigma,h}$ then its Fourier-Laplace transform $\widehat{\varphi}$ is an entire function which satisfies*

$$|\widehat{\varphi}(\eta)| \leq A_{\tau,\sigma,h} \exp\Big\{ -\Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}\Big(\Re\Big(\frac{1}{2eh\sqrt{d}}, e+|\eta|\Big)\Big)\cdot \cdot \ln^{\frac{\sigma}{\sigma-1}}(e+|\eta|) + H_K(\eta)\Big\}, \quad h>0, \eta \in \mathbf{C}^d, \quad (3.1)$$

for some $A_{\tau,\sigma,h} > 0$, where $H_K(\eta) = \sup_{y\in K} \operatorname{Im}(y\cdot\eta)$.

Conversely, if an entire function $\widehat{\varphi}$ satisfies

$$|\widehat{\varphi}(\eta)| \leq A_{\tau,\sigma,h} \exp\Big\{ -\Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}\Big(\Re\Big(\frac{2^\tau}{h}, e+|\eta|\Big)\Big)\cdot \cdot \ln^{\frac{\sigma}{\sigma-1}}(e+|\eta|) + H_K(\eta)\Big\}, \quad h>0, \eta \in \mathbf{C}^d,$$

for some $A_{\tau,\sigma,h} > 0$ then $\widehat{\varphi}(\eta)$ is the Fourier-Laplace transform of some function $\varphi \in \mathcal{D}^K_{2^{\sigma-1}\tau,\sigma,h}$.

Here we omit the proof, and the complete version of Theorem 3.1 is given in [12].

As an immediate consequence of Theorem 3.1 we obtain a Paley-Wiener theorem for $\mathcal{D}^K_{\{\tau,\sigma\}}$ and $\mathcal{D}^K_{(\tau,\sigma)}$ as follows:

Corollary 3.1. *Let $\tau > 0$, $1 < \sigma < 2$, U be open set in $\mathbf{R}^d$ and $K \subset\subset U$. If $\varphi \in \mathcal{D}^K_{\{\tau,\sigma\}}$ (resp. $\varphi \in \mathcal{D}^K_{(\tau,\sigma)}$) then its Fourier-Laplace transform is an entire function, and there exists constants $A, B > 0$ (resp. for every $B > 0$ there exists $A > 0$) such that*

$$|\widehat{\varphi}(\eta)| \leq A \exp\Big\{ -\Big(\frac{\sigma-1}{\tau\sigma}\Big)^{\frac{1}{\sigma-1}} W^{-\frac{1}{\sigma-1}}\Big(B\ln(e+|\eta|)\Big)\cdot \cdot \ln^{\frac{\sigma}{\sigma-1}}(e+|\eta|) + H_K(\eta)\Big\}, \quad h>0, \eta \in \mathbf{C}^d, \quad (3.2)$$

where $H_K(\eta) = \sup_{y\in K} \operatorname{Im}(y\cdot\eta)$.

Conversely, if there exists $A, B > 0$ (resp. for every $B > 0$ there exists $A > 0$) such that an entire function $\widehat{\varphi}$ satisfies (3.2) then it is the Fourier-Laplace transform of a function $\varphi \in \mathcal{D}^K_{\{2^{\sigma-1}\tau,\sigma\}}$ (resp. $\mathcal{D}^K_{(2^{\sigma-1}\tau,\sigma)}$).

To conclude the paper we prove the following version of Paley-Wiener theorem.

Theorem 3.2. *Let $1 < \sigma < 2$, U be open set in $\mathbf{R}^d$ and $K \subset\subset U$. Then the entire function $\widehat{\varphi}(\eta)$, $\eta \in \mathbf{C}^d$, is the Fourier-Laplace transform of*

$$\varphi \in \varinjlim_{\tau\to\infty} \mathcal{D}^K_{\tau,\sigma}$$

if and only if there exist constant $A, C > 0$ such that

$$|\widehat{\varphi}(\eta)| \leq A\exp\{-C\frac{\ln^{\frac{\sigma}{\sigma-1}}(e+|\eta|)}{\ln^{\frac{1}{\sigma-1}}(\ln(e+|\eta|))}+H_K(\eta)\}, \tag{3.3}$$

where $H_K(\eta)=\sup_{y\in K}\operatorname{Im}(y\cdot\eta)$.

Proof. Arguing in the similar way as in the proof of Proposition 1.1 (cf. [9]) we can conclude that for $0<\tau_1<\tau_2$ and $\sigma>1$ it holds

$$\mathcal{D}^K_{\{\tau_1,\sigma\}} \hookrightarrow \mathcal{D}^K_{(\tau_2,\sigma)} \hookrightarrow \mathcal{D}^K_{\{\tau_2,\sigma\}}$$

and therefore

$$\varinjlim_{\tau\to\infty} \mathcal{D}^K_{(\tau,\sigma)} = \varinjlim_{\tau\to\infty} \mathcal{D}^K_{\{\tau,\sigma\}}.$$

Hence it is sufficient to consider only the Roumieu case $\mathcal{D}^K_{\{\tau,\sigma\}}$.

Let $1<\sigma<2$ and $\varphi\in\varinjlim_{\tau\to\infty}\mathcal{D}^K_{\tau,\sigma}$. Then there exists $\tau_0>0$ such that $\varphi\in\mathcal{D}^K_{\{\tau_0,\sigma\}}$ and Corollary 3.1 implies that there exists constants $A,B,C>0$ such that

$$|\widehat{\varphi}(\eta)| \leq A\exp\Big\{-CW^{-\frac{1}{\sigma-1}}\Big(B\ln(e+|\eta|)\Big)\ln^{\frac{\sigma}{\sigma-1}}(e+|\eta|)+H_K(\eta)\Big\},$$

for every $\eta\in\mathbf{C}^d$. Using (2.5) we have

$$W^{-\frac{1}{\sigma-1}}\Big(B\ln(e+|\eta|)\Big)\ln^{\frac{\sigma}{\sigma-1}}(e+|\eta|) \asymp \frac{\ln^{\frac{\sigma}{\sigma-1}}(e+|\eta|)}{\ln^{\frac{1}{\sigma-1}}(B\ln(e+|\eta|))} \asymp$$

$$\asymp \frac{\ln^{\frac{\sigma}{\sigma-1}}(e+|\eta|)}{\ln^{\frac{1}{\sigma-1}}(\ln(e+|\eta|))},$$

as $|\eta|\to\infty$, where the second behavior follows from

$$\ln(B\ln(e+|\eta|)) \asymp \ln(\ln(e+|\eta|)), \quad |\eta|\to\infty,$$

and (3.3) follows.

Conversely, if $\widehat{\varphi}$ satisfies (3.3) for $1<\sigma<2$ and for some $A,C>0$, then Corollary 3.1 implies that $\varphi\in\mathcal{D}^K_{2^{\sigma-1}\tau_0,\sigma}$ for $\tau_0=C^{1-\sigma}\dfrac{\sigma-1}{\sigma}$. Therefore $\varphi\in\varinjlim_{\tau\to\infty}\mathcal{D}^K_{\{\tau,\sigma\}}$ and the proof is finished. □

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